Aggregation of Heavy-Tailed On-Off Flows is Multifractal

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Abstract- **In this paper we show that the aggregation of heavy-tailed on-off flows can exhibit multifractalility in nature in addition to the well-known long-range dependency. Justifications are provided through analyzing the power density spectrum of a single on-off flow, calculating the multifractal spectrum of aggregated flows, and measuring the distributions of the increment process. Effects of the on and the off period on the multifractal behavior are examined. With a new understanding of the on-off model taking into account the TCP window-based control mechanism, possible roles of the TCP window size and the RTT in producing multifractal traffic are suggested.**

I. INTRODUCTION

Self-similar and multiscaling traffic have aroused much interest because of their special nature and complex effects on network performance [1] [4] [7] [12]. Nevertheless, due to their complexity, many problems are still open. It has been known that aggregation of many on-off flows with heavy-tailed on or off distributions can exhibit long-range dependence. However, explanations for the multifrctal behavior in small scales are much more ambiguous. A well-known multifratal model is the conservative cascade [5] [11], which can generates multifractal data by multiplicatively distributing an initial quantity. Based on it, paper [5] and [11] suggest that some similar multiplicative mechanisms may lie in the network and are responsible for the multifractalility. However, it is still unclear where they exactly are. In this paper we give a different view. We show that aggregation of heavy-tailed on-off flows is in nature multifractal as well as long-range dependent. Consequently, it may be unnecessary to resort to the multiplicative mechanism to understand the multifractal behavior.

The rest of this paper is organized as follows. In Section 2 the mathematical background for multifractal processes and wavelet-based multifractal analysis is briefly reviewed. Section 3 explores the inherent multifractality of aggregated on-off flows through analyzing the power density spectrum of a single flow. Section 4 tests the multifractality by calculating the multifractal spectrum and measuring the distributions of the increment process. Section 5 reconsiders the physical implication of the on-off model for TCP traffic, and examines the effects of on and off components on the multifractal behavior. Section 6 concludes the paper.

II. PRELIMINARY

A. *Multifractal Processes*

A multifractal process is a scaling process with many scaling exponents in small time scales. Very simply speaking, a scaling process $Y(t)$ is a process that has the following property in certain time scales:

$$\Delta Y(t) = Y(t+\Delta t) - Y(t) \sim \Delta t^{\alpha(t)} \qquad (1)$$

If there is only one scaling exponent, i.e., $\alpha(t)$ is a constant $\alpha(t)=\alpha$, the process is mono-fractal with Hurst parameter $H=(\alpha+1)/2$. By contrast, the scaling structure of a multifractal process is much richer. It has multiple or even infinite scaling exponents, and they may vary with time and realization. The scaling structure can be described statistically with a multifractal spectrum. To show what the multifractal spectrum is, let us consider a normalized time range I = [0, 1]. Cut it into 2^n equal intervals with the k-th interval being $I_n^{k_n} = [k2^{-n},\ (k+1)2^{-n}]$, k = 0, 1, ..., 2^n-1. The approximate scaling exponent for the the k-th interval is

$$\alpha_{2^n}^k \doteq \frac{\log_2 |Y((k+1)2^{-n}) - Y(k2^{-n})|}{-n}$$

The exact scaling exponent at $t \in I_{2^n}^k$ can be obtained as

$$\alpha(t) = \lim_{k2^{-n}\to t} \alpha_{2^n}^k$$

Obviously, $\alpha(t)$ may take many different values in the whole range of I. Define

$$p_n^k(\alpha,\varepsilon) = \begin{cases} 1, & \alpha_{2^n}^k \in (\alpha-\varepsilon, \alpha+\varepsilon) \\ 0, & otherwise \end{cases}$$

Let

$$N_n(\alpha,\varepsilon) = \sum_{k=0}^{2^n-1} p_{2^n}^k(\alpha,\varepsilon)$$

$N_n(\alpha,\varepsilon)$ is the number of $\alpha_{2^n}^k$ taking values within $(\alpha-\varepsilon, \alpha+\varepsilon)$. The multifractal spectrum is defined as

$$f_G(\alpha) = \lim_{\varepsilon\to 0}\lim_{n\to\infty} \frac{\log_2 N_n(\alpha,\varepsilon)}{n} \qquad (2)$$

So $f_G(\alpha)$ is the frequency that $\alpha(t)$ takes value of α. It is formally called the large deviation multifractal spectrum.

In practice, a more numerically accessible multifractal spectrum, the Legendre spectrum, is often used as a substitute. It is calculated through the moments of $\Delta Y(t)$. It can be shown that for a multifractal process this holds

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$$E[|\Delta Y(t)|^q] \sim \Delta t^{\inf_\alpha (q\alpha - f_G(\alpha))}, \quad \Delta t \to 0$$

Let

$$\tau(q) = \inf_\alpha (q\alpha - f_G(\alpha)) \tag{3}$$

(3) indicates $\tau(q)$ is just the Legendre transform of $f_G(\alpha)$. It is called the structure function. It can obtained as

$$\tau(q) = \lim_{\Delta t \to 0} \frac{\log_2 E[|\Delta Y(t)|^q]}{\log_2 \Delta t} \tag{4}$$

$f_G(\alpha)$ can then be estimated through the backward Legendre transform of $\tau(q)$. Namely, let

$$f_L(\alpha) = \inf_q (q\alpha - \tau(q)) \tag{5}$$

and use $f_L(\alpha)$ as an approximation of $f_G(\alpha)$. Mathematically, $f_G(\alpha) \leq f_L(\alpha)$; when $\tau(q)$ exists and is differentiable for all real q, $f_G(\alpha) = f_L(\alpha)$ holds.

B. Wavelet-Based Multifractal Analysis

Wavelet analysis provides a very useful tool for calculating the Legendre multifractal spectrum. Suppose $Y(t)$ is the cumulative load of a traffic process $X(t)$. Then the wavelet coefficients of $X(t)$ can be viewed as samples of the increment process $\Delta Y(t)$. Thus the estimation of the $E[|\Delta Y(t)|^q]$ in (5) can be based on the wavelet coefficients. Using the Harr wavelet

$$\psi(t) = \begin{cases} 1, & 0 \leq t < 1/2 \\ -1, & 1/2 \leq t < 1 \\ 0, & otherwise \end{cases}$$

the wavelet coefficient of $X(t)$ is

$$d_{j,k} = \langle X, \psi_{j,k} \rangle = \frac{1}{\sqrt{2^j}} \int_{-\infty}^{+\infty} X(t) \psi_{j,k}(t) dt$$

Here, $\psi_{j,k}(t) = 2^{-j/2} \psi(2^{-j/2} t - k)$ is the normalized affine mapping of $\psi(t)$ at scale j (resolution 2^j) and translation k. Define

$$e_j^{(q)} = \frac{1}{n_j} \sum_k |d_{j,k}|^q \tag{6}$$

where n_j is the number of wavelet coefficients at scale j. $e_j^{(q)}$ is called the partition function. The structure function can be estimated as

$$\tau(q) = \lim_{j \to -\infty} \frac{\log_2 E[e_j^{(q)}]}{j} \tag{7}$$

Then the Legendre spectrum is calculated with (5). The expectation of the second-order partition function is exactly the energy of $X(t)$ at scale j:

$$E[e_j^{(2)}] = E[\frac{1}{n_j} \sum_k |d_{j,k}|^2] \tag{8}$$

III. Multifractal Nature of the On-Off Process

In this Section we explore the multifractality of heavy-tailed on-off flows by analyzing the power density spectrum of a flow. Assume both the on and the off periods have Pareto distributions. A Pareto random variable is represented as (b, a), where b is the scale parameter and a is the shape parameter. This covers a special case (c, ∞) which means a fixed number c. Let T_1 and T_2 are the durations of the on and the off period. Thus $T_i = (b_i, a_i)$ with mean $\mu_i = b_i a_i/(a_i - 1)$, $i = 1, 2$. Assume μ_i is finite, which requires $a_i > 1$. The on-off process is given as

$$X(t) = \begin{cases} 1, & \text{if } t \text{ is in on period} \\ 0, & \text{if } t \text{ is in off period} \end{cases} \tag{9}$$

It can be shown that an on-off process can be made stationary by adding certain delay at the initial [6]. We simply assume it is stationary here. Its autocorrelation function is

$$R_{xx}(\tau) = E[X(\tau)X(t+\tau)] = E[X(0)X(\tau)]$$

It is easy to see that $X(t)X(t+\tau) = 1$ holds only when $X(t)$ is 1 at both time t and $t+\tau$, and in all other cases $X(t)X(t+\tau) = 0$. So

$$\begin{aligned} R_{xx}(\tau) &= P[X(0) = 1, X(\tau) = 1] \\ &= P[X(\tau) = 1 \mid X(0) = 1] \cdot P[X(0) = 1] \end{aligned} \tag{10}$$

$P[\cdot]$ denotes the probability. We know

$$P[X(0) = 1] = P[X(\tau) = 1] = \frac{\mu_1}{\mu_1 + \mu_2} \tag{11}$$

Denote

$$\pi_{11}(\tau) = P[X(\tau) = 1 \mid X(0) = 1] \tag{12}$$

From renewal theory [3] we have

$$\pi_{11}(\tau) = \int_\tau^\infty \frac{\bar{F}_1(t)}{\mu_1} dt + \int_0^\tau h_{12}(t) \bar{F}_1(\tau - t) dt \tag{13}$$

Here, $\bar{F}_1(t)$ is the complementary distribution function of the on period. $h_{12}(t)$ is the renewal density of the on period given that the flow is in off state at $t = 0$. Making Laplace transform on both sides, (13) becomes

$$\Pi_{11}(s) = \frac{1}{s} - \frac{1 - L_1(s)}{\mu_1 s^2} + H_{12}(s) \frac{1 - L_1(s)}{s} \tag{14}$$

$\Pi_{11}(s)$, $L_1(s)$, and $H_{12}(s)$ are the Laplace transforms of $\pi_{11}(t)$, T_1, and $h_{12}(t)$ It can be shown

$$H_{12(s)} = \frac{L_2(s)(1 - L_1(s))}{\mu_1 s (1 - L_1(s) L_2(s))}$$

We then get the Laplace transform of $R_{xx}(\tau)$:

$$S_{xx}(s) = \frac{\mu_1}{(\mu_1 + \mu_2)s} - \frac{(1 - L_1(s))(1 - L_2(s))}{(\mu_1 + \mu_2) s^2 (1 - L_1(s) L_2(s))} \tag{15}$$

Here, $L_2(s)$ is the Laplace tranform of T_2. It can be obtained

$$L_i(s) = a_i b_i^{a_i} \tilde{\Gamma}(-a_i, s b_i) s^{a_i}$$

where $\tilde{\Gamma}(a, b)$ is the incomplete Gamma function

$$\tilde{\Gamma}(a,b) = \int_b^{\infty} x^{a-1} e^{-x} dx \, , \, b > 0$$

When $|s| \to \infty$, from (15) we have

$$S_{xx}(s) \approx \frac{\mu_1}{(\mu_1 + \mu_2)s} - \frac{1}{(\mu_1 + \mu_2)s^2} \qquad (16)$$

From the relation between the Laplace transform and the Fourier transform, we can get the power density spectrum at high frequencies ($f \to \infty$):

$$F(f) = S_{xx}(j2\pi f) \approx \frac{1}{4\pi^2(\mu_1+\mu_2)f^2} - \frac{\mu_1}{2\pi(\mu_1+\mu_2)f} i \qquad (17)$$

The magnitude of $F(f)$ is

$$A(f) \approx \frac{\sqrt{4\pi^2\mu_1^2 f^2 + 1}}{4\pi^2(\mu_1+\mu_2)f^2} = \frac{\sqrt{4\pi^2\mu_1^2 + 1/f^2}}{4\pi^2(\mu_1+\mu_2)f} \qquad (18)$$

To test whether $X(t)$ is monofractal, fit $A(f)$ to the frequency-domain scaling function, i.e., let

$$A(f) = c_f f^{-\alpha(f)}$$

c_f can be obtained by letting $f = 1$ on both sides of (18).

$$c_f = \frac{\sqrt{4\pi^2\mu_1^2 + 1}}{4\pi^2(\mu_1 + \mu_2)} \qquad (19)$$

Finally, we get

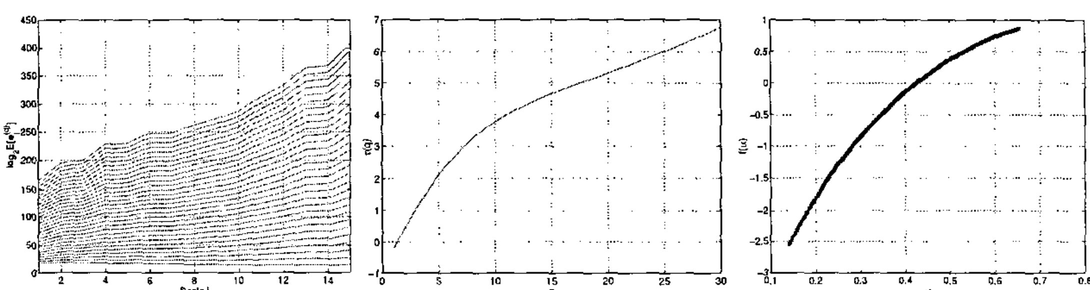
$$\alpha_{(f)} = 1 + \frac{\log_2 \frac{\sqrt{(4\pi^2\mu_1^2+1)(4\pi^2\mu_1^2+1/f^2)}}{4\pi^2(\mu_1+\mu_2)}}{\log_2 f} \qquad (20)$$

Depending on the values of μ_1 and μ_2, the second term on the right-hand side of (20) can be within (−1, 0). Then $\alpha_{(f)}$ is a scaling exponent taking value between 0 and 1 and varies with f. It is easy to see from (20) that when $\mu_2 >> \mu_1$, there is a big value space of (μ_1, μ_2) that allows $0<\alpha_{(f)}<1$. We will show in Section 5 that this is just the case of a typical TCP flow in WAN.

When there are many different on-off flows, the overall traffic has many different scaling exponents in small scales. Thus multifractality appears. Cross dependence among flows make the scaling structure even richer. In particular, if the flows arrive in a Poisson process and each lasts for a period that has a heavy-tailed distribution, the number of flows varying with time forms a M/G/∞ process [9]. The overall traffic process can be viewed as the product of a M/G/∞ process and an on-off process. The overall power density spectrum is thus the convolution of the spectra of the two processes. From the property of convolution, scaling exponents in the spectrum of the on-off process will remain in the final spectrum, but their weights will be re-assigned. New scaling exponents may also be introduced through the convolution. So the aggregated traffic has a considerably big collection of scaling exponents.

IV. TEST OF MULTIFRACTALITY

To verify the conclusion in Section 3, we calculate the multifractal spectrum for aggregated on-off flows using the wavelet-based approach described in Section 2.2. The traffic data are generated by adding N flows at each clock cycle. These flows start randomly and remain alive thereafter. A quite large number of initial data are cut out to guarantee stationarity. The final data length is M. Parameters chosen to generate the data are: $T_1 = (16, \infty)$, $T_2 = (512, 1.2)$, $N = 100$, and $M = 2^{20}$. Figure 1 gives results for the partition function $e_j^{(q)}$, the structure function $\tau(q)$, and the Legendre spectrum $f_L(\alpha)$. We see the curves of $e_j^{(q)}$ have different slopes for different q, $\tau(q)$ is concave, and $f_L(\alpha)$ includes a rich set of α. All these show the data are multifractal. As paper [2] indicated, the distributions of increment process also give information about the scaling property. If increments at any time scale j, rescaled by $2^{-j(H-1/2)}$, where H is the long-range Hurst parameter, have Gaussian distributions, and the probability density functions of all scales collapse onto a single curve, the process is monofractal. Otherwise, multiple scaling exponents are included. Figure 2 gives the probability density functions of rescaled increments for scale 1 to 6. Clearly, the smaller the scale is, the more distant the distribution is from Gaussian. They do not collapse onto a single curve.



(a) (b) (c)

Fig. 1. Partition function (a) (for q = 1 to 30 bottom-up), structure function (b), and multifractal spectrum (c) of an aggregation of heavy-tailed on-off flows

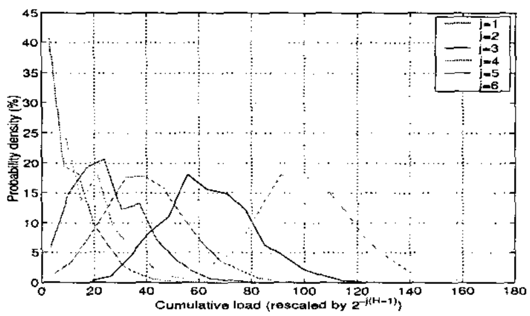


Fig. 2. Distributions of the increment process of an aggregation of on-off flows at different scales (j = 1 to 6, from left to right)

V. Physical Interpretation

On-off models have been long used to model traffic in packet-switched networks. Usually the on period is thought of as being corresponding to the packet and the off period corresponding to the inter-packet silence. While this is suitable for the UDP flow, it does not properly reflect the case of the TCP connection. Since TCP uses a window-based congestion control, packets of the same window are sent in bulk at the source node. They form a packet cluster moving in the network. Packets are tighter within a cluster than between clusters. So a TCP flow can be more accurately modeled as an embedded on-off model: a coarse on-off process profiles packet clusters and inter-cluster silences, and a finer on-off process describes the details within a single cluster. It is easy to see the coarse level actually embodies the roles of the round trip time (RTT) and the TCP window size. In a WAN setting, the packet cluster is much shorter than the inter-cluster interval, and the latter is approximately the RTT. As we have shown elsewhere [8], the cluster level is more important to the small scale behavior than the packet level. Modeling the TCP connection at only the coarse level produces a very similar result to that of the full model. Obviously, the relation $\mu_2 >> \mu_1$ holds in the coarse level on-off model of the TCP connection. This supports the rationality in Section 3. Our choice of the parameters in Section 4 also follows this guideline.

To understand the roles of packet cluster and the RTT in affecting traffic behaviors, let's see the effects of the on and the off period on the energy density and the multifractal spectrum. Figure 3 gives the logsale energy vs. time scale and the multifratal spectra for different b_l. Figure 4 shows those for different a_l. In theses figures all other parameters are the same with those given in Section 4. We can see that b_l, which can be also interpreted as an approximation of μ_l when a_l is big, controls the energy allocation between small scales and large scales. The scaling exponents become smaller accordingly when b_l decreases. a_l has a dramatic domain-dependent effect. In the domain of $a_l > 2.0$, the multifractal spectrum shrinks with the decrease of a_l. Then when $a_l < 2.0$, the multifractal spectrum completely disappears. The reason for this needs further analysis. Intuitively, if T_2 keeps still, the decrease of a_1 will enlarge the occupancy of the on period in the on-off process. This will reduce the spikiness of the process and make it more Gaussian noise like. In contrast to the active role of the on period, the off period, either b_2 or a_2, does not show significant impact on small scale behaviors. It mainly controls the long-range behavior, as much previous work has indicated [10] [14]. Results about it are omitted here.

Going back to the network context, the effects of the on and the off period suggest that the dynamics of the TCP window size governs the small scale behavior. We know that the TCP window size changes in a quasi-periodical manner [13], which is a rather conservative variation from the point of view of random processes. When this variation is approximated with the Pareto random variable T_1, a_l would be surely above 2.0. a_l being big also makes sure $\mu_2 >> \mu_1$. Thus, it is the conservative variation of the TCP window size that makes the multifractal traffic realistic. Of course, this is in the condition that the RTT is heavy-tailed. As we have known, it is the case in WAN.

VI. Conclusions

In this paper we show that the aggregation of many heavy-tailed on-off flows may exhibit multifractalility in nature. This is done through analyzing the scaling behavior of a single on-off flow, calculating the multifractal spectrum of aggregated flows, and measuring the distributions of the increment process. Effects of the on and the off period on the multifractal behavior are examined. It is shown that both the average value and the variability of the on period are critical to the multifractality. By contrast, the off period has no significant impact. We reconsider the physical implication of the on-off model taking into account the TCP control mechanism, and suggest that, if a standard, non-embedded on-off process is used to model a TCP flow, it may be more properly viewed as a description of packet clusters rather than the packets. This is critical to physical interpretation of the multifractal traffic. With this understanding, we indicate that the dynamics of the TCP window size largely decides the multifractality of the TCP traffic.

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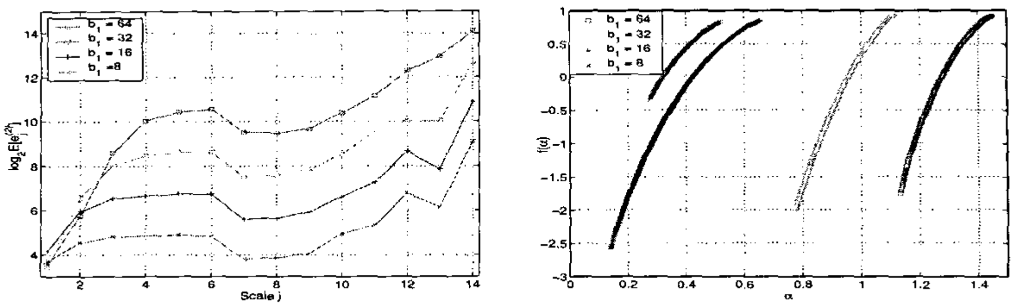
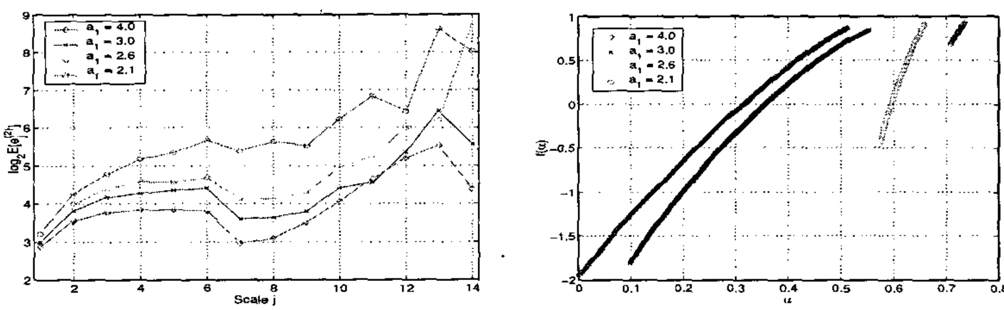


(a) (b)

Fig. 3. Logsacale energy vs. time scale (a) and multifractal spectra (b) for different minimum values of the on period



(a) (b)

Fig. 4. Logsacale energy vs. time scale (a) and multifractal spectra (b) for different variabilities of the on period

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