95-03

PROCEEDINGS

SPIE—The International Society for Optical Engineering

Wavelet Applications II

Harold H. Szu
Chair/Editor

17–21 April 1995
Orlando, Florida

SPIE PROCEEDINGS SERIES

Volume 2491
Part One of Two Parts

Image Compression Using Optimal Wavelet Basis

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ABSTRACT

We study the problem of choosing an image based optimal wavelet basis with compact support for image data compression and provide a general algorithm for computing the optimal wavelet basis. We parameterize the mother wavelet and the scaling function of wavelet systems through a set of real coefficients of the relevant quadrature mirror filter (QMF) banks. We further introduce the concept of decomposition entropy as an information measure to describe the distance between a given digital image and its projection into the subspace spanned by the wavelet basis. The optimal basis for the given image is obtained through minimizing this information measure. The resulting subspace is used for image analysis and synthesis. A gradient based optimization algorithm is developed for computing the image based optimal wavelet basis. Experiments show improved compression ratios due to the application of the optimal wavelet basis and demonstrate the potential applications of our methodology in image compression. This method is also useful for constructing efficient wavelet based image coding systems.

Keywords: Image processing, image compression, signal processing, wavelets, pattern recognition, data compression, video, medical imaging, multimedia.

1 INTRODUCTION

The last few years have witnessed extensive research interest and activities in wavelet theory and its applications in signal processing, image processing and many other fields [1, 2]. The most attractive features of wavelet theory are the multiresolution property and time and frequency localization ability. The wavelet transform decomposes a signal into its components at different resolutions. Its application actually simplifies the description of signals and provides analysis at different levels of detail. There are many applications of these properties in the fields of signal processing, speech processing and especially in image processing [3, 4, 5, 6].

It is well known that a wavelet system is usually determined by one mother wavelet function whose dilations and shifts span the signal space. Unlike *sin* and *cos* functions, individual wavelet functions are quite localized in frequency and time and they are not unique. Obviously, different wavelets $\psi(t)$ shall yield different wavelet bases. An appropriate selection of the wavelet for signal representation can result in maximal benefits of this new technique. For example, compact wavelets are better for lower accuracy approximation or for approximation of discontinuous functions such as image compression while smooth wavelets are better for solution of integral functions to achieve accurate high numerical accuracy. It is reasonable to think that if a wavelet contains enough information about an image to be represented, the wavelet system is going to be simplified in terms of the levels of required resolution, which reduces the

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computational complexity and saves CPU time. We are interested in finding an image based wavelet basis and applying the resulting wavelet system to improve the compression ratio of the image.

The key to choosing an image based optimal wavelet basis lies in the appropriate parameterization and adequate performance measure in addition to the accurate interpretation of physical phenomena. A method was proposed for choosing a wavelet for signal representation based on minimizing an upper bound of the L^2 norm of error [7, 8] in approximating the signal up to the desired scale. Coifman *et al.* derived an entropy based algorithm for selecting the best basis from a library of wavelet packets [9]. We also proposed an information measure based approach for constructing an optimal discrete wavelet basis with compact support in our earlier work on adaptive wavelet neural networks [10] and wavelet basis selection [11]. We shall illustrate the application of our methodology to image compression.

This paper is intended to demonstrate that choosing an image based optimal or suboptimal wavelet basis can improve the compression ratio of images rather than to design a complete coding system. In the rest of the paper, we first provide the definition of optimal wavelet basis for a given digital image and parameterize the basis through the corresponding quadrature mirror filter (QMF) banks. We then introduce an algorithm for constructing an optimal wavelet basis. Next, we compare the effects of different mother wavelets on image representation and provide numerical results. Finally, we summarize our conclusions.

2 OPTIMAL WAVELET BASIS

We first introduce a distance measure for optimization purpose. Inspired by the work in [9], we define an additive information measure of entropy type and the optimal basis as the following. We use $\Psi(t)$ to denote the wavelet basis spanned by dilating and shifting the mother wavelet denoted by $\psi(t)$.

Definition 2.1 *A non negative map $\mathcal{M}$ from a sequence $\{f_i\}$ to R is called an additive information measure if $\mathcal{M}(0) = 0$ and $\mathcal{M}(\sum_i f_i) = \sum_i \mathcal{M}(f_i)$.*

Definition 2.2 *Let $x \in R^N$ be a fixed vector and $\mathcal{B}$ denote the collection of all orthonormal bases of dimension N, a basis $B \in \mathcal{B}$ is said to be optimal if $\mathcal{M}(Bx)$ is minimal for all bases in $\mathcal{B}$ with respect to the vector x.*

In the definition above, vector x contains digital image data to be compressed.

The wavelet system is parameterized through using QMF banks. From the multiresolution property of wavelets due to Mallat [12], the scaling function $\phi(t)$ and the mother wavelet $\psi(t)$ are expressed as [2]

$$\phi(t) = \sqrt{2} \sum_{k=-\infty}^{\infty} c_k \phi(2t-k) \tag{1}$$

and

$$\psi(t) = \sqrt{2} \sum_{k=-\infty}^{\infty} d_k \phi(2t-k) \tag{2}$$

where the coefficient $\{c_k\}$ and $\{d_k\}$ determine a low pass filter $h_0(k) = c_k$ and high pass filter $h_1(k) = d_k$. The Fourier transforms of filter h_0 and h_1 are denoted by H_0 and H_1, respectively. We need to consider the case when $H_0(z)$ is a causal FIR filter, i.e., there are only finitely many nonzero c_k for the filter. Without loss of generality, we assume that $c_k \neq 0$ when $k \in [0, K]$ where K is a positive odd integer. The condition for wavelet basis $\Psi(t)$ generated from QMF banks to be compactly supported and orthonormal is provided by the following theorem due to Vaidyanathan [13].

Theorem 2.1 [13] *Let $H_0(z)$ and $H_1(z)$ be causal FIR filters, then the scaling function $\phi(t)$ and the wavelet function $\psi(t)$ generated by the QMF bank are causal with finite duration Kb_0. Further, if $H_0(z)$ and $H_1(z)$ satisfy the paraunitary condition, $|H_0(1)| = \sqrt{2}$ and $H_0(e^{j\omega}) \neq 0$ while $|\omega| < \pi/2$, the wavelet functions $\psi_{j,l}(t)$ are orthonormal.*

This theorem imposes constraints on parameter $\{c_k\}$ to generate a compactly supported orthonormal wavelet basis. In particular, the cross-filter orthonormality implied by the paraunitary property, is satisfied by the choice of

$$H_1(z) = -z^{-K} H_0(-z^{-1}), \text{ K odd} \tag{3}$$

or in the time domain,

$$h_1(k) = (-1)^k h_0(K-k). \tag{4}$$

As we can see from the above, both the scaling function and the wavelet function depend on the selection of $\{c_k\}$ for $k \in [0, K]$. As a consequence, the dilations and shifts of the mother wavelet depend on the selection of this set of parameters subject to the paraunitary condition imposed on the filters of the QMF bank.

Definition 2.3 *Let H be a Hilbert space which is an orthogonal direct sum*

$$H = \oplus \sum H_i, \tag{5}$$

a map $\mathcal{E}$ is called decomposition entropy if

$$\mathcal{E}(v, \Psi) = -\sum \frac{\|v_i\|^2}{\|v\|^2} \log \frac{\|v_i\|^2}{\|v\|^2} \tag{6}$$

for $v \in H$, $\|v\| \neq 0$, such that

$$v = \oplus \sum v_i, v_i \in H_i, \tag{7}$$

and we set $p \log p = 0$, when $p = 0$.

The implication of using entropy as a performance measure takes advantage of the nonuniform energy distribution of the signal or image in consideration over its energy spectrum. For a source of a finite number of independent signals, such as a digital image considered as a source of independent pixels, its entropy is maximum for uniform distribution [14] . If the entropy value is less than the maximum, then, this implies that a higher concentration of the signal energy over certain bands exists.

We introduce a cost functional to facilitate the optimization process.

$$\lambda(\Psi, v) = -\sum_j \|v_j\|^2 \log \|v_j\|^2, \tag{8}$$

which relates to the decomposition entropy through

$$\mathcal{E}(v, \Psi) = \|v\|^{-2} \lambda(\Psi, v) + \log \|v\|^2 (2M+1). \tag{9}$$

The task for constructing an image based optimal wavelet basis becomes one of finding the appropriate filter coefficient $\{c_k\}$ such that the cost functional λ is minimized for the given image. The following theorem provides the analytical gradient of the cost functional (8).

Theorem 2.2 [11] *Let $\lambda(\cdot,\cdot)$ be the additive information measure and $[0, K]$ be the compact support for $\{c_k\}$ and Ψ be the corresponding wavelet basis from dilations and shifts of the wavelet $\psi(t)$. Let $f(t)$ be a fixed signal in $L^2(R)$. Then the gradient of the information measure with respect to the parameter set $\{c_k\}$ for the given signal is described by*

$$\begin{aligned}\frac{\partial\lambda(\Psi, f(t))}{\partial c_k} &= -\sqrt{2^{-j+2}}\sum_j\sum_l \log 2\,\|f_j\|^2 \\ &\quad\cdot f_{j,l}\sum_n\Big[(-1)^{K-k}\Big\langle f(t), \phi(2^{-j+1}t-2l-n)\Big\rangle \\ &\quad+(-1)^n c_{K-n}\Big\langle f(t), \phi(2^{-j+2}t-4l-2n-k)\Big\rangle\Big].\end{aligned} \tag{10}$$

This information gradient can be used in computing the filter coefficients for the optimal wavelet basis.

3 IMAGE COMPRESSION

In extending 1-D wavelet to 2-D image applications, we follow Mallat [12] in his hierarchical wavelet decomposition. The low pass and high pass filters are applied to both horizontal and vertical directions, respectively. We then threshold the resulting wavelet coefficient; we retain those coefficients whose absolute values exceed a predetermined, adjustable threshold. The retained wavelet coefficients are used to reconstruct the image. In this process, we assume that these coefficients can be transmitted and used precisely, since our purpose is to show the improvement from using the image based wavelet basis over using an basis before optimization. We define the compression ratio to be that of the number of retained coefficients and the number of wavelet coefficients.

The intuition of our using optimal wavelet coefficients comes from linear predictive coding (LPC) of speech signals. Instead of sending the coefficients of the AR model of speech signals, we optimize and send the filter parameters and wavelet coefficients for image analysis and synthesis. The gain in compression outweighs the overhead due to implementing the optimal wavelet basis.

We are interested in starting the optimization scheme based on a low order wavelet system. The smaller the support of the wavelet, the better it can capture the feature corresponding to edges. In general, the wavelet decomposition requires less hardware implementation than does the Fourier method. With a lower order system, the cost of implementation shall be further reduced. We first tested compressing the 512 by 512 Lena image by using Daubechies 20, 12 and 4 wavelets. At the same compression ratio, 0.032, the image represented by the Daub4 basis shows comparable quality when compared against those represented by the two higher order wavelet bases. As a consequence, we select fourth order filters in optimization process.

We have identified the problem of finding an optimal wavelet basis Ψ with that of finding the corresponding parameter set $\{c_k\}$ such that the additive information measure λ is minimized; once the set $\{c_k\}$ is determined, both the scaling function ϕ and mother wavelet function ψ can be derived afterwards. The information gradient is available from the theorem above and different optimization schemes can be applied to solve this problem. Next, comes a basis selection algorithm based on the steepest descent method [11].

Algorithm 3.1 *Computation of the optimal wavelet basis*

Step 1: Set $i := 1$,
$\lambda_0 := 0$,
Initialize vector C_0;

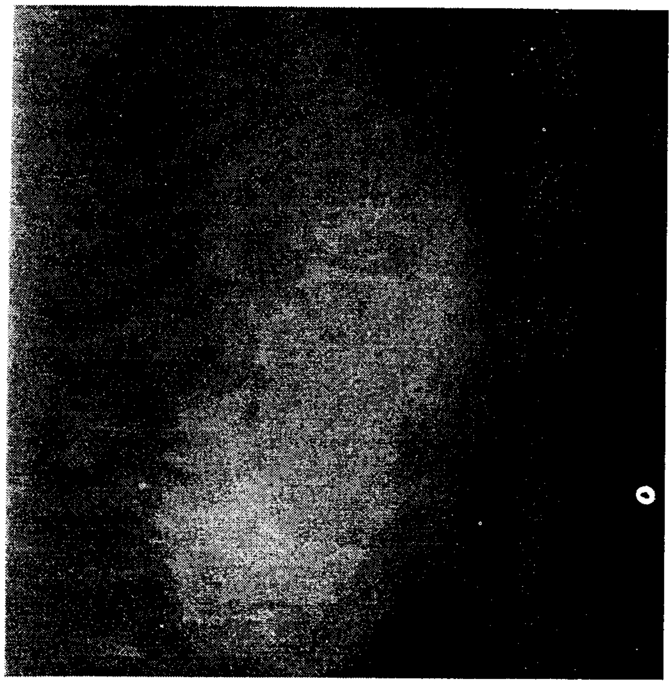

Figure 1: Original 512 by 512 mammographic image.

Input $f(t)$.
Step 2: $C_i := C_{i-1} + p_{i-1}\frac{\partial \lambda}{\partial C_{i-1}}$.
Step 3: Compute ϕ *and* ψ.
Step 4: Compute λ.
Step 5: If $|\lambda_i - \lambda_{i-1}| > \epsilon$,
$i := i + 1$, *go to Step 3.*
Step 6: Output the optimal basis Ψ *and stop.*

In the algorithm above, $f(t)$ represents the image data or signals and C denotes the parameter set $\{c_0 c_1 \cdots c_{K-l}\}$. One needs an initial parameter set as a starting point.

4 RESULTS

The optimization is applied to a digital mammographic image shown in Figure 1. This image is obtained through the Department of Radiology, Veterans Administration Medical Center in Baltimore. We choose Daubechies' fourth order wavelet coefficients as an initial parameter set to start the optimization procedure with the algorithm above. We denote Daubechies' fourth order wavelet and the optimized wavelet bases by Daub4 and Opt4, respectively. The coefficients of the two corresponding low pass filters are given in Table 1. The amplitude of the wavelet coefficients obtained with wavelet basis Opt4 is illustrated in Figure 2. The coefficients with larger amplitude concentrate on the low resolution region. The histogram in Figure 2 shows the distribution of the wavelet coefficients of the image with basis Opt4.

It is obvious that significant compression can be obtained by truncating the large number of small coefficients or by coding them with a few bits. One of the problems is to decide which nonzero wavelet coefficient corresponds to noise and which contains useful visual information to maximize the benefits

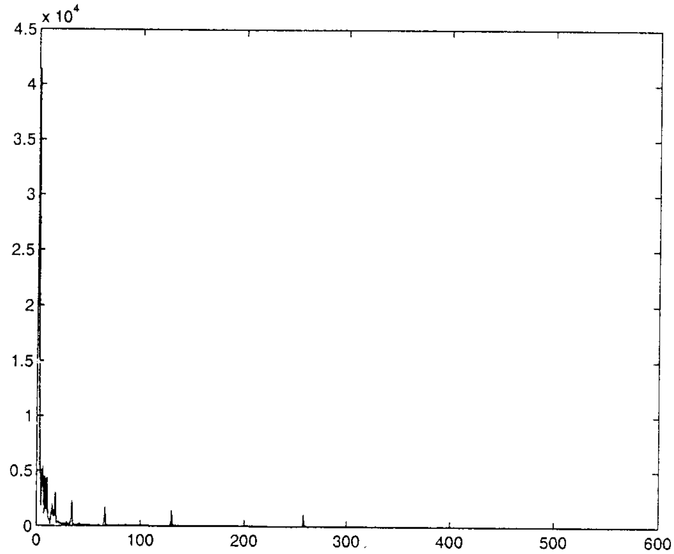

Figure 2: The amplitude of wavelet coefficients of the mammographic image using basis Opt4, listed from low resolution to high resolution components

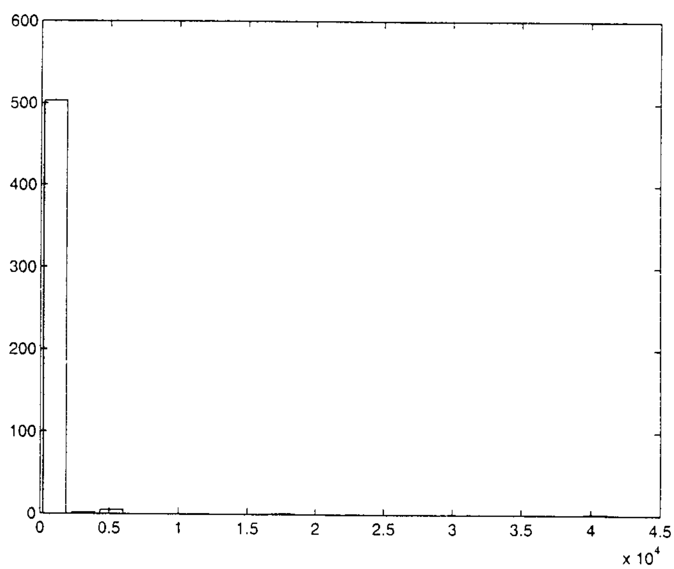

Figure 3: Histogram of the amplitude of wavelet coefficients of the mammographic image.

	Daub4	Opt4
c0	4.8296291e-01	5.2844307e-01
c2	8.3651630e-01	8.0297232e-01
c3	2.2414387e-01	1.8632579e-01
c4	-1.2940952e-01	-1.0352762e-01

Table 1: Daubechies 4 and Opt4 wavelet filter coefficients.

	λ	PSNR	Compression Ratio
Daub4	0.6995	46.3363	5.38063E-02
Opt4	0.6739	30.7888	4.87251E-02

Table 2: Entropy values, PSNR and compression ratios from employing Daub4 and Opt4 wavelet bases.

of implementing the optimal wavelet basis. As we mentioned previously, our purpose is to illustrate the effect of different wavelet bases on image compression, we truncate those wavelet coefficients whose magnitude are below an adjustable threshold and use the remaining coefficients to reconstruct the image. The threshold is selected by experiments. Different threshold values have been tested to choose one necessary to represent the image without perceptible loss in image quality. The quantitative results in terms of entropy values, peak signal noise ratios and compression ratios are listed in Table 2.

As we can see that a lower entropy value corresponds to a lower compression ratio with a certain loss in PSNR. The reconstructed images using wavelet Daub and Opt4 are illustrated in Figure 4 and Figure 5, respectively. The reconstructed image using basis Opt4 preserves the texture and edges in a level comparable to the one from using basis Daub4, but at a lower compression ratio. The improvement in the ratio is close to ten percent in this case. Although the PSNR is in favor of the Daub4 basis, the actual visual difference is not perceptible from the two images.

The reconstructed images above illustrate that finding an optimal wavelet basis can preserve the feature and improve compression. The resulting image based wavelet basis can be used to facilitate image data base search since it contains information regarding the image.

Like any gradient based optimization procedure, this method has its limitations. It often stops at a local minimum and results in a suboptimal solution. However, the suboptimal solution may still provide an acceptable parameter set. The wavelet coefficient truncation is used to reconstruct the images to compare the effect of using wavelet Daub4 and Opt4. The actual wavelet coding system design would include, in addition to finding the optimal basis, using different techniques such as the noise shaping bit allocation procedure [6] or hierarchical coding with the estimated local noise sensitivity of the human vision system(HVS) [15] among others.

5 CONCLUSIONS

This paper has provided a direct approach to construct an image based optimal orthonormal wavelet basis with compact support for image compression. The cost functional, an additive information measure, is introduced based on the decomposition entropy of the given image with respect to an initial wavelet basis. Using the resulting optimal wavelet basis improves the image compression ratio. The gain in compression outweighs the overhead due to implementing the optimal basis. The parameterization of the cost functionals described in this paper is helpful; other forms of measures or cost functions may be introduced depending on the contexts of actual problems.

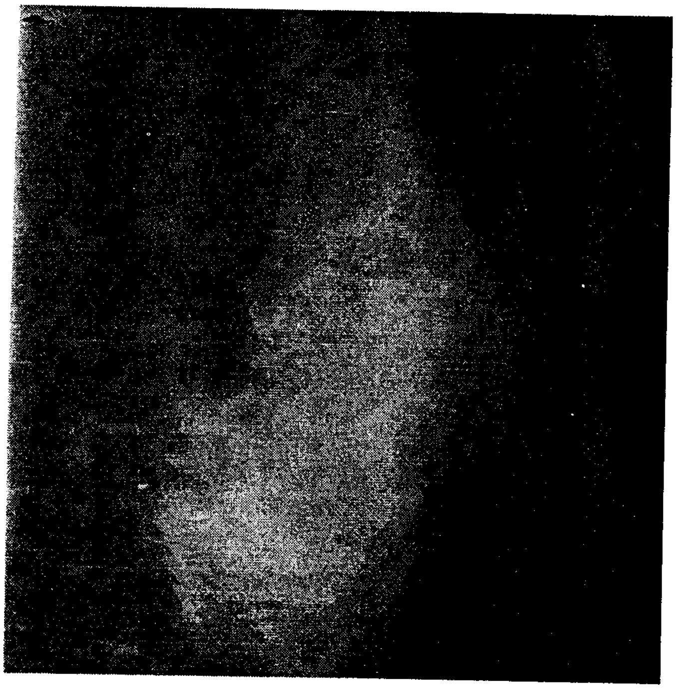

Figure 4: Reconstructed image using Daub4 wavelet, compression ratio 5.38063E-02.

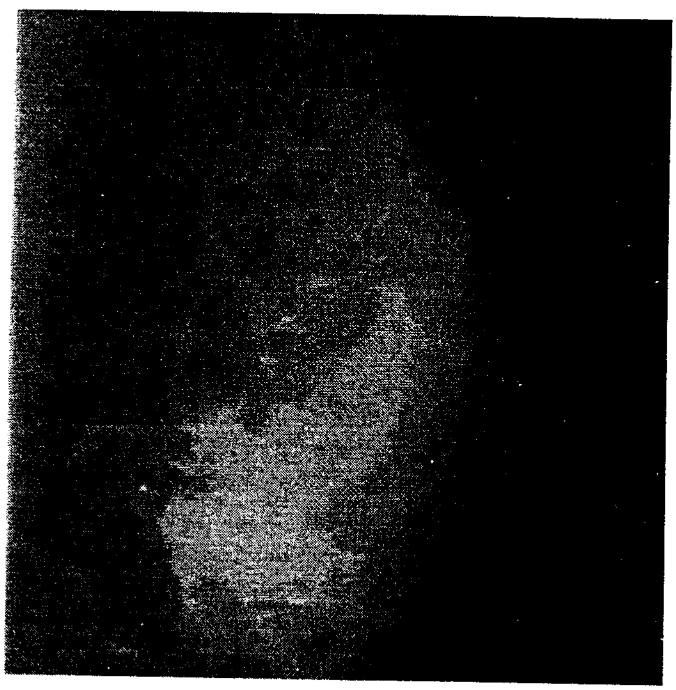

Figure 5: Reconstructed image using Opt4 wavelet, compression ratio 4.87251E-02.

This methodology of the optimal basis selection in a general setting is useful not only for image compression, signal approximation and reconstruction, but also for feature analysis, motion estimation in video and system identification. In the context of pattern recognition, it is also a way to construct the feature space and for partitioning the signal space according to its representatives.

Future work includes using the optimal wavelet basis for image feature extraction and analysis, and for designing the corresponding bit allocation scheme to maximize the benefits of implementing the signal based wavelet basis.

6 ACKNOWLEDGMENTS

The authors would like to thank Dr. E. Siegel of Department of Radiology, Veterans Administration Medical Center in Baltimore for providing the X-ray image. This research was partially supported by NSF grant NSFD CDR 8803012 through the Engineering Research Center's Program.

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