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# NONLINEAR FILTERING: The SET-MEMBERSHIP (BOUNDING) and the $H_\infty$ TECHNIQUES

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**Abstract.** We consider the problem of nonlinear filtering within the framework of deterministic uncertain systems (i.e. control systems with disturbances). We investigate the relationship between the two main approaches to the problems: the set-membership approach and the nonlinear $H_\infty$ approach. We establish an interesting connection between the two basic constructs of these two approaches: the information state and the information domain. This connection helps to establish a clearer understanding for the problem and will play a fundamental role in nonlinear robust output feedback control.

**Key Words.** nonlinear control, nonlinear filtering, nonlinear $H_\infty$, set-membership, information state.

## INTRODUCTION

At the present time, it appears, that two basic approaches have emerged for the deterministic treatment of uncertainty in the dynamics of controlled processes. The first of these is the "set-membership" or "bounding" approach based on the techniques of set-valued calculus where the uncertain items are taken to be unknown but bounded with given bounds and the performance range for the uncertain system is sought for in the form of a set [1-4]. The second one is the so-called $H_\infty$ approach based on the calculation of the minimal-norm disturbance-output map for the investigated system, the error bound for the system performance expressed through this norm and a differential game formulation [5-6,13-15]. Although formally somewhat different, these two approaches appear to show close connections, being figuratively "two sides of one coin". We shall demonstrate the specificities and the interconnections of these approaches through the treatment of the nonlinear filtering problem which will thus also serve as a "case study". The more general issue of applying the two approaches to the problem of output feedback control under uncertainty will be the topic of a separate publication. Some related results can be found also in [12].

## 1. THE NONLINEAR FILTERING PROBLEM

Consider a system described by the nonlinear ODE

$$\dot{x} = f(t,x) + c(t,x)v(t) \qquad (1.1)$$

$$y = g(t,x) + w(t) \qquad (1.2)$$

where $x \in R^n$ is the state vector, $y(t)$ the available measurement, $v(t) \in R^p, w(t) \in R^q$, the unknown "noises" or "disturbances" in the system and measurement inputs, $x(t_0) = x^0$ the unknown initial state. The functions $f(t,x), c(t,x)$ are taken to be such that they ensure existence and uniqueness of solutions.

The unknown items $\zeta(\cdot) = \{x^0, v(t),\ w(t), t_0 \le t \le \tau\}$ may be assumed to be bounded by the inequality

$$\Psi(\tau,\zeta(\cdot)) = \int_{t_0}^{\tau} \psi(t,v(t),w(t))dt + \phi(x^0) \le \mu^2 \qquad (1.3)$$

where, particularly, the bounds may be of *the quadratic integral type*, namely, such that

$$\phi(x^0) = (x^0 - a, P_0(x^0 - a)) \qquad (1.4)$$

$$\begin{aligned} \psi(t,v(t),w(t)) &= \qquad (1.5) \\ &= (v(t) - v^*(t), M(t)(v(t) - v^*(t)))^2 \\ &+ (w(t) - w^*(t), N(t)(w(t) - w^*(t)))^2 \le \mu^2 \end{aligned}$$

where $(p,q)$ $(p,q \in R^k)$, stands for the scalar product in the respective space $R^k$; $a \in R^n$ is a given vector; $v^*(t)$, $w^*(t)$ are given functions of respective dimensions, square-integrable in $t \in [t_0, \tau]$; $M(t), N(t)$ are positive definite, continuous, and $P_0 > 0$.

Another common type of restriction is given by *magnitude bounds*, a particular case of which is described by the inequalities for $t \in [t_0, \tau]$

$$I_0(x^0) = (x^0 - a, P_0(x^0 - a)) \leq \mu^2 \quad (1.6)$$

$$I_1(\tau, v(\cdot)) = \text{esssup}_t(v(t) - v^*(t), M(t)(v(t) - v^*(t)) \leq \mu^2, \quad (1.7)$$

$$I_2(\tau, w(\cdot)) = \text{esssup}_t(w(t) - w^*(t), N(t)(w(t) - w^*(t)) \leq \mu^2, \quad (1.8)$$

In this case the functional is

$$\Psi(\tau, \zeta(\cdot)) = \max\{I_0, I_1, I_2\} \quad (1.9)$$

The aim of the filtering problem could be described as follows:

(a) *Determine an estimate* $x^0(\tau)$ *for the unknown state* $x(\tau)$ on the basis of the available information: the system parameters, the measurement $y(t)$, $t \in [t_0, \tau]$, and the restrictions on the uncertain items $\zeta(\cdot)$ (if these are specified in advance).

(b) *Calculate the error bounds* for the estimate $x^0(\tau)$ on the basis of the same information.

(c) *Describe the evolution of the estimate* $x^0(\tau)$) *and the error bound in* $\tau$, preferably through a dynamic recurrence-type relation, an ODE -"filter", for example, if possible.

## 2. THE SET-MEMBERSHIP (BOUNDING) AND THE $H_\infty$ APPROACHES

Suppose that the constraints (1.3) with specified $\mu$ are given together with the available measurement $y = y^*(t)$, $t \in [t_0, \tau]$. The first, or "set-membership" approach then requires that the solution to the problem would be given through *the information domain* $X(\tau)$ of states $x$ consistent at instant $t = \tau$ with the system equations, the measurement $y^*(t)$ and the constraints on $\zeta(\cdot)$. With $X(t)$ calculated, one may be certain that for the unknown actual value $x(\tau)$ we have: $x(\tau) \in X(\tau)$, and may therefore find a certain point $x^*(\tau) \in X(\tau)$ that would serve as the required estimate for $x(\tau)$. This point $x^*$ may be particularly selected as the "Chebyshev center" of $X(\tau)$ which is the center of the smallest ball that includes the set $X(\tau)$. The inclusion $x^*(\tau) \in X(\tau)$ will be secured, as we shall see in the sequel, if $X(\tau)$ is convex. This may not be the case for the general nonlinear problem, however, when the configuration of $X(\tau)$ may be quite complicated (in fact, $X(\tau)$ may not even be a connected domain). The set $X(\tau)$ gives an unimprovable estimate of the state vector $x(\tau)$, provided the bound on the uncertain items (the number $\mu$) is given in advance.

On the other hand, in the second or $H_\infty$ approach, the value $\mu$ for *the bound on the uncertain items is not presumed to be known*, while the value of the estimation error

$$e^2(\tau) = (x(\tau) - x^*(\tau), x(\tau) - x^*(\tau)) \quad (2.1)$$

is then estimated merely through the smallest number $\gamma^2$ that ensures the inequality

$$e^2(\tau) \leq \gamma^2 \Psi(\tau, \zeta(\cdot)) \quad (2.2)$$

under restrictions (1.1), (1.2). In the linear case the smallest number $\gamma^2$ is clearly the square of *the minimal norm of the disturbance-output mapping* $T$ *e.g.* $(e(\tau) = T(\zeta(\cdot)))$ with $y = y^*(t)$ given. It obviously depends on the type of norm (the type of functional $\Psi(\zeta(\cdot))$ selected.

Both approaches have been thoroughly developed for the linear-quadratic case, while both have naturally encountered difficulties in generalization of the results to nonlinear systems [7,8], with promising results in [13-17] for the second approach. At first glance, the techniques of the two approaches may seem quite apart (as also are the communities of the scientists involved). Nevertheless, the aim of this paper is to emphasize the connections between the two approaches and to indicate, through a generalized Hamiltonian technique, a general framework that incorporates both of these, producing either of them, depending on the "a priori" information, as well as on the required accuracy of the solutions.

## 3. THE INFORMATION DOMAIN AND THE INFORMATION STATE

Assume system (1.1), (1.2) and restriction (1.3) with preassigned $\mu$ to be given and measurement $y^*(t)$, $t \in [t_0, \tau]$ to be specified.

Denote $X(\cdot) = \{x(\cdot, t_0, x^0, v(\cdot))\}$ to be *the set (the bundle) of trajectories* $x(t) = x(t, t_0, x^0, v(\cdot))$ of system (1.1) that also satisfy (1.2) for $y(t) \equiv y^*(t)$ and some $w = w(t)$, $t \in [t^0, \tau]$, whilst alltogether the triplet $\zeta(\cdot)$ satisfies the given restriction (1.3).

**Definition 3.1.** The cross section $X(\tau)$ of the tube $X(\cdot)$ at instant $t = \tau$ will be referred to as *the information domain at instant* $t = \tau$ *generated by measurement* $y = y^*(t)$, $t \in [t_0, \tau]$ *under restriction (1.3)*, (see[1-4]).

The calculation of the domains $X(\tau)$ and their evolution in time is the topic of many papers that range from theoretical schemes to numerical techniques and develop into *the theory of guaranteed state estimation* (see for example, references [3,4,9]).

Let us now introduce a scheme for describing the information domains $X(\tau)$, presuming $y^*(\cdot)$ to be given and the restriction (1.3) to be of the integral type. To start with, denote

$$\eta(\cdot) = \{x^0, v(t); t \in [t_0, \tau]\},$$
$$x(t, t_0, x_0, v(\cdot)) = x(t, t_0, \eta(\cdot))$$

and

$$\Phi(\tau, \eta(\cdot)) = (x^0 - a, P_0(x^0 - a)) + \int_{t_0}^{\tau} ((v(t) - v^*(t), M(t)(v(t) - v^*(t))) + (y^*(t) - g(t, x(t, x(t, t_0, \eta(\cdot)) - w^*(t)), N(t)(y^*(t) - g(t, x(t, x(t, t_0, \eta\cdot)) - w^*(t)))dt \quad (3.1)$$

Define

$$V(\tau, x) = \inf_{\eta(\cdot)} \{\Phi(\tau, \eta(\cdot)) \mid x(\tau, t_0, \eta(\cdot)) = x\} \quad (3.2)$$

An obvious assertion is given by

**Lemma 3.1.** *The information domain $X(\tau)$ is the level set*

$$X(\tau) = \{x : V(\tau, x) \leq \mu^2\} \quad (3.3)$$

*for the information state $V(\tau, x)$.*

The respective measurement $y^*(t) = g(t, \eta^*(\cdot)) + w^*(t)$ is the *"worst-case" realization* and the respective value $V^*(\tau) = V^0(\tau)|_{y(\cdot)=y^*(\cdot)} = 0$.

**Definition 3.2** Given the measurements $y^*(t)$, $t \in [t_0, \tau]$ and function $\Phi(\tau, \eta(\cdot))$ of (3.1), the respective function $V(\tau, x)$ will be referred to as *the information state* of system (1.1), (1.2), *relative to measurement $y^*(\cdot)$ and criterion $\Phi$.*

Therefore the main conclusion here is that:

(i) *The information domain $X(\tau)$ is the level set for the information state $V(\tau, x)$ that corresponds to the given number $\mu$.*

(ii) *The information state depends both on $y^*(\cdot)$ and on the type of functional $\Phi$.*

Let us now specify the function $V(\tau, x)$ for the case of *magnitude constraints*, presuming $\Phi$ is defined through relations (1.5) - (1.8).

Denote

$$\Lambda(\tau, \eta(\cdot), \alpha, \beta(\cdot), \gamma(\cdot)) = \{\alpha(x^0 - a, P_0(x^0 - a)) + \int_{t_0}^{\tau} (\beta(t)(v(t) - v^*(t), M(t)(v(t) - v^*(t))) + \gamma(t)(w(t) - w^*(t), N(t)(w(t) - w^*(t))))dt\}$$

**Lemma 3.2.** *The function $\Phi(\tau, \eta(\cdot))$ of (1.9), may be expressed as*

$$\Phi(\tau, \eta(\cdot)) = \sup\{\Lambda(\tau, \eta(\cdot), \alpha, \beta(\cdot), \gamma(\cdot)) \mid \alpha, \beta(\cdot), \gamma(\cdot)\} \quad (3.4)$$

*under the condition*

$$\alpha + \int_{t_0}^{\tau} (\beta(t) + \gamma(t))dt = 1, \ \alpha \geq 0; \quad \beta(t), \gamma(t) \geq 0; \ t \in [t_0, \tau] \quad (3.5)$$

The proof of an analogous result may be found in reference [4].

The crucial difficulty here is the calculation of the sets $X(\tau)$, of the function $V(\tau, x)$ and further on, of the estimate $x^*(\tau)$ for the unknown state $x(\tau)$. The calculations are relatively simple for an exceptional situation - *the linear-quadratic case* with

$$f(t, x) + c(t, x)v = A(t)x + C(t)v, \quad (3.6)$$
$$y(t) = G(t)x + w(t) \quad (3.7)$$

and $\Psi(\tau, \zeta(\cdot))$ given by (1.3).

## 4. THE HAMILTON-JACOBI TECHNIQUES (QUADRATIC CRITERIA)

Let us introduce a Dynamic Programming-type of equation treating $V(\tau, x)$ as *the value function* for the problem (3.2) with $\Phi$ given by (3.1), (1.3). Presuming the forthcoming partial derivatives existing and continuous in the corresponding variables, the respective equation is

$$\frac{\partial V}{\partial \tau} = \max_v \{(\frac{\partial V}{\partial x}, (f(t, x) + c(t, x)v)) - (v(t) - v^*(t), M(t)(v(t) - v*(t))) - (y(t) - g(t, x), N(t)(y(t) - g(t, x)))\} = 0, \quad (4.1)$$

with boundary condition

$$V(t_0, x) = (x - a, P^0(x - a)). \quad (4.2)$$

The existence of a solution to (4.1), (4.2) requires special considerations. It surely exists, however, if the system (1.1), (1.2) is linear. Presuming (3.6), (3.7), and after the elimination of $v$ equation (4.1) transforms into

$$\begin{aligned}\frac{\partial V}{\partial t} &+ (\frac{\partial V}{\partial x}, Ax + v^*) \\ &+ \frac{1}{4}(\frac{\partial V}{\partial x}, c'(t,x)M^{-1}(t)c(t,x)\frac{\partial V}{\partial x}) \\ &- (y(t) - G(t)x, \\ &\quad N(t)(y(t) - G(t)x)) = 0. \quad (4.3)\end{aligned}$$

Its solution with boundary condition (4.2) is a quadratic form

$$\begin{aligned}V(\tau,x) &= (x - z(\tau), \\ &\quad P(\tau)(x - z(\tau))) + k^2(\tau) \quad (4.4)\end{aligned}$$

where $P(t), z(t), k^2(t)$ are the solutions to the following well-known equations [3,9]

$$\begin{aligned}\dot{z} &= A(t)z + P^{-1}G'(t)N(t)(y(t) \\ &- G(t)z) + C(t)v^*, \ z(t_0) = a, \quad (4.5) \\ \dot{P} &+ PA(t) + A'(t)P \\ &+ PC'(t)M^{-1}C(t)P \\ &- G'(t)N(t)G(t) = 0, \ P(t_0) = P^0, \quad (4.6) \\ \dot{k}^2 &= (y(t) - G(t)z, \\ &\quad N(t)(y(t) - G(t)z)), \ k^2(t_0) = 0 \quad (4.7)\end{aligned}$$

An obvious consequence of the given reasoning is the following assertion.

**Lemma 4.1.** *Under restrictions (1.3), on the uncertain inputs $\zeta(\cdot) = \{\eta(\cdot), w(\cdot)\}$ the information domain $X(\tau)$ for the linear system (1.1),(1.2),(3.6),(3.7) is the level set (3.3) for the information state $V(\tau,x)$, being an ellipsoid $E(z(\tau), P(\tau))$ given by the relation*

$$\begin{aligned}X(\tau) = E(z(\tau), P(\tau)) = \{x : (x - z(\tau), \\ P(\tau)(x - z(\tau))) \leq \mu^2 - k^2(\tau)\} \quad (4.8)\end{aligned}$$

*where $z(\tau)$, $P(\tau)$, $k^2(\tau)$ are defined through equations (4.5)-(4.7).*

Formula (4.8) immediately indicates the worst-case realization of the measurement $y^*(t)$ which yields the "largest" set $X(t)$ (with respect to inclusion).

**Lemma 4.2.** *The worst-case realization of the measurement $y^*(t)$ is generated by the triplet $\{x^0 = a,\ v(t) = v^*(t),\ w(t) = w^*(t)\}$ which yields $k^2(t) = 0$ .*

In the more general case the assertion is loose:

**Lemma 4.3.** *Under existence and uniqueness assumptions for the solution to the boundary-value problem (4.1),(4.2) the level set*

$$X(\tau) = \{x : V(\tau,x) \leq \mu^2\}$$

*is the information domain for the system (1.1)-(1.3).*

In the absence of classical solutions one may apply either one of the equivalent concepts of "viscosity" or of "minmax" solutions ([10]).

## 5. THE HAMILTON-JACOBI TECHNIQUES (NONQUADRATIC CRITERIA)

In this section we indicate a Dynamic Programming - type of equation when the functional $\Phi(\tau, \eta(\cdot))$ is given by relations (1.9), (3.4). A direct derivation of the corresponding equation under obvious nondifferentiability properties is a separate topic which will not be discussed in this paper. We will follow another scheme, however, under the following assumption.

**Assumption 5.1.** *The integral*

$$\Lambda(\tau, \eta(\cdot), \alpha, \beta(\cdot), \gamma(\cdot))$$

*is convex in $\eta(\cdot) = \{x^0, v(\cdot)\}$ for any $\{x : x(\tau, \eta(\cdot)) = x\}$.*

This assumption always holds for the linear case (3.6), (3.7).

Under Assumption 5.1 the order of operations *inf, sup* may be interchanged due to the *minmax theorem*, we come to the relation

$$\begin{aligned}&V(\tau,x) = \\ &= \sup_{\alpha,\beta,\gamma} \min_{\eta(\cdot)} \Lambda(\tau, \eta(\cdot), \alpha, \beta(\cdot), \gamma(\cdot)). \quad (5.1)\end{aligned}$$

Denote $\chi(\cdot) = \{\alpha, \beta(\cdot), \gamma(\cdot)\}$. The internal minimization problem may be solved through equation (4.3) with $V(\tau,x)$ substituted by $V(\tau,x,\chi(\cdot))$ and $M(t)$, $N(t)$ by $\beta(t)M(t)$, $\gamma(t)N(t)$ respectively with boundary condition

$$V(t_0, x, \chi(\cdot)) = \alpha(x - a, P^0(x - a)). \quad (5.2)$$

This leads to

**Lemma 5.1.** *Under criteria (1.9), (3.4) and Assumption 5.1 the information state is given by*

$$V(\tau,x) = \sup\{V(\tau,x,\chi(\cdot)) \mid \chi(\cdot), (3.5)\} \quad (5.3)$$

*where $V(\tau,x,\chi(\cdot))$ is the solution to equation (4.3), under (5.2), with $M(t), N(t)$ substituted by $\beta M(t), \gamma(t)N(t)$.*

Passing to the linear case (3.6), (3.7), we observe

$$\begin{aligned}V(\tau,x,\chi(\cdot)) = (x - z(\tau,\chi(\cdot)), \\ P(\tau,\chi(\cdot))(x - z(\tau,\chi(\cdot)))) + k^2(\tau,\chi(\cdot)), \quad (5.4)\end{aligned}$$

where $P = P(t,\chi(\cdot))$, $z = z(t,\chi(\cdot))$, $k = k(t,\chi(\cdot))$

satisfy equations similar to (4.5)-(4.7) with $M(t), N(t)$ substituted by $\beta M(t), \gamma(t)N(t)$, and

$$P_{t_0} = \alpha P^0,\ z(t_0) = x^0,\ k(t_0) = 0 \quad (5.5)$$

Finally this develops into the assertion

**Lemma 5.2.** *For the linear system (1.1), (1.2), (3.6), (3.7) the information state $V(\tau, x)$ relative to measurement $y(\cdot)$ and nonquadratic criterion (1.9),(3.4) is the upper bound*

$$V(\tau, x) = \sup\{V(\tau, x, \chi(\cdot)) \mid \chi(\cdot), (3.5)\} \quad (5.6)$$

*of a parametrized family of quadratic forms $V(\tau, x, \chi(\cdot))$ of type (5.4) over the functional parameter $\chi(\cdot) = \{\alpha, \beta(\cdot), \gamma(\cdot)\}$ restricted by relations (3.5).*

As we have observed in the previous sections, the information domain $X(\tau) = E(z(\tau), P(\tau))$ is defined by $V(t, x)$ through inequality (3.3), given $\mu$. Therefore (5.6) yields the following

**Lemma 5.3.** *For the linear system (1.1), (1.2), (3.6), (3.7) with criterion (1.9), (3.4) the information set $X(\tau)$ is the intersection of ellipsoids*

$$X(\tau, \chi(\cdot)) = E(z(\tau, \chi(\cdot)), (\mu^2 - k^2(\tau))P(\tau, \chi(\cdot)))$$

*namely,*

$$X(\tau) = \{\cap E(z(\tau, \chi(\cdot)), (\mu^2 - k^2(\tau))P(\tau, \chi(\cdot))) \mid \chi(\cdot), (3.8)\} \quad (5.7)$$

*where*

$$z(t) = z(t, \chi(\cdot)) = z(t, \gamma(\cdot)),$$

$$P(t) = P(t, \chi(\cdot)), k^2(t) = k^2(t, \chi(\cdot)) = k^2(t, \gamma(\cdot)).$$

*The worst case measurement $y(t) = y^*(t)$ is generated by the triplet $x^0 = a$, $v(t) = v^*(t)$, $w(t) = w^*(t)$ and yields $k^2(\tau) = 0$.*

## 6. THE ESTIMATES AND THE ERROR BOUNDS

Consider the information domain $X(\tau)$ to be specified. Under the assumptions of this paper and the restriction (1.3) $X(\tau)$ will be closed and bounded. Let us seek an expression for the Chebyshev center of $X(\cdot)$. Applying the conventional generalized Lagrangian technique, we have

$$\min_x \max_z \{(x - z, x - z) - \lambda^2{}_\mu V(\tau, x)\} \quad (6.1)$$

$$V(\tau, x) \leq \mu^2$$

Under the assumptions made the solution to this problem exists. Summarizing the results, we have

**Lemma 6.1.** *For the set-membership estimation problem the minmax estimate $x^*(t) = z(\tau)$ (the Chebyshev center of $X(\tau)$) satisfies the property*

$$x^*(\tau) \in \mathrm{co}X(\tau)$$

*In the linear-convex case, with $\Phi(\tau, \eta(\cdot))$ of type (3.1),(1.3) or (3.4), (1.9), we have*

$$x(\tau) \in X(\tau)$$

*and $x^*(\tau) = x^*_\mu(\tau)$.*

*In the linear-quadratic case (3.1),(1.3)*

$$x^*(\tau) = z(\tau)$$

*is the center of the ellipsoid $E(\tau, P(\tau))$ described by the system (4.5)-(4.7) and does not depend on the number $\mu$ .*

In order to find the estimate $x^0(\tau)$ for the $H_\infty$ estimation problem, we have to solve the following problem:

*Find the smallest number $\gamma^2$ that ensures*

$$\min_x \max_{\zeta(\cdot)} \{(x - z, x - z) - \gamma^2 V(\tau, x)\} \leq 0$$

*under the conditions*

$$x(\tau, \eta(\cdot)) = z;\ g(t, x(t, \eta(\cdot)) \equiv y^*(t); t_0 \leq t \leq \tau$$

This, however, is equivalent to the problem of finding the smallest number $\gamma^2 = \gamma_0^2$ that ensures

$$\min_x \max_z \{(x - z, x - z) - \gamma^2 V(\tau, z)\} \leq 0 \quad (6.2)$$

It is not difficult to observe the following:

**Lemma 6.2.** *In the linear-quadratic case (3.1), (1.3) the Lagrange multiplier $\lambda_\mu$ of Lemma 6.1 satisfies the equality*

$$\lambda_\mu^2 = \gamma_0^2,\ \forall \mu$$

*and the solution $x^0(\tau)$ to (6.2) satisfies*

$$x^0(\tau) = x^*(\tau),\ \forall \mu,$$

*In the linear-convex case (1.9),(3.4) with magnitude constraints we have*

$$\lambda_\mu^2 \to \gamma_0^2,\ (\mu \to \infty)$$

*and*

$$x^*_\mu(\tau) \to x^0(\tau),\ (\mu \to \infty)$$

**Remark 6.1.** Among the conventional estimates for the nonlinear filtering problem is the following one, ([11,16,17]):

$$z^*(\tau) = \mathrm{argmin}\{V(\tau, x) \mid x \in R^n\}.$$

This selection is certainly justified for the linear-quadratic problem as in this case one has

$$z(\tau) = x^*(\tau) = x^0(\tau) = z^*(\tau),$$

so that all the estimate types coincide. However, as soon as we apply a nonquadratic functional $\Phi(\tau, \eta(\cdot))$, all the previous estimates may turn to be different (even for a linear system). This is all the more true for the nonlinear case, since always $z^*(\tau) \in X(\tau)$, while even in simple nonlinear examples one may observe that $x^*(\tau) \notin X(\tau)$.

One of the basic elements of the solution to the filtering problem is the computation of *error bounds* for the estimates. These are given in the form of sets, once the restrictions on the uncertain items $\zeta(\cdot)$ are specified in advance. Then the error set is taken to be either

$$\Omega(\tau) = X(\tau) - x^*(\tau)$$

or, more roughly,

$$\Omega_*(\tau) = \text{co}(X(\tau) - x^*(\tau))$$

The set $\Omega$ will be the *largest possible* (with respect to inclusion) if the realizations of the uncertain items $\zeta(\cdot)$ will generate *the worst-case measurement* $y^*(t)$.

As for the $H_\infty$ approach, the estimation error $e^2(\tau)$ will depend upon the number $\gamma^2$ in the inequality (2.2) (which depends in general on the measurement $y(t)$ in (2.2)).

## 7. Conclusions

This paper presents an introductory discussion on the similarities and differences in solving deterministic problems of nonlinear filtering under uncertainty in the system inputs through the two conventional approaches: the set-membership approach and the so-called $H_\infty$ techniques. The basic ideas that lead to a unified approach for the topic use the dual notions of the *information state and the information domain.* These ideas may also be readily applied to indicate the connections between the solutions of problems of output feedback control with deterministic uncertainty under the set-membership and $H_\infty$ settings.

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