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# Two-dimensional signal deconvolution: design issues related to a novel multisensor-based approach

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## ABSTRACT

We employ recent results of analysis in several complex variables to come up with a set of *compactly supported* approximate deconvolution kernels for the reconstruction of a two dimensional signal based on multiple linearly degraded versions of the signal with a family of kernels that satisfy suitable technical conditions. We discuss the question of convergence of the proposed deconvolution kernels, present simulation results that demonstrate the gain in bandwidth, and propose two data parallel grid layouts for the off-line computation of the deconvolution kernels.

## 1. INTRODUCTION

Signal deconvolution is a fundamental problem related to a variety of scientific and engineering disciplines. The traditional problem formulation can be stated as follows. We observe the output of a sensor modelled by a convolution operator with known kernel, and wish to synthesize the input signal based on output observations. In some cases it is possible to use more than one sensors, and attempt to reconstruct the common input signal by linearly combining the outputs of all available sensors. The motivation here stems from the fact that multiple operators are indeed necessary for the deconvolution problem to be well posed.<sup>2</sup> The specific application we have in mind is deconvolution for electro-optic imaging devices (Imaging Detector Arrays).<sup>2,4,6</sup>

Consider the system of figure 1. The  $f_i$ 's are distributions of compact support defined over  $\mathcal{R}^2$  and  $\mathcal{L}_{f_i}$  denotes convolution with kernel  $f_i$ . The natural question that comes up is: what is the minimum possible  $m$  and what conditions should the  $f_i$ 's satisfy so that we can uniquely determine  $s(\cdot)$  from the  $d_i(\cdot)$ 's? We are specifically interested in obtaining *linear* estimates of the input signal based on output observations from the bank of available devices. Mathematically the problem can be formulated as a convolution equation. We are looking for a family of *deconvolvers*  $h_i(\cdot)$ ,  $i = 1, \dots, m$  such that:

$$d_1 * h_1 + \dots + d_m * h_m = s \quad (1)$$

Alternatively, we need a family of entire analytic functions  $\hat{h}_i(\cdot)$ ,  $i = 1, \dots, m$  such that:

$$\hat{d}_1 \hat{h}_1 + \dots + \hat{d}_m \hat{h}_m = \hat{s} \quad (2)$$

Here  $\hat{\cdot}$  denotes Fourier Transform. Now observe that

$$\hat{d}_i = \hat{s} \hat{f}_i, \quad i = 1, \dots, m \quad (3)$$

Therefore equation 2 above is equivalent to

$$\hat{f}_1 \hat{h}_1 + \dots + \hat{f}_m \hat{h}_m = 1 \quad (4)$$

The later equation is known as the Analytic Bezout Equation (ABE). It is a well known fact that the existence of a family of deconvolvers,  $\{\hat{h}_1, \dots, \hat{h}_m\}$  that solves the Bezout Equation is completely equivalent to a coprimeness condition on the part of the  $f_i$ 's.

## 2. CONSTRUCTION OF DECONVOLVERS OF COMPACT SUPPORT

Let  $\mathcal{E}'_{\mathcal{R}^2}$  denote the space of all distributions of compact support defined over  $\mathcal{R}^2$ . Let  $\hat{\mathcal{E}}'_{\mathcal{R}^2}$  denote the Paley-Wiener space. The mapping  $\mathcal{E}'_{\mathcal{R}^2} \mapsto \hat{\mathcal{E}}'_{\mathcal{R}^2}$  given by  $f \mapsto \hat{f}$ , where  $\hat{\cdot}$  denotes Fourier transform, for all  $f \in \mathcal{E}'_{\mathcal{R}^2}$ , is 1-1 and onto the Paley-Wiener Space  $\hat{\mathcal{E}}'_{\mathcal{R}^2}$ . For convenience we drop the index  $\mathcal{R}^2$ .

**Theorem 1** <sup>5</sup> *There exists a family of functions  $\{\hat{h}_1, \dots, \hat{h}_m\}$  in  $\hat{\mathcal{E}}'$  that solves the Bezout Equation iff the family of entire functions  $\{\hat{f}_1, \dots, \hat{f}_m\}$  in  $\hat{\mathcal{E}}'$  is strongly coprime, i.e. iff:  $\sum_{j=1}^m |\hat{f}_j(\omega)|^2 \geq e^{-c\rho(\omega)}, \forall \omega \in C^2$ , for some constant  $c$ . Here,  $\rho(\omega) = |\text{Im}\omega| + \log(1 + |\omega|)$ .*

**Definition 1** *Let  $K$  be a compact subset of  $\mathcal{R}^2$ . Define the supporting function of  $K$  as follows:*

$$H_K(\xi) \triangleq \max\{x \cdot \xi \mid x \in K\}$$

where  $\cdot$  denotes inner product and  $\xi \in \mathcal{R}^2$ .

Consider a family of 2 distributions  $\{f_1, f_2\}$  of compact support in  $\mathcal{R}^2$ . Let  $H_1$  denote the supporting function of the convex hull of the union of the support sets of  $f_1, f_2$ .

**Definition 2** *A family of 2 distributions  $\{f_1, f_2\}$  of compact support in  $\mathcal{R}^2$  is well behaved if there exist positive constants  $A, B, N, K, C$  and a supporting function  $H_0$ , such that  $0 \leq H_0 \leq H_1$ , such that the common zero set,  $\mathcal{Z}$ , of the functions  $\{\hat{f}_1, \hat{f}_2\}$  is almost real i.e.  $\forall \omega \in \mathcal{Z} : |\text{Im}\omega| \leq C \log(2 + |\omega|)$ , and the number of zeros in  $\mathcal{Z}$  included in an open ball of radius  $r$  satisfies the growth condition  $n(\mathcal{Z}, r) = O(r^A)$ , and denoting*

$$|\hat{f}(z)| \triangleq \left[ \sum_{i=1}^2 |\hat{f}_i(z)|^2 \right]^{1/2}$$

the following inequality holds

$$|\hat{f}(z)| \geq \frac{Bd(z, \mathcal{Z})^K e^{H_0(\text{Im}z)}}{(1 + |z|)^N}$$

where  $d(z, \mathcal{Z})$  is the minimum of 1 and the Euclidean distance from the point  $z$  to the set  $\mathcal{Z}$ .

It can be shown that under these conditions the set  $\mathcal{Z}$  is discrete, i.e. the points  $\zeta \in \mathcal{Z}$  are isolated.

**Definition 3** A well-behaved family  $\{f_1, f_2\}$  is very well behaved if there exist constants  $M, C_1 > 0$  such that for all  $\zeta \in \mathcal{Z}$  it holds that

$$|J(\zeta)| \triangleq |\det[\frac{\partial \hat{f}_j}{\partial z_i}(\zeta)]_{ij}| \geq C_1(1 + |\zeta|)^{-M}$$

This last condition guarantees that the points in  $\mathcal{Z}$  are **well separated**, i.e. there exist constants  $M', C_2 > 0$  such that for any  $\zeta \in \mathcal{Z}$  there exists  $r = r(\zeta)$  such that

$$r(\zeta) \geq \frac{C_2}{(1 + |\zeta|)^{M'}}$$

and such that the open ball  $B_r(\zeta)$  contains no other points in  $\mathcal{Z}$ .

**Theorem 2**<sup>3</sup> Let  $\{f_1, f_2, f_3\}$  be a strongly coprime family of compactly supported distributions over  $\mathcal{R}^2$ . Suppose that the subfamily  $\{f_1, f_2\}$  is very well behaved. Suppose  $f_3$  is the kernel with the smallest support. Let  $H_0, H_1$  be as in definition 2 for the subfamily  $\{f_1, f_2\}$ . Let  $H_2$  denote the supporting function of the convex hull of the union of the support sets of  $f_1, f_2, f_3$ , and suppose  $H_2 \leq 2H_1$ . Furthermore suppose  $\exists r_0 > 0$  such that  $r_0|\theta| \leq 4H_0(\theta) - 2H_1(\theta) - H_2(\theta)$ . Then for any  $u \in C^\infty(\mathcal{R}^2)$ , with support set  $\text{sprt}u \subseteq \{x \in \mathcal{R}^n : |x| \leq r_0\}$ ,  $\hat{u}$  can be written as

$$\hat{u}(z) = \sum_{\zeta \in \mathcal{Z}} \frac{\hat{u}(\zeta)}{J(\zeta)\hat{f}_3(\zeta)} D(z, \zeta) \quad (5)$$

where  $z = (z_1, z_2)$ ,  $\zeta = (\zeta_1, \zeta_2)$ , both in  $\mathcal{C}^2$ ,

$$D(z, \zeta) \triangleq \begin{vmatrix} g_1^1(z, \zeta) & g_1^2(z, \zeta) & g_1^3(z, \zeta) \\ g_2^1(z, \zeta) & g_2^2(z, \zeta) & g_2^3(z, \zeta) \\ \hat{f}_1(z) & \hat{f}_2(z) & \hat{f}_3(z) \end{vmatrix} \quad (6)$$

$$g_1^i(z, \zeta) \triangleq \frac{\hat{f}_i(z_1, \zeta_2) - \hat{f}_i(\zeta_1, \zeta_2)}{z_1 - \zeta_1} \quad (7)$$

$$g_2^i(z, \zeta) \triangleq \frac{\hat{f}_i(z_1, z_2) - \hat{f}_i(z_1, \zeta_2)}{z_2 - \zeta_2} \quad (8)$$

and,  $J(\zeta) = \det(M(z))|_{z=\zeta}$ , where the Jacobian matrix  $M(z)$  is defined as

$$M(z) \triangleq \begin{bmatrix} \frac{\partial \hat{f}_1}{\partial z_1} & \frac{\partial \hat{f}_2}{\partial z_1} \\ \frac{\partial \hat{f}_1}{\partial z_2} & \frac{\partial \hat{f}_2}{\partial z_2} \end{bmatrix} \quad (9)$$

and

$$\mathcal{Z} = \{z \in \mathcal{C}^2 : \hat{f}_1(z) = \hat{f}_2(z) = 0\} \quad (10)$$

The significance of Theorem 2 can be demonstrated by a simple manipulation of equation (5), which yields

$$\hat{u}(z) = \hat{h}_1(z)\hat{f}_1(z) + \hat{h}_2(z)\hat{f}_2(z) + \hat{h}_3(z)\hat{f}_3(z) \quad (11)$$

with

$$\begin{aligned} \hat{h}_1(z) &\triangleq \sum_{\zeta \in \mathcal{Z}} \frac{\hat{u}(\zeta)}{J(\zeta)\hat{f}_3(\zeta)} [g_1^2(z, \zeta)g_2^3(z, \zeta) - g_1^3(z, \zeta)g_2^2(z, \zeta)] \\ \hat{h}_2(z) &\triangleq \sum_{\zeta \in \mathcal{Z}} \frac{\hat{u}(\zeta)}{J(\zeta)\hat{f}_3(\zeta)} [g_1^3(z, \zeta)g_2^1(z, \zeta) - g_1^1(z, \zeta)g_2^3(z, \zeta)] \\ \hat{h}_3(z) &\triangleq \sum_{\zeta \in \mathcal{Z}} \frac{\hat{u}(\zeta)}{J(\zeta)\hat{f}_3(\zeta)} [g_1^1(z, \zeta)g_2^2(z, \zeta) - g_1^2(z, \zeta)g_2^1(z, \zeta)] \end{aligned} \quad (12)$$

If  $u$  is designed to approximate a delta distribution, then  $\hat{u}$  will be approximately equal to one in the vicinity of the origin, in which case  $\{\hat{h}_1(z), \hat{h}_2(z), \hat{h}_3(z)\}$  will give an approximate solution to the ABE. Next we show that there exist unique distributions of compact support,  $\{h_1(t), h_2(t), h_3(t)\}$ , with corresponding FT  $\{\hat{h}_1(z), \hat{h}_2(z), \hat{h}_3(z)\}$ . Let us consider  $\hat{h}_1(z)$ . The development for the other two follows along the same lines. It suffices to show that every term of the sum over  $\zeta \in \mathcal{Z}$  is the Fourier transform of a distribution of compact support and obtain an upper bound on its support which is independent of  $\zeta$ . This is crucial. Fix  $\zeta$  and consider the following function (which is analytic in  $z$ )

$$\frac{\hat{u}(\zeta)}{J(\zeta)\hat{f}_3(\zeta)} [g_1^2(z, \zeta)g_2^3(z, \zeta) - g_1^3(z, \zeta)g_2^2(z, \zeta)]$$

For  $\zeta$  fixed, the first factor is just a scaling constant. By (lemma 1)<sup>3</sup>  $g_k^i(z, \zeta)$  is the Fourier transform of a distribution of compact support. Furthermore  $ch \text{ sprt } FT^{-1}\{g_k^i(z, \zeta)\} \subseteq ch \text{ sprt } f_i$ . Here  $ch$  denotes convex hull of a set. Therefore

$$ch \text{ sprt } FT^{-1}\{g_1^2(z, \zeta)\} \subseteq ch \text{ sprt } f_2 \quad (13)$$

$$ch \text{ sprt } FT^{-1}\{g_2^3(z, \zeta)\} \subseteq ch \text{ sprt } f_3 \quad (14)$$

By Titchmarsh Theorem<sup>7</sup> if  $f$  and  $g$  are both distributions of compact support then

$$\begin{aligned} ch \text{ sprt } f * g &= ch \text{ sprt } f \oplus ch \text{ sprt } g \\ &= \{a + b : a \in ch \text{ sprt } f, b \in ch \text{ sprt } g\} \end{aligned} \quad (15)$$

Hence

$$ch \text{ sprt } FT^{-1}\{g_1^2(z, \zeta)g_2^3(z, \zeta)\} \subseteq ch \text{ sprt } f_2 \oplus ch \text{ sprt } f_3 \quad (16)$$

Similarly

$$ch \text{ sprt } FT^{-1}\{g_1^3(z, \zeta)g_2^1(z, \zeta)\} \subseteq ch \text{ sprt } f_2 \oplus ch \text{ sprt } f_3 \quad (17)$$

Thus

$$ch \text{ sprt } FT^{-1}\{g_1^2(z, \zeta)g_2^3(z, \zeta) - g_1^3(z, \zeta)g_2^2(z, \zeta)\} \subseteq ch \text{ sprt } f_2 \oplus ch \text{ sprt } f_3 \quad (18)$$

And since the factor  $\frac{\hat{u}(\zeta)}{J(\zeta)\hat{f}_3(\zeta)}$  is a constant which does not affect the bound on the support

$$ch \text{ sprt } FT^{-1}\left\{\frac{\hat{u}(\zeta)}{J(\zeta)\hat{f}_3(\zeta)} [g_1^2(z, \zeta)g_2^3(z, \zeta) - g_1^3(z, \zeta)g_2^2(z, \zeta)]\right\} \subseteq ch \text{ sprt } f_2 \oplus ch \text{ sprt } f_3 \quad (19)$$

i.e. the bound on the summation term is independent of  $\zeta$ . Hence

$$ch\ sprt\ h_1(t) \subseteq ch\ sprt\ f_2 \oplus ch\ sprt\ f_3 \quad (20)$$

Similarly

$$ch\ sprt\ h_2(t) \subseteq ch\ sprt\ f_1 \oplus ch\ sprt\ f_3 \quad (21)$$

and

$$ch\ sprt\ h_3(t) \subseteq ch\ sprt\ f_1 \oplus ch\ sprt\ f_2 \quad (22)$$

It has to be emphasized that there are two levels of approximation here. First, we generally *choose*  $u$  to be different from  $\delta$  for reasons that are going to be discussed in section 3. This results in a family of deconvolvers that approximate the exact deconvolvers. Second, we further approximate these deconvolvers by truncating the corresponding sums. Let us call the deconvolvers of the first level of approximation the *intended* ones, and the deconvolvers of the second level of approximation the *realizable* ones. These realizable deconvolvers are going to be compactly supported by virtue of the fact that *every term* of the sums over  $\zeta \in \mathcal{Z}$  in equation (12) is the FT of a distribution of compact support whose support can be bounded independently of  $\zeta$ .

We remark that our approach is beneficial if the common zeros can be precisely localized. In this case, we can achieve good quality of deconvolution without introducing additional truncation errors, since the proposed deconvolution kernels are compactly supported and can be realized with finite delay. Otherwise, if the common zeros cannot be localized with sufficient precision, it is probably better to use the *Wiener deconvolvers*<sup>2</sup> and suffer the error due to the truncation of their duration, rather than the error induced by the incomplete knowledge of the common zero locations.

Under certain independence and stationarity assumptions, Wiener deconvolvers have been shown to be optimal in the presence of noise.<sup>2,4</sup> Numerically, the proposed deconvolvers are very close to the Wiener Deconvolvers in the Fourier transform domain, except for a certain degree of rounding up of very sharp peaks present in the Wiener deconvolvers (we attribute this to the fact that the proposed deconvolvers are analytic, and, therefore, cannot follow very sharp peaks exactly). Hence the behaviour of the proposed deconvolvers in the presence of noise is expected to be very close to optimal.

### 3. WINDOWING AND AVERAGING

Our goal is the pointwise evaluation of the FT of the intended deconvolution kernels over a suitably chosen finite grid. Here, we must strike a balance between computational feasibility and noise averaging on one hand, and quality of deconvolution on the other. The choice of  $u$  strongly affects the convergence of the realizable deconvolvers to the intended deconvolvers (because the smoother  $u$  is, the faster the decay of  $\hat{u}$  at infinity, and, therefore, the faster the convergence). Noise considerations dictate a *smooth* choice of  $u$  which in turn implies a fast decay of  $\hat{u}$  at infinity. If these issues were of no concern then we would like  $u$  to be as close to  $\delta$  as possible, or, equivalently,  $\hat{u}$  to be as close to unity as possible, in order to achieve good reconstruction of the original signal. Extensive simulations indicated that the following family of functions is a good compromise<sup>8</sup>

$$\hat{u}(z) = \left( \prod_{i=1}^2 \frac{\sin(\frac{\zeta_i}{N} z_i)}{\frac{\zeta_i}{N} z_i} \right)^N \cdot p_r(z) \quad (23)$$

Here,  $N$  is a small positive integer and  $\epsilon_1, \epsilon_2$  are small positive reals. The function  $p_r(z)$  is defined as follows:

$$p_r(z) = \begin{cases} 1, & |z_i| \leq r, \quad i = 1, 2 \\ 0, & \text{elsewhere} \end{cases} \quad (24)$$

The first factor is a two dimensional sinc-like function. The parameter  $r$  (forced cutoff in rads/sec) is to be chosen sufficiently large to include the main features of the first factor, while keeping the size of the computation reasonable.

At this point, it is useful to introduce a concrete example, in order to demonstrate the issues involved. Let  $\chi_K$  denote the characteristic function of the compact set  $K \subset \mathcal{R}^2$  and consider the following family of convolution kernels

$$\begin{aligned} f_1(t_1, t_2) &= \chi_{[-\sqrt{3}, \sqrt{3}] \times [-\sqrt{3}, \sqrt{3}]}(t_1, t_2) \\ f_2(t_1, t_2) &= \chi_{[-\sqrt{2}, \sqrt{2}] \times [-\sqrt{2}, \sqrt{2}]}(t_1, t_2) \\ f_3(t_1, t_2) &= \chi_{[-1, 1] \times [-1, 1]}(t_1, t_2) \end{aligned} \quad (25)$$

Then it is easy to verify that  $\{f_1, f_2, f_3\}$  satisfy all conditions of theorem 2. The Fourier transforms of the resulting deconvolvers are given by the infinite sums

$$\hat{h}_i(z_1, z_2) = \sum_{\zeta \in \mathcal{Z}} \frac{\hat{u}(\zeta)}{J(\zeta)\hat{f}_3(\zeta)} \cdot \frac{C_i(z, \zeta)}{(z_1 - \zeta_1)(z_2 - \zeta_2)} \quad (26)$$

with

$$C_1(z, \zeta) \triangleq \hat{f}_2(z_1, \zeta_2) \left[ \hat{f}_3(z_1, z_2) - \hat{f}_3(\zeta_1, \zeta_2) \right] - \hat{f}_2(z_1, z_2) \left[ \hat{f}_3(z_1, \zeta_2) - \hat{f}_3(\zeta_1, \zeta_2) \right] \quad (27)$$

$$C_2(z, \zeta) \triangleq \hat{f}_1(z_1, z_2) \left[ \hat{f}_3(z_1, \zeta_2) - \hat{f}_3(\zeta_1, \zeta_2) \right] - \hat{f}_1(z_1, \zeta_2) \left[ \hat{f}_3(\zeta_1, \zeta_2) - \hat{f}_3(z_1, z_2) \right] \quad (28)$$

$$C_3(z, \zeta) \triangleq \hat{f}_1(z_1, \zeta_2)\hat{f}_2(z_1, z_2) - \hat{f}_1(z_1, z_2)\hat{f}_2(z_1, \zeta_2) \quad (29)$$

Simulation results for this specific family of convolution kernels are presented in figures 2 up to 5. The sums are taken over the 3200 zeros which are located closest to the origin. A frequency step of 0.1718 rads/sec and a frequency resolution of  $256 \times 256$  points is adopted throughout the whole sequence of simulations. The magnitude of the Fourier transform of the third convolution kernel (best one) is depicted in figure 2. The bandwidth of this kernel is the available bandwidth *before any attempt is made to deconvolve the common input signal*. It is given here for comparison purposes. The magnitude of the Fourier transform of the overall system (i.e. bank of convolvers followed by bank of realizable deconvolvers, whose outputs are summed up to produce the overall system output) using the  $\hat{u}$  given by equation (23) with parameters  $\epsilon_1 = \epsilon_2 = \epsilon = 0.1$ ,  $N = 3$  is depicted in figure 3. The overall system exhibits a high degree of energy concentration along a ribbon-like neighborhood of the  $z_2$  axis. Observe that the magnitude of the response vanishes *in the vicinity of the origin!* This surprising behavior is attributed to a somewhat arbitrary (and asymptotically irrelevant) choice between two interpolation formulas, in the construction of theorem 2. The details, along with proposed solutions, can be found in<sup>8</sup>. For the example under consideration, by symmetry, a simple trick suffices to take care of the problem. The transform of each deconvolution kernel is averaged with a replica of

itself rotated by 90 degrees. It can be shown that the resulting family of transforms converges to an approximate solution of the ABE. This type of frequency averaging of the solutions results in an overall system response which is significantly better behaved than the one obtained by using the deconvolvers computed by brute force. This is especially true when it is combined with a suitable choice of the window parameters. In particular, the magnitude of the overall system response can be driven away from zero in the vicinity of the origin. The trade-off is that we suffer some loss of high frequency information. Figure 4 depicts the magnitude of the Fourier transform of the overall system using the  $\hat{u}$  given by equation (23) with parameters  $\epsilon_1 = 0.1$ ,  $\epsilon_2 = 0.5$ ,  $N = 3$ . Figure 5 depicts the magnitude of the Fourier transform of the overall system using frequency averaging of the resulting deconvolution kernels which were used in the configuration whose FT is depicted in figure 4.

#### 4. EFFICIENT COMPUTATION

The actual computation of the deconvolution kernels is a very demanding task; one basically needs to calculate pointwise approximations to an infinite sum. We have implemented all relevant computational procedures on Thinking Machines Corp. Connection Machine CM-2 system. This is a single instruction multiple data system which supports the Data Parallel Computing model.

In order to optimize the target computation with respect to work-time efficiency considerations one needs to exploit the special structure of the particular problem at hand. Let  $n$  denote the cardinality of the subset of  $\mathcal{Z}$  over which we sum, and let  $W(n)$  denote the number of operations required for a brute-force computation of the deconvolution kernels. Our implementation has been developed around the paradigm of characteristic functions over squares of suitably chosen size. It takes advantage of the symmetry of the deconvolution kernels in the transform domain, the regular structure of the nullset  $\mathcal{Z}$ , and the similarity between the deconvolution kernels, to reduce the required number of operations by a factor of 24, i.e. from  $W(n)$  to  $\frac{1}{24}W(n)$ .

Assuming "enough" processors, a time-efficient strategy is as follows. For each pair of frequencies,  $(z_1, z_2)$ , in the upper-right transform quadrant assign *one data processor to each nullpoint*  $(\zeta_1, \zeta_2)$ , and use the NEWS GRID nearest neighbor communication facility to implement the prefix sums on paths (pointer jumping) algorithm<sup>9</sup> row-wise along the grid. This scheme should be replicated for all three kernels and for all frequency pairs  $(z_1, z_2)$  in the upper right transform quadrant. The data processor grid layout would be as in figure 6, where  $\bullet$  denotes a data processor, and  $+$  denotes floating point addition. The overall algorithm inherits its work and time bounds from the underlying pointer jumping algorithm which it uses as a building block. The number of frequency points  $z$  at which we compute the corresponding sums is a finite constant, and, therefore, is hidden in the big-O notation. The pointer jumping algorithm is not work optimal (needs  $O(n \log n)$  operations instead of  $O(n)$  which is the optimal)<sup>9</sup>. On the other hand it is *very fast*, with time complexity  $O(\log n)$ . Therefore the proposed grid configuration also needs  $O(n \log n)$  operations (and hence it is not work optimal) and  $O(\log n)$  time (which is very fast). The drawback with this grid configuration is that it requires a very large number of virtual processors. This implies that each actual processor must be timeshared between a large number of tasks. In practice, only the values over a relatively small subset of frequency points,  $z$ , are simultaneously computed at any given time.

In its full configuration the Connection Machine model CM-2 employs 64K processors. With a frequency resolution of  $512 \times 512$  points we can simply assign each processor the task of sequentially computing one point value for all three kernels. This requires  $\frac{512 \times 512}{4} = 64K$  processors. This way *no interprocessor communication is needed* and the size,  $n$ , of the subset of the nullset over which we sum is not as big a concern as before, because it does not affect



the number of processors required (only affects the execution time). Thus quite large nullsets can be accommodated. Notice that since each processor computes a specific point value *for all three kernels*, the similarity between the three kernels can be easily exploited. The number of operations and execution time are both  $O(n)$  here. This grid configuration is clearly work-optimal. In a fully configured Connection Machine with 64K processors the run time (excluding I/O) is around four minutes (for the upper right hand quadrant only). A further computational step would be to use the Accelerated Cascading Strategy (ACS) to come up with a work-time optimal algorithm based on the optimal list ranking algorithm<sup>1</sup>.

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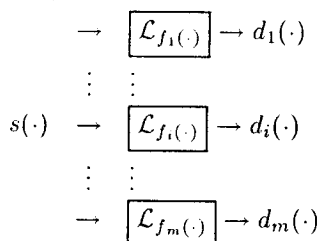


Figure 1: Multiple convolutional operators operating on a single input

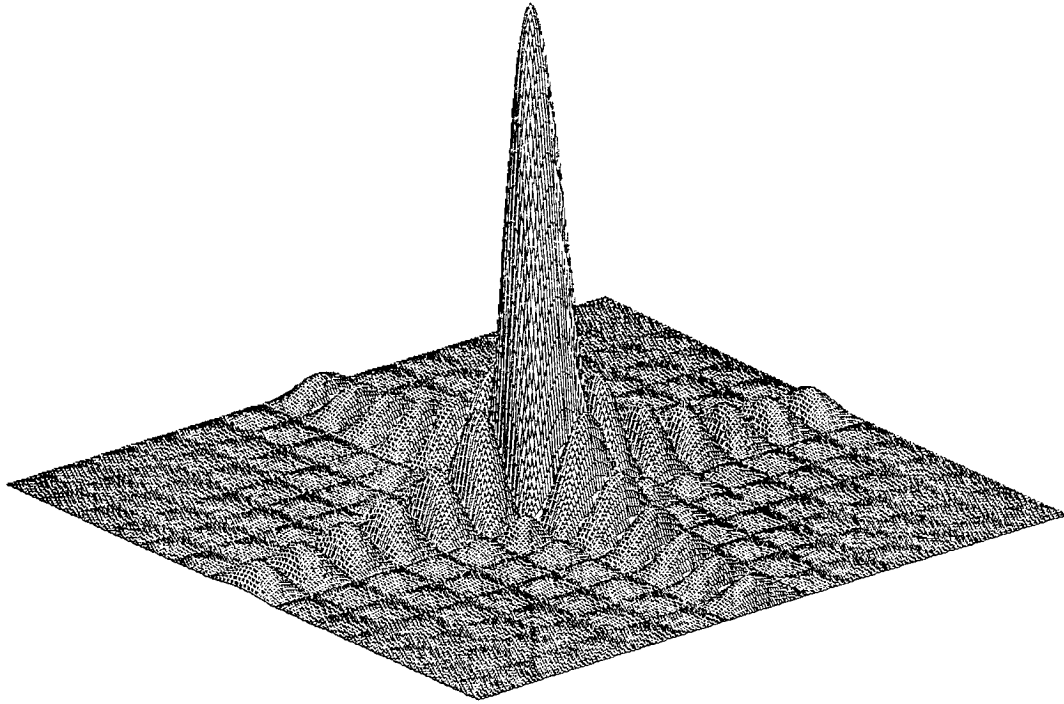


Figure 2: Magnitude of FT of convolver 3 (best one)

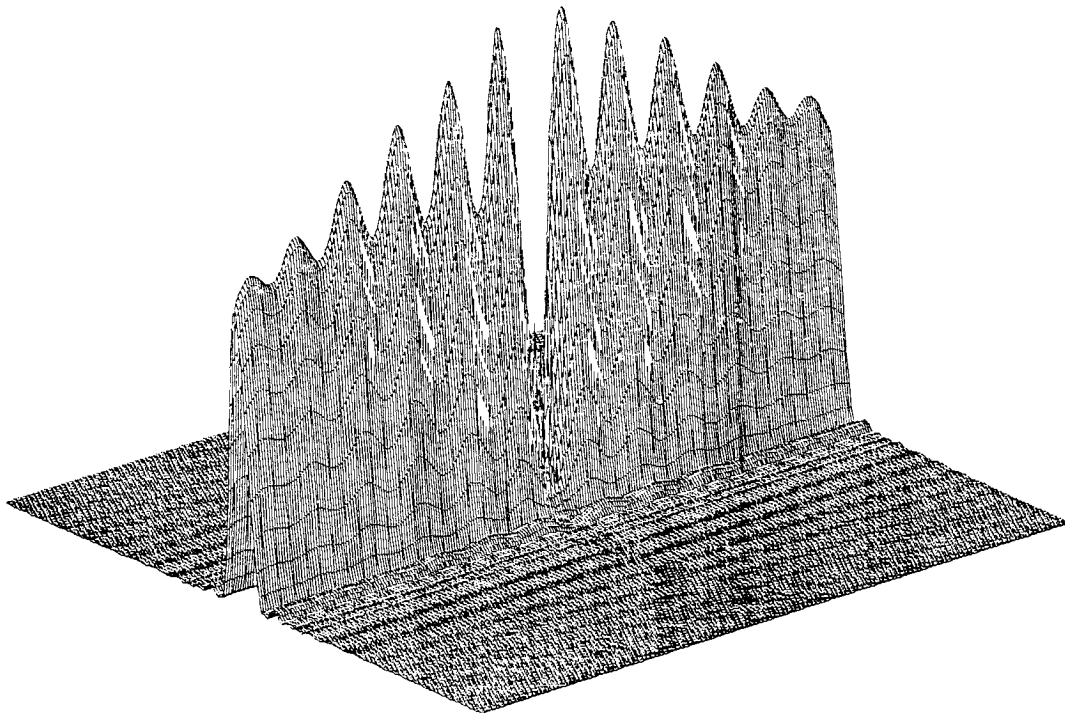


Figure 3: Magnitude of FT of overall system,  $\epsilon = 0.1$ ,  $N = 3$

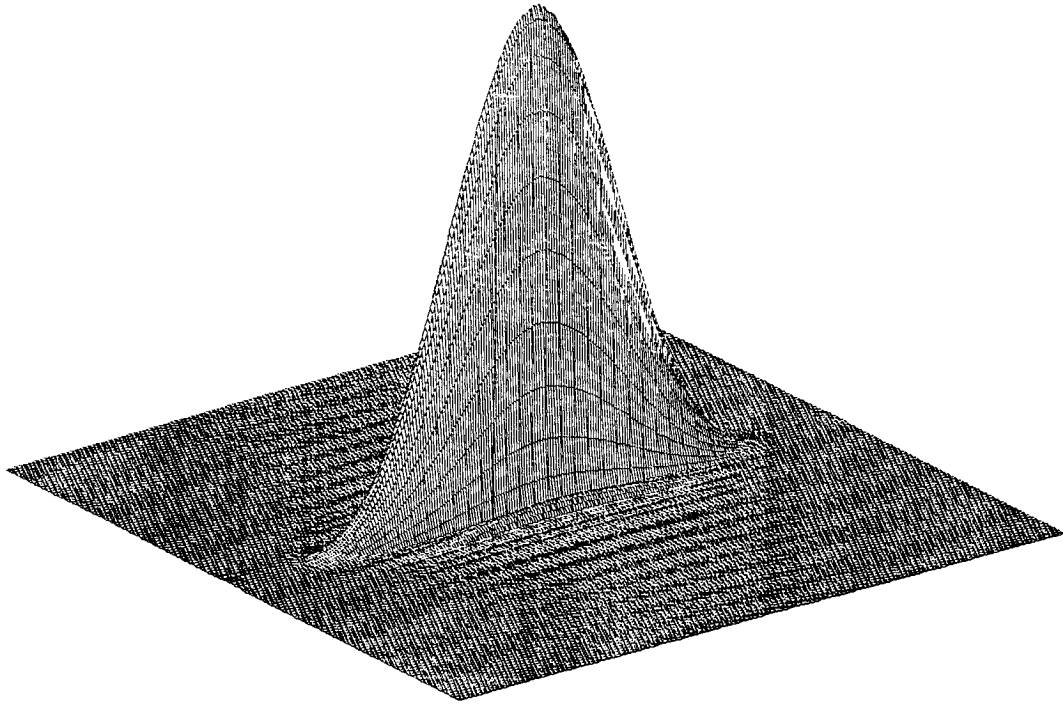


Figure 4: Magnitude of FT of overall system,  $\epsilon_1 = 0.1$ ,  $\epsilon_2 = 0.5$ ,  $N = 3$

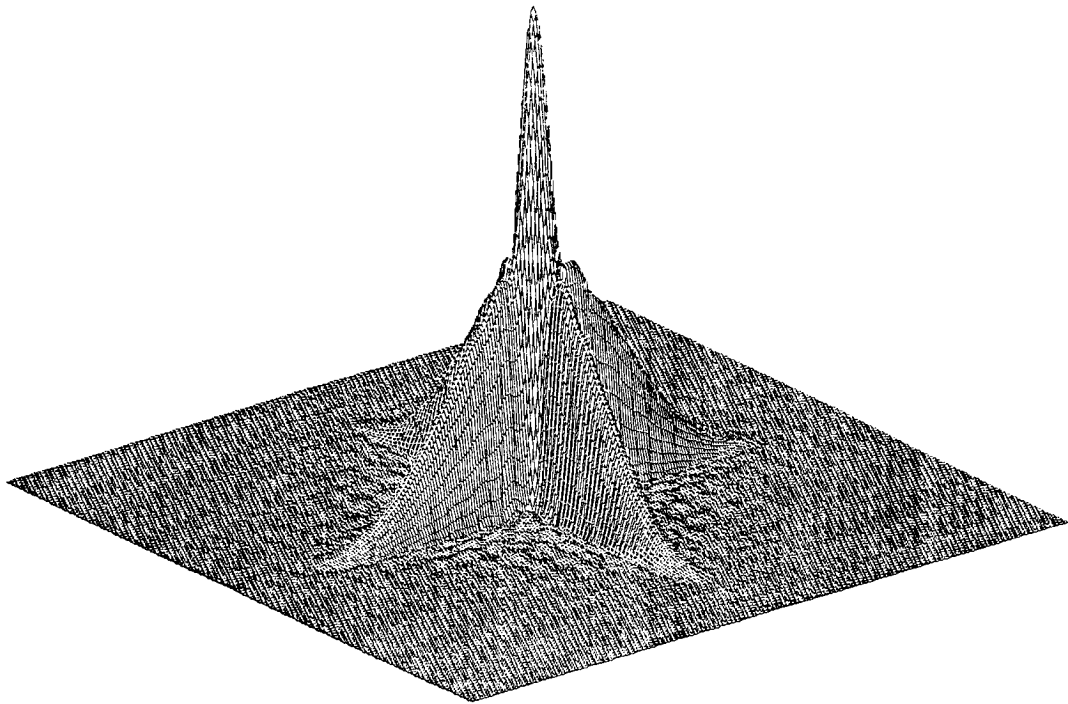


Figure 5: Magnitude of FT of overall system,  $\epsilon_1 = 0.1$ ,  $\epsilon_2 = 0.5$ ,  $N = 3$ , *averaged*

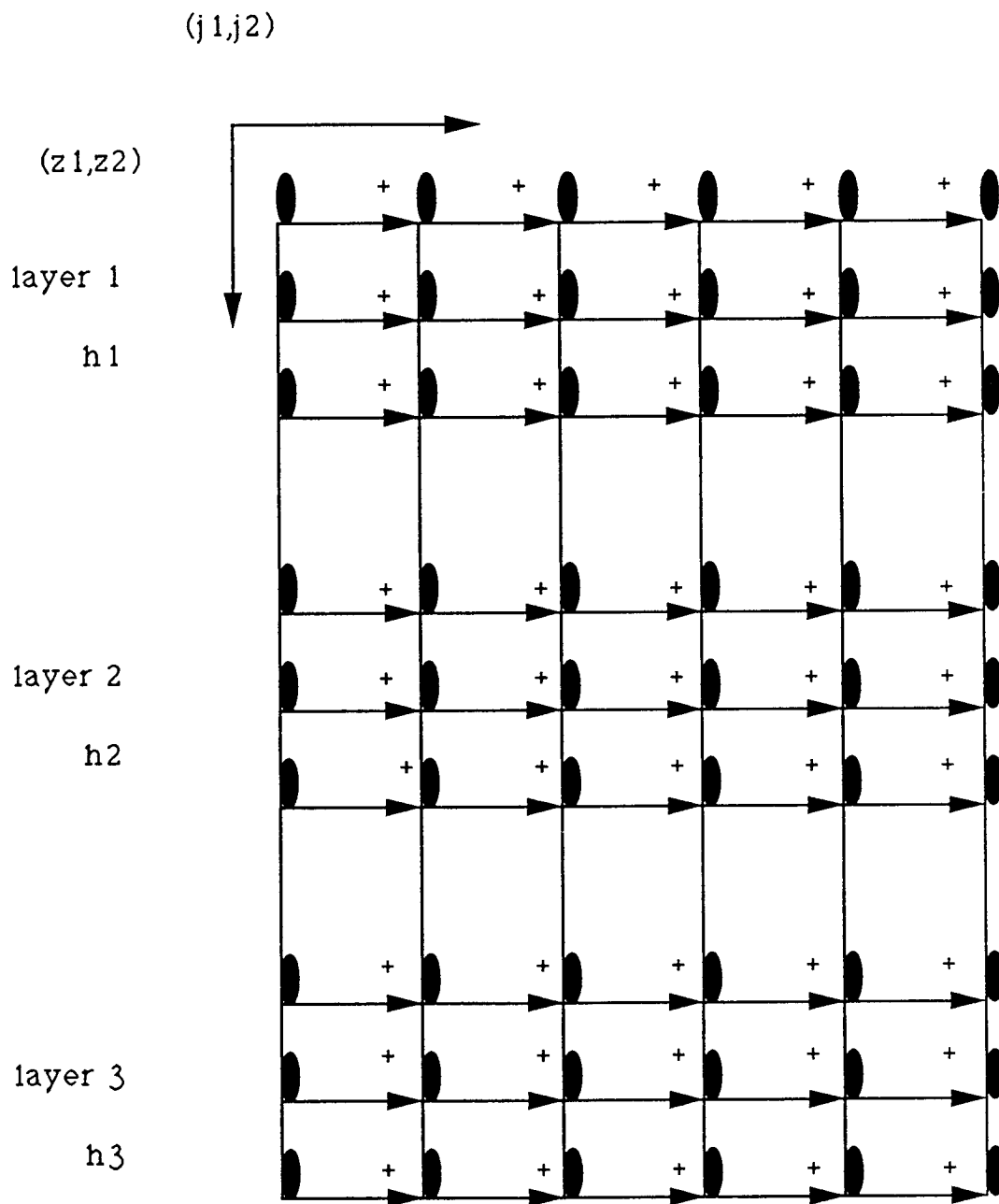


Figure 6: Layered grid configuration for distributed computation