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APPROXIMATE SOLUTION OF NONLINEAR FILTERING PROBLEMS BY DIRECT IMPLEMENTATION OF THE ZAKAI EQUATION

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Abstract

Numerical methods for the solution of the robust version of the Duncan-Mortensen-Zakai partial differential equation are considered. Both semidiscretization and complete discretization schemes are included. Direct implementation of such schemes via array processors is proposed as a design method for nonlinear filters. The importance of existence, uniqueness and tail behavior of solutions is related to properties of such implementations.

Summary

Recent studies of the nonlinear filtering problem have emphasized a pair of linear partial differential equations as central to this problem [1]. Briefly for a diffusion signal model

$$dx(t) = f(x(t))dt + g(x(t))dw(t) \qquad (1)$$

with observation

$$dy(t) = h(x(t))dt+dv(t) \qquad (2)$$

the "unnormalized" conditional probability density of $x(t)$ given $y(s)$, $s\leq t$, satisfies the linear stochastic partial differential equation

$$dU(t,x) = [a(x)U_{xx}(t,x) + b(x)U_x(t,x)+c(x)U(t,x)]dt + h(x)U(t,x)dy(t) \qquad (3)$$

$$U(o,x) = p_0(x) \ , \ 0\leq t\leq T;$$

If we normalize U to 1 we obtain the conditional density. In (3)

$$a(x) = \tfrac{1}{2}g^2(x)$$
$$b(x) = 2g(x)g_x(x) - f(x) \qquad (4)$$
$$c(x) = g_x^2(x) + g(x)g_{xx}(x) - f_x(x) - \tfrac{1}{2}h^2(x)$$

It is easier to analyze (3) indirectly via the transformation [2]

$$V(t,x) = \exp[-h(x)y(t)]U(t,x) \qquad (5)$$

which implies that V solves the linear parabolic p.d.e., for each path y,

$$\frac{\partial V(t,x)}{\partial t} = A(x)V_{xx}(t,x)+B(t,x)V_x(t,x) + C(t,x)V(t,x) \qquad (6)$$

$$V(o,x) = p_0(x), \ 0\leq t\leq t \ ;$$

usually called the robust version of (3). In (6)

$$A(x) = a(x)$$
$$B(t,x) = b(x) + 2a(x)h_x y(t)$$
$$C(t,x) = c(x) + b(x)h_x(x)y(t) + a(x)[h_{xx}(x)y(t) + h_x^2(x)y^2(t)] \qquad (7)$$

Since almost all paths of y are Hölder continuous (6) can be readily analyzed for existence, uniqueness and regularity by classical p.d.e. methods for each path. Such an approach, leading to several useful results can be found in [3].

Similarly well known efficient numerical methods for p.d.e.'s can be used to compute solutions to (6) with high accuracy. The fact that (6) is linear facilitates numerical treatment considerably. The program initiated here, is being largely motivated by the desire to develop systematic, efficient approximate solutions to nonlinear filtering problems. Recent work on analytical solutions has produced important results but most in a negative direction (see in particular the articles by Brockett, Ocone and Sussmann in [1]).

Our purpose here is not to analyze numerical schemes for (6). Rather we are interested in direct implementations of established numerical methods for p.d.e.'s of the type (6) via special purpose array processors. With the advent of VLSI technology, computer aided design of such array processors on a single chip is now a reality [4]. The development of such a design method appears as a powerfull alternative in nonlinear filtering studies. Such a study rests on the following circle of ideas; efficient stable schemes for linear parabolic p.d.e.'s, fast algorithms for the solution of the discretized or semidiscretized equations, existence-uniqueness-regularity results, theory and design of VLSI processor arrays.

The end result of such a method is a high-

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performance, VLSI implementable algorithm (with current standards a rather low-cost device) which computes recursively accurate approximations of the conditional density.

We assume that existence and uniqueness questions about solutions of (6) have been settled. For some recent results we refer to [3]. Furthermore, we assume that a uniform bound of the form

$$V(t,x) \leq M \exp(-K\phi(x)) \qquad (8)$$

has been established for the solution of (6). In (8) we assume M,K to be constants independent of the observation path and ϕ to be a non-negative function, such that $\lim_{|x|\to\infty} \phi(x) = +\infty$.

We then proceed as follows. First we choose an $\varepsilon>0$, small, induced by the accuracy requirements desired. Using (8) we define the bounded region Ω in which (6) will be solved via

$$\Omega' = \{x; V(t,x) \leq \varepsilon\} \qquad (9)$$

where ' indicates set complementation. Let Γ denote the smooth boundary of Ω.. We next use a discretization or semidiscretization scheme to solve the following "Dirichlet perturbation" of (6);

(6) together with the boundary condition $V(t,x) = 0$ on Γ (10)

If we use a space semidiscretization scheme on (10) we obtain the system of ordinary differential equations [5] [6]:

$$G_N \dot{V}_N(t) = A_N V_N(t) + B_N(t) V_N(t) + C_N(t) V_N(t)$$

with initial conditions (11)

$$V_N(0) = p_{0,N}$$

Here

$$B_N(t) = B_{N,0} + B_{N,1} y(t)$$
$$C_N(t) = C_{N,0} + C_{N,1} y(t) + C_{N,2} y^2(t) \qquad (12)$$

Furthermore G_N, A_N, $B_{N,0}$, $B_{N,1}$, $C_{N,0}$, $C_{N,1}$, $C_{N,2}$ are finite bandwidth, band matrices. This is exploited in an electronic implementation of (11) using VLSI processor arrays in a pipeline arrangement to compute the requisite matrix-vector multiplications (see [4, pp. 274-275]). The result is an approximate nonlinear filter. Similar results are obtained via full discretizations of (6), leading to digital approximate nonlinear filters.

Results on efficiency, stability, operational delay will be reported for such approximate filters.

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