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NONLINEAR FILTERING OF DIFFUSION PROCESSES

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Abstract. Representations of conditional statistics (densities) are described for the problem of nonlinear estimation of diffusion processes. Existence and uniqueness properties and algebraic structure of the Duncan-Mortensen-Zakai evolution equation are considered together with some approximation techniques.

Keywords. Nonlinear filtering; diffusion processes; partial differential equations; Lie algebras; approximation.

1. INTRODUCTION

Estimation of the values of one random process $x(t)$ in terms of observations of a related process $y(t)$ is one of the fundamental problems in engineering and other applied sciences. In particular, the construction of efficient representations for the estimates has proven to be extremely difficult in all but a few special cases. In the case of diffusion processes $x(t)$, $y(t)$ related by differential equations

$$\begin{aligned} dx(t) &= f(x(t))dt+g(x(t))dw(t) \\ dy(t) &= h(x(t))dt+dv(t) \end{aligned} \qquad (1.1)$$

(scalar processes, smooth coefficients, $w(t)$, $v(t)$ are standard Brownian motions independent from $x(0)$) one can bring to bear on the estimation problem analytic PDE methods, probability-martingale techniques, Lie algebraic-differential geometric methods, functional integration representations, group representation methods, and techniques from asymptotic analysis among others. Many of these approaches are discussed in the volume (Hazewinkel, Willems (1980)), especially in the papers by M.H.A. Davis, R.W. Brockett, E. Pardoux, S.K. Mitter, J.S. Baras, among others there, and in the papers Mitter (1980), Baras, Blankenship (1980) Mitter, Ocone (1979), Pardoux, (1979), Blankenship (1980). The paper by Davis and Marcus (1980) provides an especially useful introduction to the nonlinear filtering of diffusion processes which complements this paper.

In the remainder of this section we set up the filtering problem for diffusions. In Section 2 we discuss questions of existence and uniqueness for the evolution equation of the conditional statistics together with its algebraic structure. In Section 3 we describe certain approximation methods for treating nonlinear filtering problems.

The nonlinear filtering problem consists of recursively computing estimates $E\{\varphi(x(t))|y(s), 0\le s\le t\}$ where $x(t)$ is a scalar diffusion process and $y(t)$ a scalar observation process described by (1.1). We shall consider only scalar signal and observation processes for simplicity. All results can be easily extended to vector signal and observation processes.

The fundamental quantity in this problem is the conditional probability measure of $x(t)$ given $y(s)$, $0\le s\le t$, which we shall denote by $\nu_t(dx)$. Consider now the conditional expectation operator

$$\pi_t(\varphi) = E\{\varphi(x(t))|\mathcal{F}_t^y\} \qquad (1.2)$$

where

$$\mathcal{F}_t^y = \sigma\{y(s), 0\le s\le t\} \qquad (1.3)$$

and φ belongs to some appropriate class of functions. Clearly we can write

$$\pi_t(\varphi) = \int \varphi(x)\nu_t(dx), \qquad (1.4)$$

that is as a linear functional. One of the most significant recent developments has been the emphasis on "unnormalized" versions of conditional expectations and probability densities. As we shall see it is far more convenient to analyze an unnormalized form σ_t, which defines π_t via

$$\pi_t(\varphi) = \frac{\sigma_t(\varphi)}{\sigma_t(1)} \qquad (1.5)$$

where 1 is the constant function with values

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one everywhere. It is a direct consequence of Girsanov's theorem (Stroock, Varadhan 1979) that σ_t can be represented via the so called Kallianpur-Striebel formula

$$\sigma_t(\varphi) = \int \varphi(x(t)) \cdot \exp\left[\int_0^t h(x(s))dy(s) - \tfrac{1}{2}\int_0^t h^2(x(s))ds\right]\mu(dx) \tag{1.6}$$

where μ represents the path-space measure over the paths of the signal process $x(\cdot)$. We shall have more to say about this formula in Sections 2 and 3.

This formula is non-recursive; however, one can pass to a recursive equation in two steps. First we have a "weak" differential form of (1.6)

$$\begin{aligned} d\sigma_t(\varphi) &= \sigma_t(\mathcal{L}_0\varphi)dt + \sigma_t(h\varphi)dy(t) \\ \sigma_0(\varphi) &= \pi_0(\varphi) \end{aligned} \tag{1.7}$$

where $\mathcal{L}_0$ is the infinitesimal generator of the x process in (1.1):

$$[\mathcal{L}_0\phi](x) = \tfrac{1}{2}g^2(x)\frac{\partial^2}{\partial x^2}\phi(x) + f(x)\frac{\partial}{\partial x}\phi(x) \tag{1.8}$$

where π_0 is induced by the initial distribution of x, p_0. In a second step we consider the formal adjoint to (1.7)

$$\begin{aligned} d\nu_t &= \mathcal{L}_0^*\nu_t dt + \mathcal{L}_1\nu_t dy(t) \\ \nu_0 &= p_0 \end{aligned} \tag{1.9}$$

where

$$\begin{aligned} \mathcal{L}_0^* &= \text{formal adjoint of } \mathcal{L}_0 \\ \mathcal{L}_1 &= \text{operator "multiplication by the function h"} \end{aligned} \tag{1.10}$$

This is the Duncan-Mortensen-Zakai (DMZ) stochastic p.d.e.

For certain algebraic considerations, it has been established (Hazewinkel, Willems 1980; Mitter, 1980; Baras, Blankenship, 1980; Mitter, Ocone, 1979; Pardoux, 1980 Blankenship, 1980; Davis, Marcus, 1980) that it is more convenient to consider the Stratonovich version of (1.9) obtained after doing a Wong-Zakai correction

$$\begin{aligned} d\nu_t &= (\mathcal{L}_0^* - \tfrac{1}{2}\mathcal{L}_1^2)\nu_t dt + \mathcal{L}_1\nu_t dy(t) \\ \nu_0 &= p_0 \end{aligned} \tag{1.11}$$

We shall impose appropriate conditions of $\mathcal{L}_0$, $\mathcal{L}_1$, so that ν_t becomes a well defined probability density in an effort to minimize mathematical technicalities. We can then rewrite (1.11) using the suggestive notation

$$\begin{aligned} \frac{\partial\nu_t}{\partial t} &= (\mathcal{L}_0^* - \tfrac{1}{2}\mathcal{L}_1^2 + \dot{y}(t)\mathcal{L}_1)\nu_t \\ \nu_0 &= p_0 \end{aligned} \tag{1.12}$$

We consider equation (1.12) to be the fundda-mental equation of nonlinear filtering and in essence the nonlinear filtering problem is the study of this equation in an "invariant" fashion.

Relative to equation (1.12) it is worth pointing out one other fact. Clearly what is most desirable is the fundamental solution of equation (1.12). This can be obtained by functionally integrating (1.12) via a Feynman-Kac formula

$$G_t(\alpha,\beta) = \int \exp\left(\int_0^t h(x_s)dy_s - \tfrac{1}{2}\int_0^t h^2(x_s)ds\right)\mu_{\alpha\beta}(dx)\ , \tag{1.13}$$

where $\mu_{\alpha\beta}$ is the conditioned x-process measure, the conditioning being $x_0=\alpha$ and $x_t=\beta$.

We would like to discuss one other facet of nonlinear filtering, viz. robust filtering (Hazewinkel, Willems 1980). From the theory of conditional expectations, there exists a functional $\psi: C(0,T) \to \mathbb{R}$ such that

$$E[\varphi(x_t)|\mathcal{F}_t^y] = \psi(y)\ ,\ \mu_y - \text{a.s.} \tag{1.14}$$

and μ_y has the same null sets as Wiener measure. Hence the set of functions of bounded variation is a null set. Now in physical situations all observations will have finite bandwidth and hence will be of bounded variation and ψ will be undefined for all physical observations. Thus, what is required is an extension of the functional ψ from $C(0,T;\mu_y)$ to all of $C(0,T)$. This cannot be done unless we eliminate the Itô-integral from (1.6). This can be achieved by an "integration by parts", and we can express $\sigma_t(\varphi)$ as

$$\sigma_t(\varphi) = \int T^y_{0,t}B_{y(t)}\nu_0\varphi dx \tag{1.15}$$

where $T^y_{s,t}$ is the y-dependent semi-group associated with a certain multiplicative functional transformation (Hazewinkel, Willems 1980) and $B_{y(t)}$ is the operator multiplication by $\exp[h(x)y(t)]$. Computing the generator of $T^y_{s,t}$ amounts to obtaining a "robust" form of the Zakai equation, an equation where there are no stochastic differentials.

Computing the generator of $T^y_{s,t}$ is of some interest and it is useful to introduce the ideas of multiplicative functionals of Markov processes, and some special multiplicative functions. The ideas here are due to Davis (Hazewinkel, Willems 1980) and Mitter (Hazewinkel, Willems 1980).

One then computes the generator of $T^y_{s,t}$ as

$$\mathcal{L}^y_s\varphi = \mathcal{L}_0\varphi + y(s)(\mathrm{ad}_h\mathcal{L}_0)\varphi + \tfrac{1}{2}y^2(s)(\mathrm{ad}_h^2\mathcal{L}_0)\varphi - \tfrac{1}{2}h^2 \tag{1.16}$$

where (using the Lie brackett notation]

$$\{(\mathrm{ad}_h\mathcal{L}_0)\varphi\}(x)=[h,\mathcal{L}_0]\varphi(x)=h(x)(\mathcal{L}_0\varphi)(x)- -\mathcal{L}_0(h\varphi)(x) \qquad (1.17)$$

and $\mathrm{ad}_h^2\mathcal{L}_0$ is defined recursively.

Indeed the above considerations suggest that the Lie algebra of operators $\mathcal{L}A\{\mathcal{L}_0^*-\frac{1}{2}\mathcal{L}_1^2,\mathcal{L}_1\}$ has an essential role to play in nonlinear filtering theory. This is explored further in Section 2.

2. THE FUNDAMENTAL P.D.E: EXISTENCE, UNIQUENESS AND ALGEBRAIC STRUCTURE.

In this section we use methods from the theory of partial differential equations to analyze the basic stochastic p.d.e. in nonlinear filtering theory of diffusion processes, i.e. the Duncan-Mortensen-Zakai equation (1.9). We consider diffusions in unbounded domains. The existence of boundaries introduces further complications and for this reason such cases are not treated here.

In order to analyze (1.9) with elementary methods we need some definitions and several basic facts from the classical theory of p.d.e.'s.

Definition 2.1: $u(x,t)$ is a regular solution of a p.d.e. if u is continuous in $\mathbb{R}\times[0,T]$, if the derivatives of u which appear in the p.d.e. are continuous and satisfy the p.d.e. at every point of $\mathbb{R}\times(0,T]$.

We will be primarily interested in classical solutions here. Extensions to distributional solutions etc. are rather straightforward (see for example Pardoux, 1979). On the other had we wish to include, unbounded f,g,h (c.f. (1.1) in the theory due to important practical examples of such type. Furthermore since $u(\cdot,\cdot)$ is a probability density it clearly belongs to $L^\infty(\mathbb{R}\times(0,T))$, $u(\cdot,t)\in L^1(\mathbb{R})$, for each $t\in[0,T]$ and is nonnegative.

We will follow an indirect route in the analysis of the stochastic p.d.e. (1.9). To avoid consideration of a stochastic p.d.e., we introduce following (1.14)-(1.16), an integrating factor and consider instead "pathwise solutions" (Davis, Marcus, 1980). Thus letting

$$\rho(x,t) = \exp(-h(x)y(t))\nu(x,t) \qquad (2.1)$$

we see that ρ satisfies formally the p.d.e.

$$\frac{\partial\rho(x,t)}{\partial t} = \frac{1}{2}a^*(x)\frac{\partial^2}{\partial x^2}\rho(x,t)+(y(t)a^*(x)h'(x)+ a^{*\prime}(x)-f(x))\frac{\partial\rho}{\partial x}(x,t)+\{\frac{1}{2}y^2(t)a^*(x)h'(x)^2+ y(t)(\frac{1}{2}a^*(x)h''(x)+a^{*\prime}(x)h'(x)-f(x)h'(x)) -\frac{1}{2}h^2(x)+\frac{1}{2}a^{*\prime\prime}(x)-f'(x)\}\rho(x,t)$$

$$\rho(x,0)=p_0(x), \qquad (2.2)$$

where we have made the simplifying assumption $y(0)=0$. To simplify later computations we introduce the notation

$$a^*(x): = g^2(x)$$
$$b^*(x,t): = y(t)a^*(x)h'(x)+a^{*\prime}(x)-f(x)$$
$$c^*(x,t): = \frac{1}{2}y^2(t)a^*(x)h'(x)^2 + y(t)(\frac{1}{2}a^*(x)h''(x)+a^{*\prime}(x)h'(x)- - f(x)h'(x)) -\frac{1}{2}h^2(x)+\frac{1}{2}a^{*\prime\prime}(x)-f'(x). \qquad (2.3)$$

The significance of the transformation (2.2), which brings the fundamental equation to its so called "robust form", is that we are now faced with a classical p.d.e. parametrized by the observation paths. In particular for a given sample path $y\in C[0,T]$ (2.3) is a linear parabolic p.d.e., and therefore classical p.d.e. methods can be applied in analyzing existence, uniqueness, regularity of solutions.

We can now make precise what we shall understand as a solution to the stochastic p.d.e. (1.9) or (1.12).

Definition 2.2: $\nu(x,t)$ is a regular solution of (1.9) or (1.12), if for each $y\in C[0,T]$,

(i) it belongs to $L^\infty(\mathbb{R}\times(0,T))$ and to $L^1(\mathbb{R})$ for each $t\in[0,T]$;
(ii) it is related via an exponential transformation (2.1) to a non-negative regular solution of (2.2).

This route to analyzing the unnormalized conditional density in nonlinear filtering of diffusion processes has been followed previously by Davis (1980) and Pardoux (1979), although they emphasized different constructions than ours. Furthermore their results cover cases where the system parameters, i.e. the functions f,g,h are uniformly bounded. So in a certain sense the results presented here are extensions to unbounded coefficients.

Classical p.d.e. results useful in nonlinear filtering can be found in Aronson and Besala (1967), which are extensions of earlier results by Krzyzański (1962), Szybiak (1959) and Bodanko on existence and uniqueness of regular solutions to linear parabolic p.d.e. with unbounded coefficients.

Using such results we see that under some widely used conditions on f,g,h of (1.1), existence, uniqueness and regularity of (2.2) hold. Specifically consider

Hypothesis 1: The coefficient functions f,g,h of (1.1) satisfy ($a^*(x)=g^2$ as in (2.3)):

(i) $f,g,h,f_x,g_x,h_x,g_{xx},h_{xx}$ are Hölder continuous on compact subsets of $\mathbb{R}$;
(ii) For some $0\leq\lambda\leq 2$, there exist positive constants $G,G_1,G_2,F,F_1,H,H_1,H_2$ such that

$$|g(x)|\le G(|x|^2+1)^{(2-\lambda)/4},\ |g_x(x)|\le G_1(|x|^2+1)^{\lambda/4},$$
$$|g_{xx}(x)|\le G_2(|x|^2+1)^{(3\lambda-2)/4},\ |f(x)|\le F(|x|^2+1)^{1/2},$$
$$|f_x(x)|\le F_1(|x|^2+1)^{\lambda/2},\ |h(x)|\le H(|x|^2+1)^{\lambda/4},$$
$$|h_x(x)|\le H_1(|x|^2+1)^{\frac{(\lambda-1)}{2}},\ |h_{xx}(x)|\le H_2(|x|^2+1)^{\lambda-1} \quad (2.4)$$

for all $(x,t)\in\mathbb{R}\times[0,T]$.

We have then the following existence result for (2.2).

Theorem 2.1 (Baras, Mitter, Ocone (1980)) Suppose Hyp. 1 holds and that $g(x)\neq 0$ in $\mathbb{R}$. Then for some $T_0>0$, the fundamental solution $\Gamma(x,t;\xi,\tau)$ of (2.2) exists for all $x,\xi\in\mathbb{R}$, $0\le\tau\le t\le T_0$ and

$$\int_{\mathbb{R}}\Gamma(x,t;\xi,\tau)d\xi\le Me^{G(x)}$$

where

$$G(x)=\begin{cases}[k\ln(|x|^2+1)+1]^2, & \text{if } \lambda=0\\ k(|x|^2+1)^{\lambda/2}; & \text{if } 0\le\lambda\le 2\end{cases}$$

and M,k are positive constants. Moreover $\Gamma(x,t;\xi,\tau)\ge 0$ and it is the fundamental solution of the adjoint equation to (2.2) as a function of (ξ,τ).

Theorem 2.2 (Baras, Mitter, Ocone (1980)): Suppose hypothesis 1 holds and that $g(x)\ge\gamma(|x|^2+1)^{(2-\lambda)/4}$ for some $\gamma>0$ and all $(x,t)\in\mathbb{R}\times[0,T]$. If $\rho_1(x,t)$, $\rho_2(x,t)$ are two non-negative regular solutions of (2.3) on $\mathbb{R}\times(0,T]$ with the same initial data then $\rho_1(x,t)=\rho_2(x,t)$ in $\mathbb{R}\times(0,T]$.

We note that theorems 2.1, 2.2, cover the linear case, where $f(x)=\alpha x$, $g(x)=\beta$, $h(x)=\gamma x$, α,β,γ constants, while for example the results in Hazewinkel, Willems (1980); Pardoux (1979), do not. In Baras, Mitter, Ocone (1980), vector analogs of these results are given.

There are of course several theorems of this type, that one can give based on classical p.d.e. results. An important point to make, however, is that existence of solutions (1.9) (1.12) can be established quite easily for fairly general coefficients by probabilistic arguments. This route is based on certain martingale properties of the Kallianpur-Striebel path integral representation of solutions (1.6).

Then let

$$\mathcal{G}_t=\sigma\{w(s),v(s),x(0),0\le s\le t\}\ . \quad (2.5)$$

It is a direct consequence of the Kallianpur-Striebel formula that

$$\nu(x,t)=E_x\{\nu_0(x(t))\exp[\int_0^t h(x(s))dy(s) - \frac{1}{2}\int_0^t h^2(x(s))ds]\mu(dx)\} \quad (2.6)$$

It is not difficult to show that under reasonable assumptions on $x(\cdot)$ and h, $\nu(x,t)$ is a.s. finite.

Theorem 2.3 (Baras, Blankenship 1980). Suppose (1.1) has a weak solution on $\mathbb{R}\times[0,T]$, and that $P(\int_0^T h^2(x(s))ds)=1$. Then the stochastic Feyman-Kac formula (2.6) provides a well defined a.s. finite representation for $\nu(x,t)$.

Note that $\nu(x,t)$, as given by (2.6) will satisfy the DMZ equation (1.9) in a weak sense. There is no claim for uniqueness in Theorem 2.3.

A uniqueness result different than that of theorem 2.2 can be stated. It is clear that via appropriate transformations it suffices to consider the case where f=0, g=1. Then by a rather standard exponential transformation, and a classical maximum principle argument one can show that solutions to (1.12) are unique provided

$$|h(x)|/|h'(x)|\to\infty \text{ as } |x|\to\infty$$
$$|h^2(x)|/|h''(x)|\to\infty \text{ as } |x|\to\infty \quad (2.7)$$

Note that this condition includes polynomial nonlinearities, in the observations.

Instead of computing the conditional density from (1.9) (1.12), one may only be interested in computing some conditional "statistic" $s(t)=E\{\varphi(x(t))|\mathcal{F}^y_t\}$. There are several ways one can compute such statistics, but the most widely understood way is via a finite dimensional stochastic system driven by the observations:

$$dz(t)=F(z(t))dt+G(z(t))dy(t),\ s(t)=H(z(t)). \quad (2.8)$$

We shall call (2.8) a finite dimensional nonlinear filter. A natural question is then: Given (1.1) when can we say that there exists (or not) a statistic s(t) that can be described by a nonlinear filter like (2.8)? For a special type of statistic, Brockett (1978) was able to provide an interesting answer. The argument goes as follows. Suppose we restrict our attention to analytic functionals of the process, meaning s(t) is an analytic functional of $\rho(x,t)$. In (2.8) z(t) is usually vector valued. Then we can view the map from observations y to the statistic s as an input-output map, which has two realizations. One is via the conditional density equation (1.9) and the other is via the filter (2.8), or

$$dz(t)=\{F(z(t))-\frac{1}{2}G_z(z(t)G(z(t))]dt+G(z(t))dy(t) \quad (2.9)$$

Brockett (1978) then, by analogy with finite dimensional linear analytic realization theory, conjectured that the two Lie algebras:

$$\mathcal{L}.A\{\mathcal{L}_0^*-\frac{1}{2}\mathcal{L}_1,\mathcal{L}_1\}\overset{hom}{\to}\mathcal{L}.A\{F-\frac{1}{2}G_zG,G\} \quad (2.10)$$

should be homomorphic. Moreover once the homomorphism is known a statistic, evolving according (2.8) can be constructed. This observation created a stream of recent work, which aims at understanding the ideal structure of the so called estimation algebra (the left hand side of (2.10) (see Hazewinkel, Willems 1980).

3. APPROXIMATION METHODS

Synthesis of nonlinear estimators using limited computational resources is a critical problem in engineering and the applied sciences. While it is possible that the algebraic structure of the DMZ-equation will reveal novel finite dimensional, recursive filters in specific cases, it is not unlikely that most of these will be mathematical curiosities that do not contribute to the synthesis problem in any major way. Rather the algebraic structure of the DMZ-equation in particular, and of nonlinear estimation problems in general will be of value to the extent that it enables the construction of useful approximations to specific nonlinear filters.
Given this prolegomenon, we shall examine two approximation methods for nonlinear filters: (i) approximate evaluation of the function space integral for the conditional density; and (ii) techniques for the asymptotic analysis of filtering problems containing parameters.

A. Accurate evaluation of stochastic function space integrals

In the system

$$\begin{aligned} dx(t)&=f(x(t)dt+g(x(t)dw(t) \\ dy(t)&=h(x(t)dt+dv(t), \quad 0\le t\le T \end{aligned} \tag{3.1}$$

suppose that $g(x)\neq 0$ everywhere so that $x(t)$ is equivalent under a change of measure (Girsanov transformation) to a Wiener process, say $z(t)$. Let $(z(t),y(t))$ be the process in the transformed system. An estimate of z given $\mathcal{F}_t^y$ is evidently equivalent to an estimate of x. Since z is a Wiener process the function space integral (equation (1.13)) which represents the conditional density is a Wiener integral, that is, an integral versus Wiener measure on $C([0,T])$. Hence, its evaluation or approximation is especially simple. To see this consider the stochastic Wiener integral

$$I = E\{\exp[\int_0^T V(x(s)dy(s)]\} \tag{3.2}$$

Here V is a smooth function, y is an Itô process, and the expectation is relative to Wiener measure. (So I is a random variable with sample space the path space of y.) Now if F is a functional on $C([0,T])$ the Wiener integral $I_0=E\{F(x)\}$ is defined as the sequential limit

$$I_0= \lim_{\substack{\max|t_j-t_{j-1}|\to 0 \\ 1\le j\le n}} \int_{R^n} da_1 \ldots da_n F(z_{sx}) \cdot \prod_{j=1}^{n} \frac{\exp[-(a_j-a_{j-1})^2/2(t_j-t_{j-1})]}{[2\pi(t_j-t_{j-1})]^{n/2}} \tag{3.3}$$

where $0<t_1<t_2<\ldots<t_n=T$ and z_{sx} is a polynomial function on $[0,T]$ passing through x at t=0. If one chooses quadratic interpolation formulas for the Brownian paths (an idea due in this context to A. Chorin), expands the functional F in a Taylor series, and then uses the Gaussian-Wiener measure to explicitly evaluate the terms in the expansion of the integral, the approximation

$$E\{F(x)\}=\pi^{-n/2}\int_{R^n} F_n(u_1,\ldots,u_n)\exp(-u_1^2-\ldots-u_n^2) \cdot du_1\ldots du_n+O(n^{-2}) \tag{3.4}$$

follows. The analysis in (Blankenship, Baras 1980) is an extension of this process to Wiener integrals of the stochastic functional $F(x)=\exp[\int_0^T V(x(s)dy(s)]$. From Theorem 4 of (Blankenship, Baras 1980)

$$E\{\exp[\int_0^T V(x(s))dy(s)]\}$$
$$=(2\pi)^{-n/2}\int_{R^n} \exp[\sum_{i=1}^{n} V(x_{i-1}+vT/(2n)^{\frac12}\Delta y_{i-1}]\} \cdot\exp[(-u_1^2-\ldots-u_{n-1}^2-v^2)/2]du_1\ldots du_{n-1}dv+e_n \tag{3.5}$$

where $(E|e_n|^2)^{\frac12}=O(n^{-2})$ (expectation over the paths of y). This formula together with the results of Chorin (1973) for the case $dy(s)\to ds$ lead to simple accurate approximations in filtering problems. We illustrate this with two generic examples.

Conisder

$$\begin{aligned} dx(t)&=\tfrac12 x(t)dt+x(t)dw(t), x_0>0 \\ dy(t)&=x(t)dt+dv(t), \quad 0\le t\le T \end{aligned} \tag{3.6}$$

This violates $g(x)\neq 0$, but it is accessible to a simple transformation.

Let $z=\ln x$. Then, using the Itô calculus

$$\begin{aligned} &dz(t)=dw(t), \quad dy(t)=e^{z(t)}dt+dv(t), \\ &z_0=\ln x_0 \text{ with density } q_0 \end{aligned} \tag{3.7}$$

The DMZ equation in the new coordinates is

$$d\xi_t=\tfrac12\frac{\partial^2}{\partial z^2}\xi_t dt+e^z\xi_t dy(t), \xi_0(z)=q_0(z) \tag{3.8}$$

Because the Laplacean appears in (3.8) the underlying measure μ in the representation formula (1.13) is Wiener measure. In the "backwards" form (Pardoux 1979)

$$\xi_t(z)=E\{\exp[\int_t^T e^{(z(s)+z)}d\tilde{y}(s)- \tfrac12\int_t^T e^{2(z(s)+z)}ds]\cdot\phi(z(T)+z)\} \tag{3.9}$$

Here $\xi_T(z)=\phi(z)$ and $d\tilde{y}$ is a backwards Itô increment and the expectation is over Brownian paths z with z(t)=z. It is not obvious that (3.9) is well-defined; however, using results onstochastic integrals with respect to semi-martingales, one can show that (3.9) is finite almost surely (samples of y). Using (3.5) and Chorin's formula it is clear how to approximate $\xi_t(z)$ arbitrarily well. We will omit the obvious expression.

A second example considered in Blankenship, Baras (1980) is

$$dx(t)=f(x(t))dt+dw(t),\ dy(t)=x(t)dt+dv(t),\quad 0\le t\le T. \tag{3.10}$$

B. Asymptotic Analysis of Filtering Problems

Suppose now that the operators $\mathcal{L}_0, \mathcal{L}_1$ in (1.9) depend on a parameter $\epsilon>0$ and that the operator limit $(\mathcal{L}_0^\epsilon, \mathcal{L}_1^\epsilon)\to(\mathcal{L}_0^0, \mathcal{L}_1^0)$ is "natural" as $\epsilon\downarrow 0$. This is clearly the case when the message process x is weakly stochastic $(g(x)\sim\epsilon g(x))$, or the observation noise is small $(v(t)\sim\epsilon v(t))$, etc., in (3.1). In the papers (Blankenship 1980; Blankenship, 1978) a methodology is developed for an asymptotic analysis of such problems, especially when the limit system $(\mathcal{L}_0^0, \mathcal{L}_1^0)$ corresponds to a Kalman-Bucy system. We will illustrate this analysis with a single example-weak parametric noise.

Consider the scalar system

$$\begin{aligned} dx^\epsilon(t) &= ax^\epsilon(t)dt+bdw(t)+\sqrt{\epsilon}x^\epsilon(t)dn(t) \\ dy^\epsilon(t) &= cx^\epsilon(t)dt+dv(t),\quad 0\le t\le T \end{aligned} \tag{3.11}$$

where (a,b,c) are constants, $\epsilon>0$, and v,w,n are independent, standard Brownian motions. As $\epsilon\downarrow 0$ the pair (x^ϵ, y^ϵ) approaches, in distribution, a Gauss-Markov process (x,y) to which the Kalman-Bucy algorithm may be applied. To estimate $x^\epsilon(t)$ given $\mathcal{F}_t^{y^\epsilon}$ for ϵ small, we use a perturbation argument based on this "limiting filter".

The DMZ equation for (3.11) is

$$d\sigma_t^\epsilon=\mathcal{L}_0^{\epsilon*}\sigma_t^\epsilon dt+\mathcal{L}_1\sigma_t^\epsilon dy^\epsilon(t),\quad \mathcal{L}_1 f=(cx)\cdot f \tag{3.12}$$

$$\mathcal{L}_0^{\epsilon*} f=\frac{1}{2}\frac{\partial^2}{\partial x^2}f-a\frac{\partial}{\partial x}(xf)+\epsilon\frac{1}{2}\frac{\partial^2}{\partial x^2}(x^2 f)\triangleq\mathcal{L}_0^{0*}+\epsilon A$$

Take the ansatz $\sigma_t^\epsilon=\sigma_t^0+\epsilon\sigma_t^1+\ldots$ and substitute in(3.12). Equating powers of ϵ, this gives

$$d\sigma_t^0=\mathcal{L}_0^{0*}\sigma_t^0+\mathcal{L}_1\sigma_t^0 dy^\epsilon(t) \tag{3.13}$$

which is the Kalman-Bucy algorithm with y^ϵ as an input, and

$$d\sigma_t^k=[\mathcal{L}_0^{0*}\sigma_t^k dt+\mathcal{L}_1\sigma_t^k dy^\epsilon(t)]+A\sigma_t^{k-1}dt,\quad k=1,2,\ldots \tag{3.14}$$

Evidently, σ_t^0, a Gaussian density, is the fundamental solution of (3.14). Therefore, (3.14) can be integrated explicitly in terms of various moments of a Gaussian for any k. We will omit the self-evident expression; see Blankenship (1980) for details.

Convergence of the asymptotic expansion for σ_t^ϵ can be shown by writing the evolution equation for $\theta_t^{\epsilon k}=\sigma_t^\epsilon-(\sigma_t^0+\epsilon\sigma_t^1+\ldots+\epsilon^k\sigma_t^k)$ and exploiting simple estimates based on the Gaussian density σ_t^0.

While it is evident that an asymptotic analysis can yield new information about certain nonlinear filtering problems, the practical and computational limits of the methodology are, as yet, unknown.

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