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LINEAR FILTERING WITH
QUANTUM MECHANICAL MEASUREMENTS

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ABSTRACT

Filtering problems arising in quantum electronics are investigated in the quantum mechanical framework. In particular the minimum variance linear estimator based on optimal selection of present quantum measurement and optimal linear processing of past and present measurement outcomes is found for vector signal processes. Under certain natural assumptions a separation theorem for the filter is proven which leads to considerable simplification of the complexity of the filter. Examples from optical communication systems provide recursive filters and illustrate the results.

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## Summary

With the advent of lasers detection and estimation problems in quantum electronics became of primary importance [1, 2, 3]. These studies explicitly demonstrated the importance of correctly formulating these problems incorporating quantum mechanics. Such work applies directly, e.g., to establishing fundamental limitations in optical communication systems [4]. More recently the analogue of filtering a random signal sequence has been considered [5, 6, 13, 14, 15, 16, 17]. Here we would like to summarize and present the most important of the results obtained in this work. The problems are novel and nonstandard due to the differences between classical probability theory and the mathematical description of quantum mechanical measurements and statistics.

Consider the problem of estimating $x_k$, a member of a discrete time signal sequence $\{x_0, x_1, \dots, x_k, \dots\}$ of vector random variables, utilizing past and present quantum mechanical measurements. To be chosen is the optimal measurement at time k and the optimal linear combination of present and past measurements at times $i = 0, 1, \dots k$. The optical communication setting for this problem is as follows: At time k a laser field modulated in some fashion by $x_k$ is received in a cavity containing otherwise only an electromagnetic field due to thermal noise: the total field is in a state described by a density operator $\rho(x_k)$ that depends on $x_k$ (but not otherwise on k). Since the signal process is an $R^N$ vector process $\{x(k)\}$ this problem is much more delicate and difficult than the scalar problem [16] because of the fundamental "compatibility" constraint in quantum mechanics. That is, only "compatible" measurements can be made simultaneously [7]. However by adjoining auxiliary aparatus to the original system (see Holevo [1, 2]) one can perform "compatible" measurements on the augmented system which correspond statistically to "non-compatible" measurements on the original system. By optimal is meant <u>minimum mean-square error</u> and the implied average is over the (classical) distributions of $\{x_k\}$ and the distributions due to quantum mechanical measurements.

The state of the received field is described quantum mechanically [7] by a density operator (d. o.) $\rho(x_k)$, on a Hilbert space $\mathcal{H}$, which is self-adjoint, positive, and has trace one. Thus instead of considering measurements as represented by projection valued measures (and thus self-adjoint operators) we have to consider representations via positive operator valued measures (p. o. m.). Such measurements are called extended measurements and are described as mappings

$$M: \mathcal{B}^N \longrightarrow B(\mathcal{H}) \tag{1}$$

(where $\mathcal{B}^N$ is the Borel $\sigma$-algebra of $R^N$ and $B(\mathcal{H})$ the algebra of bounded operators on $\mathcal{H}$) such that

i) $\forall B \in \mathcal{B}^N, \ M(B) \geq 0$

ii) $\forall$ partition $\{B_i\}$ of $R^N$, $B_i \in \mathcal{B}^N$; $\sum_i M(B_i) = I$

As pointed out by Holevo [2], the extension is well justified in view of Naimark's theorem: $\exists$ a Hilbert space $\mathcal{H}_e$, a state $\rho_e$ on $\mathcal{H}_e$ and a simple measurement (i. e. corresponding to a projection valued measure) $M_V$ on $\mathcal{H} \otimes \mathcal{H}_e$ such that $\forall B \in \mathcal{B}^N$ and $\forall \rho$ on $\mathcal{H}$

$$\mathrm{Tr}\{(\rho \otimes \rho_e) M_V(B)\} = \mathrm{Tr}\{\rho M(B)\} \tag{2}$$

The triple $\{\mathcal{H}_e, \rho_e, M_V\}$ is called a realization of the measurement represented by the p. o. m. M.

Considering now a sequence of measurements represented by the p. o. m.'s $M_i$, with vector outcomes $v_i$ we have that the joint distribution function characterizing the outcomes of the first k+1 measurements is given by

$$F_{v_0, \dots v_k}(a_0, \dots, a_k) = \int \dots \int F_{v_0|x_0}(a_0, \xi_0) \dots F_{v_k|x_k}(a_k, \xi_k) F_{x_0, \dots x_k}(d\xi_0, \dots d\xi_k) \tag{3}$$

where

$$F_{v_i|x_i}(a_i, \xi_i) = \mathrm{Tr}[\rho(\xi_i) M_i \Big( \prod_{i=\ell}^{N} (-\infty, a_{i\ell}] \Big)] \tag{4}$$

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$\pi$ denotes cartesian product, and $a_{i\ell}$ is the $\ell^{th}$ coordinate of the vector $a_i$. In (3) we utilize the assumption that we cleanse the cavity after each measurement which implies that the measurement outcomes conditioned on the signal are independent.

The corresponding filtering problem then is to find a p.o.m. $M_k$ and $N \times N$ matrices $C_i(k)\ i = 0,1, \ldots k$ as to minimize

$$\mathrm{MSE} = E\{\|x_k - \hat{x}_k\|^2_{R^N}\} \tag{5}$$

where

$$\hat{x}_k = \sum_{i=0}^{k} C_i(k) v_i \tag{6}$$

The average in (5) is with the statistics of (3). We have then the following results (for details see [17], [14]).

<u>Theorem 1:</u> There exist p.o.m. $\hat{M}_k$ and matrices $\hat{C}_i(k)$, $i = o, \ldots k-1$ which minimize the MSE in (5).

<u>Theorem 2:</u> Necessary and sufficient conditions for $\hat{C}_0(k), \ldots \hat{C}_k(k)$ and $\hat{M}_k$ to be the optimal processing coefficient matrices and the optimal p.o.m. at time k are

i) $$\langle \mathfrak{J}_{\hat{C}(k)}, X \rangle_{R^N} \geq \langle \mathfrak{J}_{\hat{C}(k)}, \hat{M}_k \rangle_{R^N} \tag{7}$$

for every other p.o.m. X

and ii)
$$E\left\{\begin{bmatrix} v_0 \\ \vdots \\ v_k \end{bmatrix} [v_0^T \ldots v_k^T]\right\} \begin{bmatrix} \hat{C}_0(k)^t \\ \vdots \\ \hat{C}_k(k)^t \end{bmatrix} = \begin{bmatrix} Ev_0 x_k^t \\ \vdots \\ Ev_0 x_k^t \end{bmatrix} \tag{8}$$

where $\langle \mathfrak{J}_{\hat{C}(k)}, X \rangle_{R^N}$ is the trace-integral [2] of the operator valued function $\mathfrak{J}_{\hat{C}(k)}$ with respect to the p.o.m. X over $R^N$, and for $u \in R^N$ we have:

$$\mathfrak{J}_{\hat{C}(k)}(u) = \Lambda_k - 2u^t \hat{C}_k(k) \delta_k + u^t \hat{C}_k(k)^t \hat{C}_k(k) u\, n_k$$
$$+ 2 \sum_{i=0}^{k-1} u^t \hat{C}_k(k)^t \hat{C}_i(k) \zeta_{k,i} - 2 \sum_{i=0}^{k-1} \mathrm{Tr} \hat{C}_i(k)^t \pi_{k,i} \tag{9}$$

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where

$$
\begin{aligned}
\Lambda_k &= \int x_k^t \, x_k \, \rho(x_k) \, F_{x_k}(dx_k) \\
n_k &= \int \rho(x_k) \, F_{x_k}(dx_k) \\
\delta_k &= \int x_k \, \rho(x_k) \, F_{x_k}(dx_k) \\
\zeta_{k,i} &= \int E\{v_i \mid x_i\} \, \rho(x_k) \, F_{x_i, x_k}(dx_i, dx_k) \\
\pi_{k,i} &= \int x_k \, E\{v_i \mid x_i\}^t \, \rho(x_k) \, F_{x_i, x_k}(dx_i, dx_k)
\end{aligned}
\tag{10}
$$

Here we have also a "separation" of the filter under additional conditions as described in the sequel. We restrict to p.o.m.'s with a base [2], that is p.o.m.'s that can be expressed as

$$
x(B) = \int_B P(u) \, \mu(du) \tag{11}
$$

for some positive-operator valued function P and a measure $\mu$ on $\mathcal{B}^N$. Let $\hat{M}_i$, $i = 0, \ldots k$ be the optimal measurements when we use post processing (with outcomes $v_i$) and let $\hat{Z}_i$, $i = 0, \ldots k$ be the optimal measurements (with outcomes $z_i$) when we do not use post processing. Then we have:

Theorem 3: Suppose that the vector signal sequence $\{x_i\}$ is pairwise Gaussian, and that $z_i, x_i$ are jointly Gaussian for $i = 0, 1, \ldots k$. Then the measurement outcomes $z_i$, $i = 0, \ldots k$ are a sufficient statistic for the linear mean square estimate of $x_k$.

This "separation" is best illustrated in Figure 1, which demonstrates the tremendous reduction in the complexity of the filter.

As a multiparameter recursive filtering example consider the problem of estimating the two dimensional dynamical state $x_k$ which is transmitted as the in-phase $x_{1k}$ and quadrature $x_{2k}$ amplitudes of a laser (assumed monochromatic)

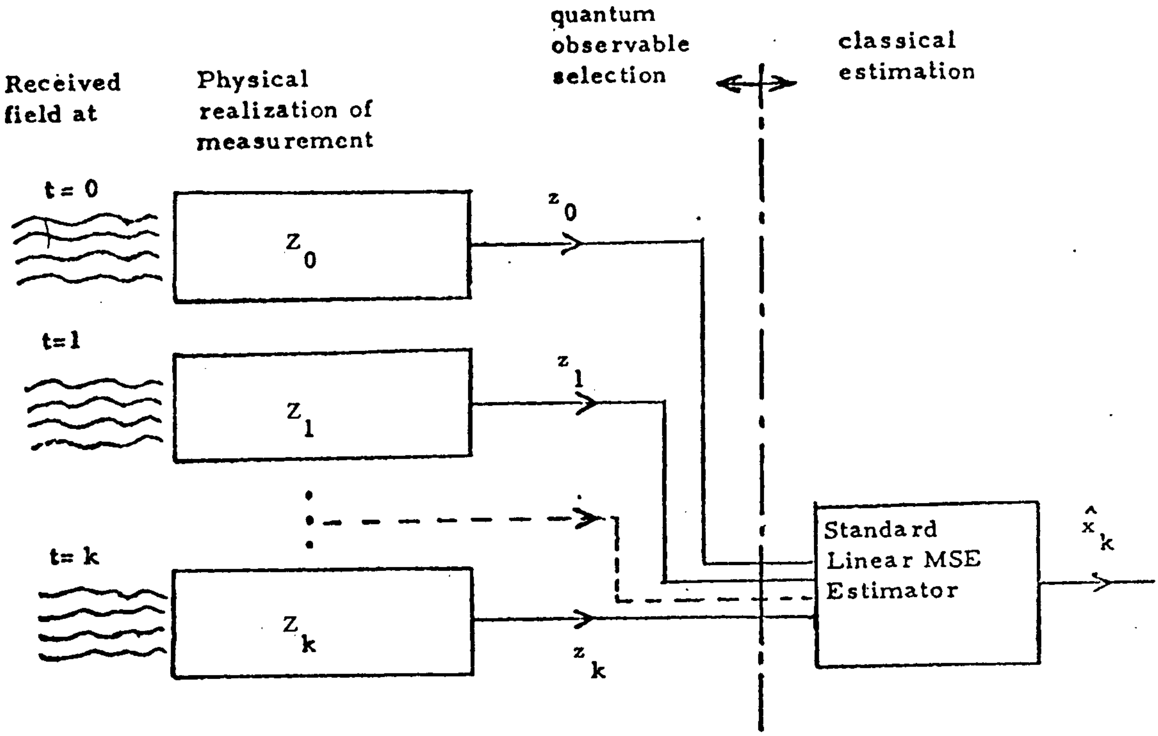


Figure 1.

and received along the thermal noise, in a single mode cavity. Suppose that $x_k$ satisfies the recursion

$$x_{k+1} = \phi_k x_k + w_k$$

where $\phi_k$ is a sequence of NxN matrices and $w_k$ a white noise sequence with covariance $Q_k$. Then the filter becomes [17,14]:

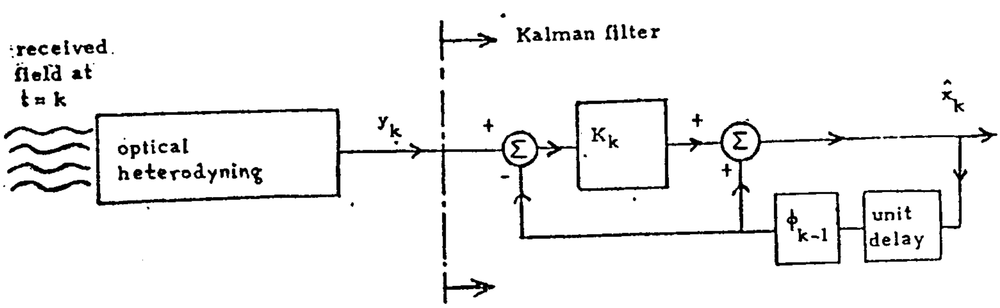


Figure 2.

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where $$K_k = P_k \left[P_k + \left(\frac{n_0}{2} + \frac{1}{2}\right) I_2\right]^{-1}$$

$$P_k = \phi_{k-1} \left[P_{k-1} - K_{k-1} P_{k-1}\right] \phi_{k-1}^{t} + Q_{k-1}$$

Finally in another direction, recursive filters have been obtained in [15,17] for finite memory vector signal processes.

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