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BILINEAR DELAY - DIFFERENTIAL SYSTEMS

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Abstract

We analyze certain controllability properties of systems of the type

$$\frac{dx(t)}{dt} = \left(A + \sum_{i=1}^{p} u_i(t)B_i\right)x(t) + Cx(t-\tau)$$

where $x(t) \in \mathbb{R}^n$, u_i, $i=1,\ldots p$ are scalar functions, measurable and bounded on finite intervals, and A, B_i, C, $i=1,\ldots p$ are nxn matrices. In particular we derive criteria for local accessibility, and a "band-bang" theorem for these systems. These results generalize those existing for bilinear systems without delays and for linear delay-differential systems.

1. Introduction

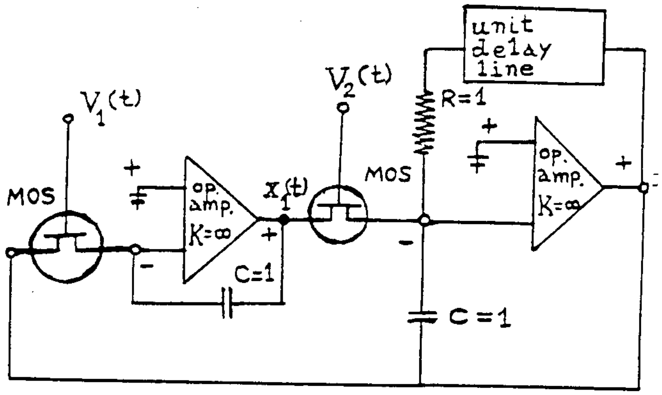
In the last few years the class of finite dimensional bilinear systems has been the subject of intense study by various investigators [1] - [4]. Bilinear systems being the simplest type of nonlinear systems, with a rich structure, provide insight for the analysis of more complex nonlinear systems. On the other hand delay-differential systems have been heavily studied, due to the importance of delays in practical applications [5] - [8]. The theory of linear delay-differential systems has reached a certain degree of completeness to date.

There are however many problems where the dynamics depict both bilinearity (due mainly to variable structure) and hereditary behavior. Here are some representative examples:

Example 1: The prime elements of integrated circuits are resistors, capacitors, transistors (TFT, MOS, FET, MOSFET) and RC lines, while one of the most important generating element is the operational amplifier [11, pp. 8-34]. It is well known [11, p. 13] that MOS transistors can be used as voltage-controlled resistors when operated in the region below pinch-off (i. e. with drain to source voltage $V_d < V_p$ where V_p is the pinch-off voltage), and that the drain to source conductance is approximately given by [12, p. 229]

$$g_d = M(V_g - V_p)$$

where M is a constant depending on the particular transistor. We consider the following network, which consists of an integrated network coupled with a delay line:



The equations for this network are:

$$\dot{x}_1 = -M(V_1(t) - V_p)x_2(t)$$

$$\dot{x}_2(t) = -M(V_2(t) - V_p)x_1(t) - x_2(t-1)$$

or

$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = MV_p\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} - MV_1(t)\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

$$- MV_2(t)\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1(t-1) \\ x_2(t-1) \end{bmatrix}.$$

Example 2: A widely studied class of systems is the linear delay differential systems:

$$\frac{d}{dt}x(t) = Ax(t) + Dx(t-\tau) + Bu(t)$$

$$y(t) = Cx(t)$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^p$, $y(t) \in \mathbb{R}^m$.
A usual control law is that of time varying output feedback $u(t) = K(t)y(t)$. Then the original system becomes

$$\frac{d}{dt}x(t) = Ax(t) + BK(t)Cx(t) + Dx(t-\tau) =$$

$$= Ax(t) + \sum_{i=1,j=1}^{p,m} K_{ij}(t)BE_{ij}Cx(t) + Dx(t-\tau)$$

where E_{ij} is a $p \times m$ matrix with the only non zero element being the ij^{th} one, which is 1.

Example 3: This example originates from the theory of slowing down of neutrons [13]. Consider a homogeneous medium of infinite extension in which, per second, Q neutrons of energy E_0 are produced. The energy is changed by collisions and we want to calculate the stationary energy distribution $\rho(x)$ where $x = E_0/E$. A related quantity is the average number of collisions a neutron experiences in the interval between x and x+ dx, K(x), where with no capture

$$\rho(x)\frac{dx}{x} = Q[\ell(v)/v]\,K(x)\,dx/x$$

with $\ell(v)$ = mean free path for scattering and v = velocity. Let $k(x,x')\frac{dx}{x}$ denote the probability that a neutron shall be in the interval dx after one collision, if its energy before the collision was E_0/x'. This function will depend on the interaction and on the material which causes the slowing down. For example [13, p. 428] if the medium contains nuclei of different atomic weights M_s the energy distribution after one collision is determined by:

$$k(x,x') = \mu\left(\frac{x'}{x}\right)\frac{x'}{x}\sum_s \frac{C_s(x')}{1-p_s}\,\mu\left(\frac{p_s x}{x'}\right)$$

where $\mu(u) = 1$ for $u < 1$ and 0 for $u > 1$, and $C_s(x') = \frac{\ell(x')}{\ell_s(x')}$ is a measure of concentration of nuclei of type s, since ℓ is the total mean free path while ℓ_s the mean free path for scattering at a nucleus of type s. Moreover $p_s = \left(\frac{M_s - 1}{M_s + 1}\right)^2$. Letting $y = \ln x$ after some calculations we derive the equation

$$K(y) = 1 + \sum_s \int_{y-\tau_s}^{y} u_s(\sigma)\,K(\sigma)\,d\sigma$$

where $u_s(\sigma) = \frac{p_s}{1-p_s}C_s(\sigma)$ and $\tau_s = -\ln p_s$ or

$$\frac{dK(y)}{dy} = \Sigma_s\, u_s(y)K(y) - u_s(y-\tau_s)K(y-\tau_s)$$

We can control the slowing down by choosing the concentration of the various types of nuclei and hence the functions u_s. Similar equations can be derived for slowing with capture [13].

The purpose of this paper, is to present several initial results about bilinear delay-differential systems. For simplicity of exposition we chose to consider only systems of the simple type

$$\frac{dx(t)}{dt} = (A + \sum_{i=1}^{p} u_i(t)B_i)x(t) + Cx(t-\tau) \tag{1}$$

where $x(t) \in \mathbb{R}^n$, $u_i(\cdot)$ are scalar functions (the controls) measurable and bounded on finite intervals and A, B_i, C are nxn matrices. However, all the results reported here extend to multiple delays and more complicated hereditary behavior. These will be presented in subsequent papers. We are mainly going to investigate properties of the reachable sets, controllability and accessibility properties and "bang-bang" theory for this class of systems.

2. Dynamical characteristics of bilinear delay-differential systems

Let $\mathcal{U}$ denote the set of admissible controls, which in our case are $\mathbb{R}^p$-valued functions, which are bounded and measurable on every finite time interval. Following a method similar to that used in [5] or [9] it is easy to see that for this class of controls the bilinear delay differential system (1) has a unique absolutely continuous solution on $[t_0, \infty)$ given an initial function $\varphi \in C([t_0-\tau, t_0];\mathbb{R}^n)$ (the space of continuous $\mathbb{R}^n$-valued functions on $[t_0-\tau, t_0]$). We let u denote the column vector with components u_i, $i=1,\dots p$ and $x(t;t_0,\varphi,u)$ the trajectory of (1) for $t > t_0$, using controls u and which satisfies $x(\theta) = \varphi(\theta)$ for $\theta \in [t_0-\tau, t_0]$ and $\varphi \in C([t_0-\tau, t_0];\mathbb{R}^n)$. The trajectory is described explicitely by [9,10]:

$$x(t;t_0,\varphi,u) = K(t,t_0;u)\varphi(t_0) + \int_{t_0-\tau}^{t_0} K(t,s+\tau;u)C\varphi(s)ds \tag{2}$$

where the nxn matrix valued function K(t, s;u) is defined on $[t_0-\tau, t] \times [t_0, t]$ as the unique solution of the differential equation

$$\left.\begin{aligned} &\frac{\partial}{\partial t}K(t,s;u) = (A + \sum_{i=1}^{p} u_i(t)B_i)K(t,s;u) + CK(t-\tau,s;u) \quad \text{for } t \geq s \\ &K(t,s) = I, \text{ for } t = s \\ &K(t,s) = 0, \text{ for } t_0-\tau \leq t < s \end{aligned}\right\} \tag{3}$$

Note that over one delay interval $t-\tau < s \leq t$ the first of (3) reduces to the fundamental matrix equation of a linear timevarying system since $K(t-\tau, s;u) = 0$ then.

Definition: The matrix K(t, s;u) defined above will be called the fundamental matrix of the bilinear delay-differential system (1) corresponding to controls $u_1, \dots u_p$.

The natural state space for (1) is a subset of $C([-\tau, 0];\mathbb{R}^n)$, and we will denote as usual [5] by

$$x_t(\theta) = x(t+\theta); \quad \theta \in [-\tau, 0] \tag{4}$$

the state of the bilinear delay-differential system (1) at time t, whenever x(t) describes a euclidean trajectory of (1). Since we are primarily interested in controllability properties some related notational conventions and definitions are in order.

The reachable set in $\mathbb{R}^n$ from initial condition φ, <u>at</u> time $t > 0$ will be denoted by $R(t, \varphi)$, and it is the set of all $y \in \mathbb{R}^n$ such that $x(t;0,\varphi, u) = y$ for some admissible control u. The reachable set in $\mathbb{R}^n$ from initial condition φ, <u>in</u> time $t > 0$ will be denoted by $\mathbb{R}(t, \varphi)$ and it is the set $\mathbb{R}(t, \varphi) = \bigcup_{0 \le s \le t} R(s, \varphi)$. The reachable set in $\mathbb{R}^n$ from initial condition φ will be denoted by $\mathbb{R}(\varphi)$ and is the set $\mathbb{R}(\varphi) = \bigcup_{t \ge 0} \mathbb{R}(t, \varphi)$. We have similar notions for function space reachability. For ease of notation we let $\mathcal{C}$ denote $C\{[-\tau,0];\mathbb{R}^n\}$ and $\mathcal{C}^1$ denote $C^1\{[-\tau,0];\mathbb{R}^n\}$. Then the reachable set in $\mathcal{C}^1$ from initial condition φ, <u>at</u> time $t>0$, will be denoted by $R_C(t, \varphi)$, and it is the set of all $\lambda \in \mathcal{C}^1$ s.t. $\lambda(\theta) = x_t(\theta)$, $\theta \in [-\tau, 0]$ for some admissible control u. Similarly the reachable set in $\mathcal{C}^1$ from initial condition φ, <u>in</u> time $t > 0$, is the set $\mathbb{R}_C(t, \varphi) = \bigcup_{0 \le s \le t} R_C(s, \varphi)$ and the reachable set in $\mathcal{C}^1$ from initial condition φ, is the set $\mathbb{R}_C(\varphi) = \bigcup_{t \ge 0} \mathbb{R}_C(t, \varphi)$.

We have now the following set of definitions (see also [2]):

<u>Definition 1:</u> Let $\lambda(\cdot) = x(\cdot;0,\varphi,u)$ be a trajectory of the system. The system has the <u>local accessibility property along λ, in $\mathbb{R}^n$,</u> at time t_1 if there exists an $\mathbb{R}^n$-neighborhood of $x(t_1;0,\varphi, u)$ which is included in $R(t_1, \varphi)$.

<u>Definition 2:</u> Let λ as above. The system has the <u>local accessibility property along λ, in function space</u>, at time t_1 if there exists a $\mathcal{C}$-neighborhood of x_{t_1} which is included in $R_C(t_1, \varphi)$.

<u>Definition 3:</u> The system is euclidean controllable (resp. at time t_1, in time t_1) from initial condition φ if $\mathbb{R}(\varphi) = \mathbb{R}^n$ (resp. $R(t_1, \varphi) = \mathbb{R}^n$, $\mathbb{R}(t_1, \varphi) = \mathbb{R}^n$).

<u>Definition 4:</u> The system is <u>function space controllable</u> to a subspace $H \subseteq \mathcal{C}^1$ (resp. at time t_1, in time t_1) from initial condition φ if $H \subseteq \mathbb{R}_C(\varphi)$ (resp. $H \subseteq R(t_1, \varphi)$, $H \subseteq \mathbb{R}_C(t_1, \varphi)$).

<u>Definition 5:</u> The system is <u>completely euclidean controllable</u> (at time t_1, in time t_1) if it is euclidean controllable (at time t_1, in time t_1) from every initial condition φ.

<u>Definition 6:</u> The system is <u>completely function space controllable</u> to the subspace $H \subseteq \mathcal{C}^1$ (at time t_1, in time t_1) if it is function space controllable to H (at time t_1, in time t_1) from every initial condition φ.

<u>Definition 7:</u> The system has the <u>euclidean accessibility property from φ</u> (resp. the <u>accessibility property in function space from φ</u>) if $\mathbb{R}(\varphi)$ (resp. $\mathbb{R}_C(\varphi)$) has non empty interior in $\mathbb{R}^n$ (resp. in $\mathcal{C}$).

<u>Definition 8:</u> The system has the <u>euclidean accessibility property</u> (resp. <u>the accessibility property in function space</u>) if it has the euclidean accessibility property (resp. the accessibility property in function space) from every initial condition φ.

<u>Definition 9:</u> If we replace $\mathbb{R}(\varphi)$ (resp. $\mathbb{R}_C(\varphi)$) with $R(t, \varphi)$ (resp. $R_C(t, \varphi)$ for some $t > 0$ in Definitions 7 and 8 we have the <u>strong euclidean accessibility property</u> (resp. <u>strong accessibility property in function space</u>) from initial condition φ. Similarly for every φ.

It is a consequence of the definitions given, that the system has the strong euclidean accesibility property (resp. strong accessibility property in function space) from initial condition φ if and only if it has the local euclidean accessibility property (resp. in function space) along all trajectories emanating from φ at some time $t > 0$, the same for all trajectories. It may help to note that whether we are in $\mathbb{R}^n$ or in function space if the system has the local accessibility property along some trajectory at some time t_1, it certainly has the local accesibility along the same trajectory at any time $t_2 \ge t_1$. So conditions guaranteeing local accessibility a long all trajectories imply strong accessibility. Local accessibility is very strongly related to controllability of linearized equations. Notice that controllability implies accessibility.

<u>3. Euclidean Accessibility Properties</u>

Consider the general nonlinear differential delay system

$$\dot{x}(t) = f(t, x(t), x(t-\tau), u(t)) \tag{5}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^p$, f is continuously differentiable in all arguments and $f(t,0,0,0)=0$. We first show that local euclidean accessibility of (5) is implied by controllability of the linearized system about the trajectory $x_0(t) = x(t;0,\varphi,u_0)$:

$$\dot{y}(t)=A(t)y(t)+C(t)y(t-\tau)+B(t)u(t) \tag{6}$$

where

$$\left.\begin{aligned} A(t) &= \frac{\partial}{\partial x} f(t,x(t),x(t-\tau),u(t))\Big]_{x_0,u_0} \\ C(t) &= \frac{\partial}{\partial x_{-\tau}} f(t,x(t),x(t-\tau),u(t))\Big]_{x_0,u_0} \\ B(t) &= \frac{\partial}{\partial u} f(t,x(t),x(t-\tau),u(t))\Big]_{x_0,u_0} \end{aligned}\right\} \tag{7}$$

where $u_0 \in \mathcal{U}$ and $x_{-\tau}(t)=x(t-\tau)$. The method is a variation of that used previously by Weiss [o].

Theorem 1: Suppose that system (6) is completely euclidean controllable at time t_1. Then the nonlinear delay differential system (5) has the local euclidean accessibility property along x_0 at t_1.

Proof: We let

$$z(t)=x(t)-x_0(t)=-x_0(t)+\varphi(0)+\int_0^t f(\sigma,x(\sigma),x(\sigma-\tau),u(\sigma))d\sigma \tag{8}$$

and introduce a parameter $\xi \in \mathbb{R}^n$ in (8) via

$$u^{\xi}(t)=u_0(t)+B^T(t)K^T(t_1,t)\xi\ ;\ 0 \le t \le t_1 \tag{9}$$

where $K(t,s)$ is the fundamental matrix of (6). Then let the solution of (8) due to control $u^{\xi}(t)$ be $z(t;0,\xi)$ and define

$$J(t) = \frac{\partial z}{\partial \xi}(t;0,\xi)\Big]_{\xi=0} \tag{10}$$

Then since $z(t;0,0)=0$ and $u^0(t)=u_0(t)$ we have from (8)

$$J(t)=\int_0^t \left[A(\sigma)J(\sigma)+C(\sigma)J(\sigma-\tau)+B(\sigma)\frac{\partial u^{\xi}(\sigma)}{\partial \xi}\right]_{\xi=0} d\sigma \tag{11}$$

So $\dot{J}(t)=A(t)J(t)+C(t)J(t-\tau)+B(t)B^T(t)K^T(t_1,t)$ and since $J(\theta)=0$ for $\theta \in [-\tau,0]$ we have

$$J(t_1)=\int_0^{t_1} K(t_1,\sigma)B(\sigma)B^T(\sigma)K^T(t_1,\sigma)d\sigma$$

and from the complete euclidean controllability assumption we have [6]:

$$\det J(t_1) \ne 0 \tag{12}$$

Consider now the map

$g: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$

$$g(\xi,y) = x(t_1;0,\varphi,u^{\xi})-y$$

Then clearly $g(0,x(t_1;0,\varphi,u^0))=g(0,x_0(t_1))=0$ and the Jacobian with respect to ξ is of full rank (12). Then by the implicit function theorem there exists an open neighborhood N_0 of $x_0(t_1)$ in $\mathbb{R}^n$, such that for every open neighborhood of $x_0(t_1)$, $N \subset N_0$ there exists a unique continuous map $\pi: N \to \mathbb{R}^n$ such that $g(\pi(y),y)=0$ for all $y \in N$. But this is precisely the statement of local euclidean accessibility along x_0 at t_1.

Certainly theorem 1 applies to bilinear delay differential systems. We next utilize it to obtain local euclidean accessibility criteria for (1). Let us denote

$$\hat{B}_x(t) \triangleq [B_1x(t);B_2x(t);\ldots;B_px(t)] \tag{13}$$

Then (1) becomes

$$\frac{d}{dt}x(t)=Ax(t)+\hat{B}_x(t)u(t)+Cx(t-\tau) \quad \text{and therefore}$$

$$\left.\begin{aligned} &\frac{\partial}{\partial x}[Ax(t)+\hat{B}_x(t)u(t)+Cx(t-\tau)]\Big]_{x_0,u_0} = A+\sum_{i=1}^{p} u_{i0}(t)B_i \triangleq \\ &\qquad \triangleq \hat{A}(t) \\ &\frac{\partial}{\partial x_{-\tau}}[Ax(t)+\hat{B}_x(t)u(t)+Cx(t-\tau)]\Big]_{x_0,u_0} = C \\ &\frac{\partial}{\partial u}[Ax(t)+\hat{B}_x(t)u(t)+Cx(t-\tau)]\Big]_{x_0,u_0} = \hat{B}_{x_0}(t) \end{aligned}\right\} \tag{14}$$

So the linearized system is

$$\frac{d}{dt}x(t)=\hat{A}(t)x(t)+Cx(t-\tau)+\hat{B}_{x_0}(t)u(t) \tag{15}$$

Theorem 2: Let $x_0(t)=x(t;0,\varphi,u_0)$ be a trajectory of (1), Let

$$P_0(t)=\hat{B}_{x_0}(t)$$
$$P_1(t)=-\hat{A}(t)P_0(t)+\dot{P}_0(t)$$
$$\vdots$$
$$P_k(t)=-\hat{A}(t)P_{k-1}(t)+\dot{P}_{k-1}(t)$$
$$Q_C(t;x_0,k)=[P_0(t);P_1(t);\ldots P_{k-1}(t)]$$

Suppose there exist $t_1 > \tau$ and an integer $k > 0$ such that the derivatives needed in the definition of Q_C exist and are continuous and rank $Q_C(t_1;x_0,k)=n$. Then the bilinear delay differential system (1) has the local euclidean accessibility property along x_0, at t_1.

Proof: Observe that the fundamental matrix of the linearized system (15) $K_\ell(t,s)$ is identical to the fundamental matrix of the bilinear system (1) corresponding to controls u_0, $K(t,s;u_0)$. Notice also that the derivatives needed in the hypothesis will exist and be continuous if the controls u_{i0}, $i=1,\ldots p$ and the initial condition φ have k-2 continuous derivatives, or if the time $t_1>(k-1)\tau$). Then proceeding in a manner first used by Buckalo [14] for linear delay systems we show that the hypothesis of the theorem imply that rank $\int_0^{t_1} K_\ell(t_1,\sigma)\hat{B}_{x_0}(\sigma)\hat{B}^T_{x_0}(\sigma)K^T_\ell(t_1,\sigma)d\sigma = n$. Because if not, there exits $\eta \in \mathbb{R}^n$, $\eta \neq 0$ such that

$$\eta^T K_\ell(t_1,\sigma)\hat{B}_{x_0}(\sigma) \equiv 0 \quad \text{for } \sigma \in [0,t_1] \quad \text{and so}$$

$$\eta^T K_\ell(t_1,\sigma)\hat{B}_{x_0}(\sigma) \equiv 0 \quad \text{for } \sigma \in [t_1-\tau, t_1] \qquad (16)$$

Differentiating (16) we have

$$0=\eta^T\Big[\frac{\partial}{\partial\sigma}K_\ell(t_1,\sigma)\hat{B}_{x_0}(\sigma)+K_\ell(t_1,\sigma)\dot{\hat{B}}_{x_0}(\sigma)\Big]=$$

$$=\eta^T\Big[-K_\ell(t_1,\sigma)\hat{A}(\sigma)\hat{B}_{x_0}(\sigma)+K_\ell(t_1,\sigma)\dot{\hat{B}}_{x_0}(\sigma)\Big]=$$

$$=\eta^T\Big[K_\ell(t_1,\sigma)P_1(\sigma)\Big]$$

Similarly $0\equiv \eta^T[K_\ell(t_1,\sigma)P_m(\sigma)]$, $m=0,1,\ldots k-1$ for $\sigma \in [t_1-\tau,t_1]$. Therefore $0\equiv\eta^T K_\ell(t_1,\sigma)Q_C(\sigma;x_0,k)$, $\sigma\in[t_1-\tau,t_1]$ and so rank $Q_C(t_1;x_0,k)\neq n$, which is a contradiction to the hypothesis. Then since rank $\int_0^{t_1}K_\ell(t_1,\sigma)\hat{B}_{x_0}(\sigma)\hat{B}^T_{x_0}(\sigma)K^T_\ell(t_1,\sigma)= n$ implies complete euclidean controllability at t_1 of the linearized system (15) [6], the result follows from Theorem 1.

The following theorem is an improvement of Theorem 2, and utilizes a result of Weiss [15].

Theorem 3: Let $x_0(t)=x(t;0,\varphi,u_0)$ be a trajectory of (1). Define the matrices $Q_i^j(t)$ via the equations:

$$Q_0^0(t)=\hat{B}_{x_0}(t)$$

$$Q_i^j(t)=\dot{Q}_i^{j-1}(t)-\hat{A}(t+i\tau)Q_i^{j-1}(t)-C\,Q_{i-1}^{j-1}$$

$$i=1,\ldots m,\ j=i,\ldots,k-1$$

and $Q_i^j=0$ for $i<0$ or $i>j$. Let

$$Q(t)=[Q_0^0(t),\ldots Q_1^{k-1}(t),Q_1^1(t-\tau),\ldots Q_1^{k-1}(t-\tau),\ldots$$
$$\ldots Q_m^{k-1}(t-m\ \tau)]$$

Suppose there exist integer $k>0$ and time $t_1 \in [m\tau,(m+1)\tau)$ such that all the derivatives needed in the formation of Q exist and are continuous and rank $Q(t_1)=n$. Then the bilinear delay-differential system (1) has the local euclidean accessibility property along x_0 at t_1.

Proof: We follow Weiss [15], and show that the hypothesis imply that rank

$$\int_0^{t_1}K_\ell(t_1,\sigma)\hat{B}_{x_0}(\sigma)\hat{B}^T_{x_0}(\sigma)K^T_\ell(t_1,\sigma)\,d\sigma = n .$$

For if not, then there exists $\eta\in\mathbb{R}^n, \eta\neq 0$ such that

$$\eta^T K_\ell(t_1,\sigma)\hat{B}_{x_0}(\sigma)\equiv 0\ ;\ \sigma\in[0,t_1] \qquad (17)$$

Or equivalently (17) holds for $\sigma\in[t_1-(i+1)\tau, t_1-i\tau]$ $i=0,1,\ldots m-1$ and for $\sigma\in[0,t_1-m\tau]$. Let $i=0$ and differentiate (17) repeatedly to obtain

$\eta^T K_\ell(t_1,\sigma)Q_0^p(\sigma)=0;\ \sigma\in[t_1-\tau,t_1];\ p=0,1,\ldots k-1.$

Therefore $\eta^T Q_0^p(t_1)=0\ ;\ p=0,1,\ldots k-1 \qquad (18)$

$$\eta^T K_\ell(t_1,t_1-\tau)\,Q_0^p(t_1-\tau)=0;\ p=0,1,\ldots k-1 \qquad (19)$$

Let $i=1$ and differentiate (17) repeatedly to obtain

$$\eta^T[K_\ell(t_1,\sigma)Q_0^p(\sigma)+K_\ell(t_1,\sigma+\tau)Q_1^p(\sigma)+K_\ell(t_1,\sigma+2\tau)Q_2^p(\sigma)]=0$$

for $\sigma\in[t_1-2\tau,t_1-\tau]$, $p=0,1,\ldots k-1$. Therefore

$$\eta^T Q_1^p(t_1-\tau)=0;\quad p=1,\ldots k-1 \qquad (20)$$

and $\eta^T[K_\ell(t_1,t_1-2\tau)Q_0^p(t_1-2\tau)+K_\ell(t_1,t_1-\tau)Q_1^p(t_1-\tau)]=0$

Similarly we have

$$\eta^T Q_m^p(t_1-m\tau)=0;\ p=m,\ldots k-1 \qquad (21)$$

Then (18), (20), (21) imply $\eta^T Q(t_1)=0$ and therefore rank $Q(t_1)\neq n$, contradiction. Then the hypothesis imply that the linearized system (15) is completely euclidean controllable and the result follows from Theorem 1.

Note that the matrix Q_C of Theorem 2 is part of the matrix Q of Theorem 3. Again the derivatives in the hypothesis will exist and be continuous if either the controls u_0 and the initial condition φ have k-2 continuous derivatives or the controls u_0 have k-2 continuous derivatives and $t_1>(k-1)\tau$. Notice that if the reference trajectory corresponds to controls $u_0=0$ (i.e. force free), then the matrix $\hat{A}$ in the statements of Theorems 2, 3 becomes A and we need no differentiability assumption about the controls, which is very satisfactory from practical considerations. Indeed in that case we only need $t_1 > (k-1)\tau$ and the criteria are easily computed. In that case it is easily checked that whenever $C = 0$ (no delays) both criteria, reduce to the well known ones for bilinear systems without delays [2], [16]. Moreover one can then express the matrices Q_C or Q in terms of commutators (or Lie brackets) of the parameter matrices A, B_i, C of the system.

4. Bang-Bang Control

In this section we generalize the results of Sussman [17] to bilinear delay - differential systems. Following [17] we let $U(T)$= set of all measurable functions defined on $[0, T]$ with values in the cube $\{(u_1, u_2, \dots u_p): -1 \le u_j \le 1, j = 1, 2, \dots p\}$; $UB(T) = \{u \in U(T): |u_i(t)| = 1, i = 1, \dots p\}$; $UBP(T) = \{u \in UB(T): u(t)$ is piecewise constant$\}$. Then we know from Lemma 1 of Sussman [17] that $UBP(T)$ is weakly dense in $U(T)$ (in the weak L_2 sense). According to whether we use controls from $U(T)$ or $UB(T)$ or $UBP(T)$ we have for a given initial condition φ, the reachable sets $R(T,\varphi)$, $\mathbb{R}(T,\varphi)$, $RB(T,\varphi)$, $\mathbb{R}B(T,\varphi)$, $RBP(T,\varphi)$, $\mathbb{R}BP(T,\varphi)$. Then we have as a straightforward generalization of Lemma 2 of [17]:

<u>Lemma 1:</u> Let the functions u_k converge weakly to u. Then $x(\cdot;0,\varphi,u_k)$ converge uniformly to $x(\cdot;0,\varphi,u)$ for $0 \le t \le T$.

<u>Proof:</u> For each $v \in U(T)$

$$x(t;0,\varphi,v) = \varphi(0) + \int_0^t \Big[A + \sum_1^p B_i v_i(\sigma)\Big] x(\sigma;0,\varphi,v)\,d\sigma + \int_0^t C x(\sigma-\tau;0,\varphi,v)\,d\sigma$$

now since the functions A, B_i, C are bounded φ is bounded and $|v_i(t)| \le 1$ then there exists constants C_1, C_2, such that

$$\|x(t;0,\varphi,v)\| \le \|\varphi(0)\| + C_1\int_0^t \|x(\sigma;0,\varphi,v)\|\,d\sigma + C_2\int_0^t \|x(\sigma-\tau;0,\varphi,v)\|\,d\sigma$$

Now if $0 \le t \le \tau$ we have

$$\|x(t;0,\varphi,v)\| \le D_1 + C_1\int_0^t \|x(\sigma;0,\varphi,v)\|\,d\sigma$$

where $D_1 = \|\varphi(0)\| + C_2\tau \sup_{\sigma\in[-\tau,0]}\|\varphi(\sigma)\|$. Hence

$$\|x(t;\varphi,v)\| \le D_1 e^{C_1 t} \quad \text{for all } v \text{ and } 0 \le t \le \tau$$

Similarly for $\tau \le t \le 2\tau$

$$\|x(t;0,\varphi,v)\| \le \|\varphi(0)\| + C_1\int_0^t \|x(\sigma;0,\varphi,v)\|\,d\sigma + C_2\int_0^t D_1 e^{C_1(\sigma-\tau)}\,d\sigma \le D_2 + C_1\int_0^t \|x(\sigma;\varphi,v)\|\,d\sigma$$

with obvious identification of constants. So again $\|x(t;0,\varphi,v)\| \le D_2 e^{C_1 t}$ for all v and $\tau \le t \le 2\tau$. By a finite argument (since T is finite) we deduce

$$\|x(t;0,\varphi,v)\| \le D e^{C_1 t} \quad \text{for all } v \text{ and } 0 \le t \le T.$$

thus the functions $x(\cdot;0,\varphi,v_k)$, $v_k \in U(T)$ are uniformly bounded. But then their derivatives are also uniformly bounded, (1). By the Ascoli Arzela theorem every subsequence has a subsequence that converges uniformly to some function. Thus our lemma will be proved if we can show that if u_k converges weakly to u and if $x(\cdot;0,\varphi,u_k)$ converges uniformly to $x(\cdot)$ then $x(\cdot) = x(\cdot;0,\varphi,u)$. But

$$x(t;0,\varphi,u_k) = \varphi(0) + \int_0^t \Big(A + \sum_1^p B_i u_{k_i}(\sigma)\Big)\big[x(\sigma;0,\varphi,u_k) - x(\sigma)\big]\,d\sigma + \int_0^t C\big(x(\sigma-\tau;0,\varphi,u_k) - x(\sigma-\tau)\big)\,d\sigma$$
$$+ \int_0^t \Big[\Big[A + \sum_1^p B_i u_{k_i}(\sigma)\Big] x(\sigma) + C x(\sigma-\tau)\Big]\,d\sigma$$

Using the weak convergence of u_k to u and the uniform convergence of $x(\cdot;0,\varphi,u_k)$ to $x(\cdot)$ it follows that

$$x(t) = \varphi(0) + \int_0^t \Big[\Big(A + \sum_1^p B_i u_i(\sigma)\Big) x(\sigma) + C x(\sigma-\tau)\Big]\,d\sigma$$

Thus

$x(t) = x(t;0,\varphi,u)$ and we are done. The following theorem is then the analogue of Corollaries 1-3 of [17] in our setting.

<u>Theorem 4:</u> The sets $R(T,\varphi)$ and $\mathbb{R}(T,\varphi)$ are compact. The sets $RBP(T,\varphi)$ and $\mathbb{R}BP(T,\varphi)$ are dense in $R(T,\varphi)$ and $\mathbb{R}(T,\varphi)$ respectively.

<u>Proof:</u> Observe that Lemma 1 implies that the map $u \to x(\cdot;0,\varphi,u)$ is continuous from $U(T)$ with the weak topology into $C([0,T];\mathbb{R}^n)$ with the uniform topology. Then the result follows from the weak compactness of $U(T)$ and from the fact that $UBP(T)$ is weakly dense in $U(T)$.

We consider now the reachable sets in function space $R_C(T,\varphi)$, $R_CB(T,\varphi)$, $R_CBP(T,\varphi)$ and $\mathbb{R}_C(T,\varphi)$, $\mathbb{R}_CB(T,\varphi)$, $\mathbb{R}_CBP(T,\varphi)$. Let $x_t(\varphi,u)$ be the state at time t starting at φ and using control u, i.e.

$$x_t(\varphi, u)(\theta) = x(t+\theta; 0, \varphi, u);\ \theta \in [-\tau, 0]$$

Now suppose that $u_k \to u$ weakly, then $x_{(\cdot)}(\varphi, u_k)$ converges uniformly to $x_{(\cdot)}(\varphi, u)$ in the space $C([0,T];\mathcal{C})$ (the $\mathcal{C}$ inside the parameter is $C([-\tau, 0]; \mathbb{R}^n])$. Indeed

$$\|x_{(\cdot)}(\varphi, u_k) - x_{(\cdot)}(\varphi, u)\|_{C([0,T];\mathcal{C})} =$$

$$= \sup_{t\in[0,T]} \|x_t(\varphi, u_k) - x_t(\varphi, u)\|_{\mathcal{C}} =$$

$$= \sup_{t\in[0,T]} \left(\sup_{\theta\in[-\tau,0]} \|x(t+\theta; 0, \varphi, u_k) - x(t+\theta; 0, \varphi, u\|\right)_{\mathbb{R}^n}$$

and we are done by Lemma 1. So we have the analogue of Theorem 4 in function space.

Theorem 5: The sets $R_C(T,\varphi)$ and $\mathbb{R}_C(T,\varphi)$ are compact. The sets $R_C BP(T,\varphi)$, $\mathbb{R}_C BP(T,\varphi)$ are dense in $R_C(T,\varphi)$ and $\mathbb{R}_C(T,\varphi)$ respectively.

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