JB 75-07

Paper Title:

Nonclassical Filtering Problems in Quantum Electronics

From the Proceedings:

The 1975 IEEE Decision and Control Conference

pp. 804-806

Houston, Texas
December 1975

NONCLASSICAL FILTERING PROBLEMS IN QUANTUM ELECTRONICS

John S. Baras
Electrical Engineering Department
University of Maryland
College Park, Maryland 20742

Robert O. Harger
Electrical Engineering Department
University of Maryland
College Park, Maryland 20742

Abstract

Filtering problems arising in quantum electronics are investigated. In particular the minimum variance linear filtering problem incorporating quantum mechanical measurements is solved for both scalar and vector signal processes. The simplification of the filter under certain assumptions is demonstrated. Examples from optical communication systems illustrate the results.

Summary

With the advent of lasers detection and estimation problems in quantum electronics became of primary importance [1, 2]. More recently linear filtering of a random signal sequence utilizing quantum measurements has been considered [3, 4, 8, 9]. The basic problem we consider is the linear filtering of a random sequence $\{x_k\}$, which influences a quantum field, based on quantum mechanical measurements. The following optical communication problem gives a concrete example. At each time k a laser modulated in some fashion by $\{x_j\}$ is received in a cavity and a device is used to perform a measurement on the captured field. Then the cavity is cleansed and reopens to repeat the process at time k+1. We wish to select optimally the measuring device at each time along with the postprocessing scheme of past and current measurement outcomes, in order to estimate $\{x_k\}$. This is clearly a nonclassical filtering problem.

There are two cases, depending on whether $\{x_k\}$ is a scalar or vector process, with marked difference in the complexity of the analysis. First we describe our results for the scalar case. For details we refer to [3, 4]. The formulation is a follows. The quantum state of the captured field is described by a density operator [5] $\rho(x_k)$ on a Hilbert space $\mathcal{H}$, which does not depend explicity on k due to the cleansing of the cavity. The device which measures one scalar physical quantity at time j is represented by a selfadjoint operator V_j on $\mathcal{H}$ (an observable) [5]. The outcome v_j of this measurement is a random variable. The statistics of the outcomes $\{v_0, \ldots v_k\}$ are described by the joint probability measure

$$\mu_k(B_0 x \ldots x B_k) = \int \cdots \int \cdots \mathrm{Tr}\{\rho(x_i) M_i(B_i)\} \cdots F_{x_0, \ldots x_k}(dx_0, \ldots d_{x_k}) \tag{1}$$

where M_i is the projection valued measure associated with V_i [5], $F_{x_0, \ldots x_k}$ is the joint distribution function of $\{x_0, \ldots x_k\}$ and $B_i \in \mathcal{B}$, the Borel σ-algebra of $\mathbb{R}$. Given the outcomes $v_0, \ldots v_{k-1}$ choose the measurement V_k at time k and the coefficients $c_i(k)$, so that to minimize $E\,|x_k - \hat{x}_k|^2$ where

$$\hat{x}_k = \sum_{i=0}^{k} c_i(k) v_i \tag{2}$$

The MMSE estimate of x_k without postprocessing (i.e. set $c_i(k)=0$, $i=0, \ldots k-1$, in (2)) is given [6] by the outcome τ_k of the measurement represented by T_k which is the solution of:

$$n_k T_k + T_k n_k = 2\delta_{kk} \tag{3}$$

$$\text{where } n_k \triangleq E_{\underset{\sim}{x}}\{\rho(x_k)\} \text{ and } \delta_{kj} \triangleq E_{\underset{\sim}{x}}\{x_k \rho(x_j)\} \tag{4}$$

The solution to our filtering problem is ([3, 4]):

<u>Theorem 1:</u> The optimum observable $\hat{V}_k$ and optimal processing coefficients $\hat{c}_i(k)$, $i = 0, 1, \ldots k-1$ ($\hat{c}_k(k)=1$ always) exist and are given as solutions of the equations:

$$\hat{V}_k = T_k - \sum_{i=0}^{k-1} \hat{c}_i(k) \sigma_{ki} \tag{5}$$

$$\sum_{j=0}^{k-1} \hat{c}_j(k) \mathrm{Tr}\{\zeta_{ji} V_j - \zeta_{ki} \sigma_{kj}\} = \mathrm{Tr}\{\delta_{ki} V_i - \zeta_{ki} T_k\}; i=0, \ldots k-1 \tag{6}$$

$$\text{where } n_k \sigma_{ki} + \sigma_{ki} n_k = 2\zeta_{ki} \tag{7}$$

$$\text{and } \zeta_{ij} = \begin{cases} E_{\underset{\sim}{x}}\{(\mathrm{Tr}\{\rho(x_j)V_j\})\rho(x_i)\}, & i \neq j \\ n_i V_i, & i = j \end{cases} \tag{8}$$

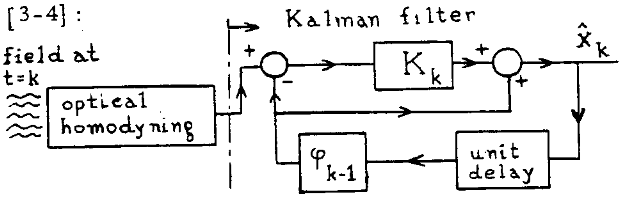
This general filter is not very satisfactory from practical considerations. The number of measuring devices needed for implementing the filter is large and $\hat{V}_k$ may depend in a significant structural way on k and on new data. In addition if x_k satisfies a recursion like

$$x_{k+1} = \varphi_k x_k + w_k \tag{9}$$

where φ_k is a sequence of scalars and $\{w_k\}$ is a sequence of independent, Gaussian random variables with zero mean and variance Q_k, a recursive filter is highly desirable. Additional assumptions on the problem produce highly simplified filter structures. A crucial result in this direction is the following "separation" theorem from [3, 4]:

Theorem 2: Suppose the signal process $\{x_k\}$ is pairwise Gaussian, the measurements V_i, $i = 0, 1, \ldots k$, are optimally chosen (according to Theorem 1) and that the outcomes τ_i of the measurements represented by T_i have the property that (τ_i, x_i) are jointly Gaussian for each i. Then the quantum mechanical linear filtered estimate of x_k (i.e. $\hat{x}_k$) is equal to the classical linear MMSE estimate of x_k given the random variables $\{\tau_i, i = 0, 1, \ldots k\}$.

Consider now the problem where x_k satisfies a recursion like (9) and is transmitted as the (real) amplitude of a laser (assumed monochromatic) and received, along with thermal noise, in a single mode cavity. Then the optimal filter becomes [3-4]:



$$K_k = P_k\left[P_k + \left(\frac{N}{2} + \frac{1}{4}\right)\right]^{-1}, \; P_k = \varphi_{k-1}^{\,2}\{P_{k-1}[1 - K_{k-1}]\} + Q_{k-1} \tag{10}$$

Next we describe our results when the signal process is an $\mathbb{R}^N$ vector process. This case is more delicate and difficult because of the fundamental "compatibility" constraint in quantum mechanics. That is, only "compatible" measurements can be made simultaneously [5]. However by adjoining auxiliary aparatus to the original system [1, 7] one can perform "compatible" measurements on the augmented system which correspond satistically to "non compatible" measurements on the original system. The effect of this on our problem is that instead of considering measurements as represented by projection valued measures (and thus self adjoint operators) we have to consider representations via positive operator valued measures (p.o.m.). Such measurements are called <u>extended measurements</u> [1]. Considering now a sequence of measurements represented by the p.o.m.'s M_i, we have that the probability measure characterizing the joint statistics of the vector measurement outcomes $\{v_0, \ldots v_k\}$ is given by (1) with the appropriate changes from scalar to vector processes. The corresponding filtering problem is to find a p.o.m. M_k and NxN matrices $C_i(k)$, $i = 0, 1, \ldots k$ so as to minimize $E\{\|x(k) - \hat{x}(k)\|^2_{\mathbb{R}^N}$

$$\text{where } \hat{x}_k = \sum_{i=0}^{k} C_i(k) v_i \tag{11}$$

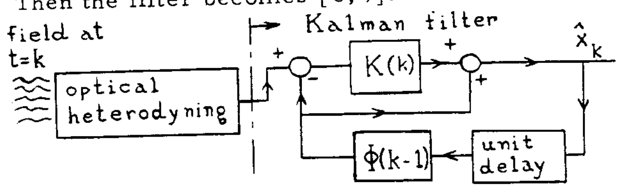
The existence of solution to this optimization problem has been established in [8, 9] and various necessary or sufficient conditions have been found. Here we have also a "separation" of the filter under additional assumptions as described in the sequel. We restrict to p.o.m.'s <u>with a base</u>[1], that is p.o.m.'s that can be expressed as

$$x(B) = \int_B P(u)\mu(du) \tag{12}$$

for some positive-operator valued function P and a measure μ on $\mathcal{B}^N$. Let $\hat{M}_i$, $i = 0, \ldots k$ be the optimal measurements when we use post processing (with outcomes v_i) and let $\hat{Z}_i$, $i = 0, \ldots k$ be the optimal measurements (with outcomes z_i) when we do not use postprocessing. Then we have [8, 9]:

Theorem 3: Suppose that the vector signal sequence $\{x_j\}$ is pairwise Gaussian, and that (z_i, x_i) are jointly Gaussian for $i = 0, \ldots k$. Then the quantum mechanical linear filtered estimate of x_k is equal to the classical linear MMSE estimate of x_k given the random variables $\{z_i, i = 0, 1, \ldots k\}$.

As a multiparameter recursive filtering example consider the problem of estimating the two dimensional state x_k which is transmitted as the in-phase x_{1k} and quadrature x_{2k} amplitudes of a laser (assumed monochromatic) and received along with thermal noise, in a single mode cavity. Suppose that x_k satisfies the vector analog of (9). Then the filter becomes [8, 9]:



$$\text{where } K(k) = P(k)\left[P(k) + \left(\frac{n_0}{2} + \frac{1}{2}\right)I_2\right]^{-1}$$

$$P(k) = \Phi(k-1)[P(k-1) - K(k-1)P(k-1)]\Phi(k-1)^t + Q(k-1)$$

References

1. A.S. Holevo, "Statistical Decision Theory for Quantum Systems", Journal of Multivariate Analysis, Vol. 3, pp. 337-394, 1973.

2. C.W. Helstrom, J.W. Liu, and J.P. Gordon, "Quantum Mechanical Communication Theory", Proceedings of the IEEE, Vol. 58, pp. 1578-1598, October 1970.

3. J.S. Baras and Y.H. Park, "Estimation of Random Signals Based on Quantum Mechanical Measurements", Proceedings of Eight Asilomar Conference on Circuits, Systems, and Computers, December 1974, pp. 533-539.

4. J.S. Baras, R.O. Harger and Y.H. Park, "Quantum Mechanical Linear Filtering of Random Signal Sequences", to be published IEEE Trans. on Information Theory.

5. E. Prugovecki, Quantum Mechanics in Hilbert Space, New York: Academic Press, 1971.

6. S.D. Personick, "Application of Quantum Estimation Theory to Analog Communication Over Quantum Channels", IEEE Trans. on Information Theory, Vol. IT-17, pp. 240-246, May 1971.

7. C.W. Helstrom and R.S. Kennedy, "Noncommuting Observables in Quantum Detection and Estimation Theory", IEEE Trans. on Information Theory, Vol. IT-20, pp. 16-24, January 1974.

8. J.S. Baras and R.O. Harger, "Multiparameter Filtering with Quantum Measurements", Proc. 19th Conference on Information Sciences and Systems, pp. 178-184, April 1975.

9. J.S. Baras, R.O. Harger, "Quantum Mechanical Linear Filtering of Vector Signal Processes", to be submitted for publication.