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FREQUENCY RESPONSE METHODS IN MULTIVARIABLE INFINITE DIMENSIONAL LINEAR SYSTEMS

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Abstract

Recent results on the analysis of models and structural properties of linear distributed systems are presented. The presentation emphasizes the role played by harmonic analysis in these studies. The conclusions are that a careful selection of mathematical methods makes possible a satisfactory classification and detailed analysis of distributed systems models. These methods provide simple models that reflect input-output data of engineering importance.

SUMMARY

Modeling distributed parameter systems one finds a number of intrinsic problems that do not appear in lumped parameter systems modeling. Typically a linear distributed system is modeled by a differential equation

$$\left.\begin{aligned} \frac{dx(t)}{dt} &= A\,x(t) + Bu(t) \\ y(t) &= C\,x(t) \end{aligned}\right\} \qquad (1)$$

Here for a great variety of problems it suffices to assume that x(t) is in a Hilbert space $\mathcal{X}$ [1]. The operator A arises from a formal partial differential or integrodifferential operator and it may include boundary conditions through the definition of its domain $\mathcal{D}(A)$. In all situations A is assumed to generate a strongly continuous semigroup of bounded operators on $\mathcal{X}$. This last statement is an abstract phrasing of the usual assumption that the system of equations under study be well-posed. The controls u are for us square integrable $\mathbb{C}^n$-valued functions and the outputs y are square integrable $\mathbb{C}^m$-valued functions. So $u \in L^2_n$ and $y \in L^2_m$. Certainly other input and output function spaces can be utilized. It turns out however that the L^2 topology gives rise to a particularly rich theory. This does not state that other function spaces can not provide theories with similarly rich structures. The latter remains to be proved however. It is fair to say that to date other theories (based on distributions for example [3]) have not produced detailed results like the ones we describe here.

Describing the properties of the operators B and C in (1) above is more intricate. Indeed there are various possibilities that are due to the following facts: in distributed systems we can (a) apply distributed control, that is control distributed in the spatial domain of our partial differential operator or, (b) apply boundary control, that is control through the boundary conditions of our p.d.e. system; in distributed systems we can, (c) have as outputs linear functionals of the whole solution

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x, that is distributed observations (in the form of a weighted average) or, (d) have as outputs linear functionals of the boundary values of the solution and (or) its derivatives, that is boundary observations. In (1) $B: \mathbb{C}^n \to \mathcal{X}$ and $C: \mathcal{X} \to \mathbb{C}^m$. In the case of distributed control B is bounded, appears in (1) directly from the physical description of the system and usually Range $(B) \subseteq \mathcal{D}(A)$. Similarly with distributed observation C is bounded. In case of boundary obervation C turns out to be typically unbounded. The usual situation however is that C is A-bounded [4]. That is its domain $\mathcal{D}(C) \supseteq \mathcal{D}(A)$ and $\|Cx\|_{\mathbb{C}^m} \leq k_1 \|Ax\|_{\mathcal{X}} + k_2 \|x\|_{\mathcal{X}}$ for some positive k_1, k_2 and for all $x \in \mathcal{D}(A)$. The situation with boundary control is a little more subtle. In such cases the physical description of the system does not result directly in a model like (1). Typically one has a p.d.e. $\frac{dx(t)}{dt} = \sigma x(t)$ and a boundary partial differential operator τ, which gives the control via $\tau x(t) = u(t)$, with τ being σ - bounded. One has to work further to bring the original description into the form of (1). At the end of this construction one ends with an operator B that is "unbounded", in the sense that B now maps $\mathbb{C}^n$ into $V' \supset \mathcal{X} \supset V$ where V' is the dual of V (note here that V is included in X as a set and not as a Hilbert space, the inner products in $\mathcal{X}$ and V may be considerably different) (see [1] or [2] for details). However as a map from $\mathbb{C}^n$ into V' B is clearly bounded. These ideas have been used formally in engineering problems when replacing boundary controls with delta-function type distributed controls.

We mainly analyze here models that have both operators B and C bounded. We would like to point out however that most of the results can be extended to the other cases with additional work required by the more complex mathematical technicalities. The basic ideas remain the same. The matrix valued function $T(t) = Ce^{At}B$ associated with (1) is the weighting pattern of the system. The Laplace transform of T is the transfer function $\hat{T}(s) = C(Is-A)^{-1}B$ which is originally well defined in some right half plane. The triple (A, B, C) is a regular realization for T or $\hat{T}$, when B and C are bounded and $T(t) = Ce^{At}B$. This last equation is a representation for the function T, and thus we expect classical function theoretic representation results to be quite useful here. We shall see that this is indeed the case.

It is clear that spectral properties of the generator A are crucial for the analysis of systems like (1). Utilization of spectral information can provide structural and qualitative analysis of great detail. On physical grounds it is desirable that the spectral properties of A "faithfully represent input-output measurements". Let us make the last statement more precise. Clearly $\hat{T}(s) = C(Is-A)^{-1}B$ can be analytically continued in $\rho_0(A)$, the connected component of the resolvent set of A that contains $+\infty$. To simplify the discussion we assume that $\rho(A)$ is connected. Then if we let $\sigma(\hat{T})$ denote the set of nonanalyticity of $\hat{T}$ we have the spectral inclusion property [5], $\sigma(\hat{T}) \subseteq \sigma(A)$. A realization (A, B, C) is spectrally minimal [5] if $\sigma(\hat{T}) = \sigma(A)$, for some analytic continuation of $\hat{T}$, and with multiplicities counted whenever meaningful. Our position is that spectrally minimal realizations are very useful and natural models for linear distributed systems. Afterall physicists and engineers usually measure things like natural frequencies, spectral lines, radiation modes that are reflected in the singularities of $\hat{T}$. We want then to investigate existence of such models, find simple models of this type and study relations between such models. What follows is a very brief summary of results in this direction. For details and further references we refer to [5] [6] [7].

A regular realization (A, B, C) is reachable whenever $B^* e^{A^*t}x=0$ for $t \geq 0$ implies $x=0$; is observable whenever $C e^{At} x = 0$ for $t \geq 0$ implies $x = 0$; is canonical whenever it is reachable and observable; is exactly reachable whenever the limit $\lim_{t_1 \to \infty} \int_0^{t_1} e^{At} BB^* e^{A^*t} dt$ exists as a bounded and boundedly invertible operator; is exactly observable whenever the limit $\lim_{t_1 \to \infty} \int_0^{t_1} e^{A^*t} C^* C e^{At} dt$ exits as a bounded and boundedly invertible operator. First notice that the existence of regular realizations implies certain properties for T. Indeed we have:

<u>Theorem 1:</u> Let T be an mxn matrix weighting pattern. If T has a regular realization then T is continuous and of exponential order. On the other hand if T is locally absolutely continuous and its derivative T is of expontial order, T has a regular realization.

To proceed in the analysis we need to use the theory of Hardy functions H^2_{ℓ}, H^{∞}_{mxn}, H^2_{mxn}, $H^2(\mathcal{L}(\mathbb{C}^k, N))$ (see [7] for notations). Then

<u>Theorem 2:</u> Let $\hat{T}$ be analytic in Res > 0. If $\hat{T}(i\omega) = C(i\omega)^* B(i\omega)$ a.e. with $C \in H^2(\mathcal{L}(\mathbb{C}^m, N))$, $B \in H^2(\mathcal{L}(\mathbb{C}^n, N))$ where N is an auxiliary Hilbert space, then $\hat{T}$ has a regular realization.

This latter realization is given by

$$\left.\begin{aligned} &\mathcal{X} = H^2(N) \\ &G = \mathbb{C}^n \to \mathcal{X};\ (Gu)(i\omega) = B(i\omega)u \\ &e^{Ft}x = P_{H^2(N)} M_{e^{i\omega t}} x \\ &H: X \to \mathbb{C}^m;\ Hx = \frac{1}{2\pi}\int_{-\infty}^{\infty} C^*(i\omega)x(i\omega)\,d\omega \end{aligned}\right\} \qquad (2)$$

where $M_{e^{i\omega t}}$ is the operator 'multiplication by $e^{i\omega t}$'. This is the <u>translation realization</u>.

It is interesting to ask when does the factorization condition of the previous theorem become necessary? Then

<u>Theorem 3:</u> Let $\hat{T}$ be a transfer function matrix. If either

(a) $\hat{T}$ has a dissipative (i.e. for $x \in \mathcal{D}(A)$, $\langle Ax, x\rangle + \langle x, Ax\rangle \leq 0$) globally assymptotically stable (i.e. $\lim_{t\to\infty} \|e^{At}x\| = 0$, $\forall x \in \mathcal{X}$) regular realization, or

(b) $\hat{T} \in H^2_{mxn}$ and has a reachable and exactly observable regular realization,

then the factorization condition of Theorem 2 is also necessary.

We would like to analyze case (b) a little further. Note that the square integrability assumption is inessential. The Hankel operator is then well defined:

$$(H_T u)(t) = \int_0^{\infty} T(t+\sigma)u(\sigma)d\sigma \qquad (3)$$

or in the frequency domain

$$H_{\hat{T}}\hat{u} = P_{H^2_m} M_{\hat{T}} \sigma \hat{u} \qquad (4)$$

where $\sigma\hat{u}(i\omega) = \hat{u}(-i\omega)$. Then the following is a well defined regular realization:

$$\left.\begin{aligned} &\mathcal{X} = \overline{\text{Range }(H_{\hat{T}})} \subset H^2_m \\ &e^{At}x = P_{\mathcal{X}} M_{e^{i\omega t}} x \\ &(Bu)(i\omega) = \hat{T}(i\omega)u \\ &Cx = \frac{1}{2\pi}\int_{-\infty}^{\infty} x(i\omega)d\omega \end{aligned}\right\} \qquad (5)$$

(5) is the <u>restricted translation realization</u>. But we know [6] [8] that if (A, B, C) and (F, G, H) are two regular, reachable and exactly observable realizations of the same weighting pattern T, then there exists a bounded and boundedly invertible operator P so that PA = FP, PB = G, C = HP. So it suffices to analyze the restricted translation realization for this class of weighting patterns (and thus systems). Note that this is an extremely

simple model and that the Fourier transform (which is a classical function theoretic representation theorem) was utilized in its construction. Now $\overline{\text{Range}\,(H_{\hat{T}})}$ is a left translation invariant subspace, and therefore $\overline{\text{Range}\,(H_{\hat{T}})} = (Q_r H_k^2)^{\perp}$, $k \leq m$, where $Q_r(i\omega)$ is isometric a.e. The important case is when $k = m$. Then Q_r is inner and the subspace of full range. This fact must reflect some properties of $\hat{T}$. The relevant property is that of existence of a pseudomeromorphic continuation of bounded type in the open left half plane. A transfer function matrix $\hat{T}$ analytic in $\text{Re}\,s > 0$, has the above mentioned property if there exists a matrix function G and a scalar function g, both bounded and analytic in $\text{Re}\,s < 0$ so that $\hat{T}(i\omega) = G(i\omega)/g(i\omega)$ a.e. on the $i\omega$-axis. This is a generalization of the concept of regular analytic continuation. Then the following are equivalent:

(a) $\hat{T}$ has a meromorphic pseudocontinuation of bounded type in $\text{Re}\,s < 0$.

(b) $(\text{Range}\,(H_{\hat{T}}))^{\perp} = Q_r H_m^2$, Q_r inner.

(c) $\hat{T}$ has a right coprime factorization $\hat{T}(i\omega) = Q_r(i\omega) P_r^{*}(i\omega)$, with Q_r inner and $P_r \in H^{\infty}_{mxn}$.

Now Q_r determines the spectrum of A in the restricted translation realization (5) with multiplicities: $\sigma(A) = \{\mu \in \text{OLP}$ such that $Q_r^{*}(-\bar{\mu})$ has non null kernel$\} \cup \{$points on $i\omega$-axis through which Q_r cannot be continued analytically$\}$. Q_r also determines the singularities of the pseudocontinuation of $\hat{T}$ and $\sigma(\hat{T}) = \sigma(A)$, multiplicities counted. Note that, except for pathological cases, the pseudocontinuation will be a true analytic continuation for $\hat{T}$. Thus we have:

<u>Theorem 4:</u> Suppose $\hat{T} \in H^2_{mxn} \cap H^{\infty}_{mxn}$, $\hat{T}$ has a meromorphic pseudo-continuation of bounded type in O.L.P., and $\hat{T}$ has a reachable and exactly observable regular realization. Then i) the restricted translation realization is spectrally minimal, ii) any other reachable and exactly observable realization is spectrally minimal.

Note also that Q_r gives a precise state space decomposition for this class via the Jordan model theory of Nagy-Foias [10, ch. III]. Similar results for discrete time systems can be found in [8], [9], and the references therein. We would like to remark again that all of the above can be extended to the other cases, i.e. B or C or both being unbounded. What is involved is a careful analysis of the restricted translation realization (5) (which can formally be written for any H^{∞}_{mxn} function) in order to make the various operators well defined.

This is as far, invariant subspace theory and Hardy spaces go. There are however inportant classes of distributed systems that arise from engineering and physics that are not included here. To produce examples one needs only consider transfer functions with branch points. For a simple example consider heat transfer along a long bar:

$$\left.\begin{aligned}
&\frac{\partial x(t,z)}{\partial t} = \frac{\partial^2 x(t,z)}{\partial z^2} - x(t,z)\\
&x(0,z) = 0\\
&x(t,0) = u(t)\\
&\lim_{z\to\infty} x(t,z) = 0\\
&y(t) = (\text{temperature at } z = 1) = x(t,1)
\end{aligned}\right\} \qquad (6)$$

Then $T(t) = e^{-t} \dfrac{1}{2\sqrt{\pi t^3}} e^{-1/4t}$ and $\hat{T}(s) = e^{-\sqrt{s+1}}$.

One can write a translation realization for this T

$$\left.\begin{aligned}
&\mathcal{X} = L_2[0,\infty)\\
&e^{At} = \text{left translation an } [0,\infty)\\
&Bu = \left(T - \frac{d}{dt}\,T\right)u\\
&Cx = \int_0^{\infty} g(t)x(t)dt,\ g(t) = e^{-t}
\end{aligned}\right\} \qquad (7)$$

This is a canonical regular realization. However $\sigma(A)$ = closed L.P. while $\sigma(\hat{T})$ = $\{$branch cut from -1 to $-\infty\}$. Thus no spectral minimality. But certainly (7) is an unnatural model for (6), because it ignores the great internal symmetry of (6). One has to use other means. In particular many problems from mathematical physics lead to models like (1) where A is selfadjoint or normal. Then one can show by the use of the spectral theorem that if (A, B, C) is a canonical regular realization for T, and $A = A^*$ this realization is spectrally minimal [6]. The internal symmetry of the system results to additional properties for T, which then can be utilized to construct simple models. A simple example, which illustrates the point, and also indicates how classical function representation results can be used here, is provided by the well known to electrical engineers completely monotonic and positive definite functions. These arise naturally from lumped distributed RC networks. A function ϕ is completely monotonic, if it is C^∞ on $[0, \infty)$ and $(-1)^n \phi^{(n)}(t) \geq 0$ for $t > 0$, and is positive definite on $(-\infty, \infty)$ if $\sum_{i,j} \phi(t_i - t_j) a_i \bar{a}_j \geq 0$, for every set of real numbers $\{t_i\}$ and complex numbers $\{a_i\}$. Then we have [6]:

Theorem 5: A weighting pattern T is completely monotonic if and only if it has a regular realization (A, b, b) with $A = A^*$ and stable. T has a positive definite extension on $(-\infty, \infty)$ if and only if it has a regular realization (A, b, b) with $A = -A^*$.

One uses Bernstein's representation of completely monotonic functions and Bochner's representation of positive definite functions to construct simple models.

Under such symmetry the state space isomorphism theorem can be improved. It is important to note that spectral minimality results from assumptions on A alone (like $A = A^*$, A normal), while the state space isomorphism theorem requires additional symmetry:

Theorem 6 ([6]): Let (A, B, C) and (F, G, H) be canonical regular realizations of T, with $A = A^*$, $B = C^*$, $F = F^*$, $G = H^*$. Then they are similar via a unitary map.

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