JB 74-04

Paper Title:

# Recursive Filtering of Operator Valued Processes in Quantum Estimation

From the Proceedings:

1974 IEEE International Symposium on Information Theory
pp. 5

Notre Dame, Indiana
October 1974

# RECURSIVE FILTERING OF OPERATOR VALUED PROCESSES IN QUANTUM ESTIMATION

by

J. Baras, R.O. Harger, A. Ephremides
Electrical Engineering Department
University of Maryland
College Park, Maryland 20742

## Abstract

The unified approach to estimation of a stochastic signal satisfying a dynamical equation through the use of martingale and multiplicity theory is applied, in its infinite dimensional version, to the problem of estimation of a signal via quantum mechanical measurements in an optical communication system. The "state" is an operator valued stochastic process representing a density operator in the quantum mechanical sense acting in an appropriate Hilbert space. The aim is the derivation of recursion relations for the optimum measurement operators that minimize a quadratic cost function. The Markovian nature of the density operator indexed by the signal process allows for optimum or suboptimum recursive filtering relations that are analogous to the ones in the classical case.

## Summary

The study of optical communication systems based on quantum mechanical measurements has led to a number of results in signal detection and parameter estimation problems [1-3]. The quantum theoretical framework in which these problems are described usually requires departure from classical mathematical techniques. The problem of estimating a discrete time stochastic signal $x_t$ via quantum measurements has been considered in [4].

The scope of filtering and estimation theory has been recently enlarged through the use of the theory of martingales, the theory of multiplicity, and the extended, infinite dimensional Kalman filtering applied to operator valued stochastic processes [5-9].

In this paper some preliminary work has been done towards formulating the quantum filtering prpblem in the context of infinite dimensional Kalman filtering. The recursive formulas for the estimators obtained in classical cases by using the multiplicity (linear innovations) approach [10, 11] or the markov-martingale (general innovations) approach [12, 13] are expected to be generalized in the quantum case.

The information carrier is a scalar or vector random process $x_t$, $t = 1, 2, \ldots$, which is received by an apperture that opens and closes periodically in the usual fashion [1]. The received field in the cavity after the apperture closes is described by a density operator $\rho^s_{x_t}$, indexed by the value of the signal $x_t$, that acts on a separable Hilbert space $H^s$, and

is positive definite, self-adjoint, and of trace one [14]. Through the dependence on the process $x_t$, the sequence $\rho^s_{x_t}$ becomes an operator valued stochastic process, in the space of Hilbert - Schmidt operators $HS(H^s, H^s)$, that, besides, under the trace norm evolves on the unit sphere of the space $TC(H^s, H^s)$ [14].

The assumption that $\rho^s_{x_t}$ depends only on the value of the signal $x_t$ at time t and not on time t explicity bypasses the need to consider the Schrödinger evolution equation, and is justified by the assumed periodic opening and closing pattern of the apperture by which the receiver operates.

The signal process $x_t$ is assumed to be Markov, satisfying a dynamical difference equation. It is then shown that $\left(\rho^s_{x_t} =\right) \rho^s_t$ is also a Markov (infinite dimensional) process that satisfies an operator difference equation of the form

$$\rho^s_{t+1} = \varphi(\rho^s_t) + w_t$$

where $w_t$ is an operator valued stochastic process with independent, or othogonal, increments (depending on whether the Markov property is strict sense or wide sense). The map $\varphi$ as well as the properties of $w_t$ depend on the dynamics of the signal process $x_t$ and on the quantum physical nature of $\rho^s_t$.

The measurements at time t are described by self-adjoint operators $V_t$, in the usual quantum mechanical sense [1-3]. To allow for measurements that do not correspond to physical observables, as well as in order to by-pass the non-commutativity of operators corresponding to different observables [2, 3] it is now accepted that an extended, product Hilbert space $H = H^S \otimes H^A$ must be considered ($H^S$ corresponding to the original physical system, and $H^A$ to an apparatus added to the system). The operators $\rho_t$ are assumed to be the extended ones that act on the enlarged product space ($\rho_t = \rho_t^S \otimes \rho_t^A$). The extended operators continue to posses the Markovian property.

The problem is to choose $V_t$ in order to minimize the mean-squared error

$$E\left[\hat{x}_t - x_t\right]^2 = \int p(x_t) \cdot \mathrm{Tr}\left\{\rho_{x_t} (V_t - x_t I)(V_t^* - x_t^* I)\right\} dx_t$$

where $p(x_t)$ is the probability density function of $x_t$, and $*$ denotes adjoint or complex conjugate.

First, conditions are seeked for the optimum measurements to satisfy recursion relations. Secondly, assuming a suboptimal recursive structure of the form

$$V_t = f(V_{t-1}, V_{t-2}, \ldots, V_{t-k})$$

the questions of obtaining f and of interpreting it are investigated.

The theory is applied to special cases such as the coherent estimation of an amplitude modulated signal by an ideal receiver with a single mode corresponding to a harmonic oscillator [1, 2].

## References

1. C. Helstrom, "The Minimum Variance of Estimates in Quantum Signal Detection", IEEE Trans. on Inf. Th. IT-14, p. 234, 1968.

2. C. Helstrom and R.S. Kennedy, "Noncommuting Observables in Quantum Detection and Estimation Theory", IEEE Trans. on Inf. Th. IT-20, p. 16, 1974.

3. H.P. Yuen and M. Lax, "Multiple-Parameter Quantum Estimation and Measurement of Nonselfadjoint Observables", IEEE Trans. on Inf. Th. IT-19, p. 740, 1973.

4. R.O. Harger and Y.H. Park, "Linear Estimation of Random State with Quantum Measurement", Proc. of 7th Princeton Conf. on Information Sciences and Systems, p. 383, 1973.

5. G. Kallianpur and V. Mandrekar, "Multiplicity and Representation Theory of Purely Non-Deterministic Stochastic Processes", Theory of Prob. and Appl. Vol. X, 4, p. 553, 1965.

6. K.Y. Lee and V. Mandrekar, "Multiplicity and Martingale Approach to Linear State Estimation", Proc. of 7th Princeton Conf. on Information Sciences and Systems, p. 136, 1973.

7. K.Y. Lee, " Optimal Estimation of Operator-Valued Stochastic Processes and Applications to Distributed Parameter Systems", Proc. of 1972 IEEE Conf. on Decision and Control, p. 94.

8. P.L. Falb, "Infinite-Dimensional Filtering: The Kalman-Bucy Filter in Hilbert Space", Information and Control, 11, p. 102, 1967.

9. A.V. Balakrishnan and J.L. Lions, "State Estimation for Infinite-Dimensional Systems", Journal of Comp. and System Sciences, 1, p. 391, 1967.

10. T. Kailath, "The Innovations Approach to Detection and Estimation Theory", Proc. of IEEE, Vol. 58, 5, p. 680, 1970.

11. A. Ephremides and L.H. Brandenburg, "On the Reconstruction Error of Sampled Data Estimates", IEEE Trans. on Inf. Th., IT-19, 3, p. 365, 1973.

12. R. Boel, P. Varaiya and E. Wong, "Martingales on Jump Processes I: Representation Results, II: Applications", Electronics Res. Lab. Memoranda No. ERL-M407, ERL-M409, University of California, Berkeley.

References (cont'd)

13. A. Segall, "A Martingale Approach to Modeling, Estimation and Detection of Jump Processes", Techn. Report No. 7050-21, Information Systems Laboratory, Stanford University.

14. E. Prugovecki, Quantum Mechanics in Hilbert Space, Academic Press.