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LUMPED-DISTRIBUTED NETWORK SYNTHESIS AND INFINITE DIMENSIONAL REALIZATION THEORY

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Abstract

We study the relations between infinite dimensional realization theory, and the synthesis of networks using lumpted and distributed elements. In particular we indicate how results from operator theory and especially the theory of invariant subspaces in H^p spaces can be applied to both these problems. We discuss synthesis by reactance extraction and cascade synthesis.

1. INTRODUCTION

The interelations between system theory and network theory are more than apparent. The interactions between the two disciplines have produced many results of important theoretical and practical value. One of the major problems in network theory is that of synthesis. The so called state space theory of linear finite dimensional systems proved to be a very effective tool in solving the synthesis problem for a network with lumped components. Especially the realization theory of Kalman [1], has been used repeatedly to produce elegant synthesis procedures. For an excellent exposition of these methods see the recent books by Anderson-Vongpanitlerd [2] and Saeks [3].

Due to developments in integrated circuit technology and microwave circuits, the synthesis and design of networks with distributed and lumped elements is currently attracting the research efforts of many network theorists, see Youla [4]. The goal here is to develop precise synthesis procedures for nonrational impedance or scattering matrices.

On the other hand due to the limitations of the classical (to date) linear finite dimensional theory, system theorists study intensively distributed parameter systems, of which there is an abundance in practical problems. Recently there has been considerable success in developing a realization (modeling) theory for distributed parameter systems using fairly recent results from operator theory and invariant subspace theory. The interested reader should see Baras and Brockett [5], Baras [6], Fuhrmann [7, 8], Helton [9].

Since lumped-distributed networks are distributed parameter systems, the question naturally arises, to whether or not there is a link between infinite dimensional realization theory and lumped-distributed network synthesis. This is the major theme of this paper. In addition we investigate the effectiveness and relevance of certain results from the Nagy-Foias operator theory [10], and from the theory of Hardy spaces of functions [11], with respect to lumped-distributed network synthesis. It turns out that these mathematical tools are particularly suited for studying both the infinite dimensional realization problem and the network synthesis problem. Our methods are single variable, in contrast with the common multivariable approach to distributed networks.

The organization of the paper is as follows. In section 2 we summarize the recent results in infinite dimensional realization theory and discuss their relevance to distributed network systhesis. In section 3 we show how a lumped-distributed network synthesis problem is equivalent to an infinite dimensional realization problem, providing thus a link between these two areas. We also discuss possible extensions and indicate how similar methods can be developed for other cases as well. Finally in section 4 we study cascade synthesis and demonstrate the effectiveness of the mathematical methods mentioned above in the investigation of the problems we consider.

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2. RECENT RESULTS IN INFINITE DIMENSIONAL REALIZATION THEORY

Many distributed effects can be very effectively described by linear partial differential equations, where the solution belongs to some Hilbert space. The Hilbert space is usually induced by some kind of energy inner-product. Thus quite naturally one is lead to investigate systems with dynamical equations of the form

$$\frac{d}{dt} x(t) = Ax(t) + Bu(t) \qquad (1)$$
$$y(t) = Cx(t)$$

where $u(t) \in U$, $y(t) \in Y$ and $x(t) \in X$, all being Hilbert spaces. Here

$B: U \to X$ and bounded
$C: X \to Y$ and bounded
$A: X \to X$

are linear operators and A generates a strongly continuous semigroup of bounded operators on X [5]. In cases of distributed parameter systems with boundary observations one considers a dynamical model of the form [12]:

$$\frac{d}{dt} x(t) = A\,x(t)$$
$$B'x(t) = u(t) \qquad (2)$$
$$y(t) = C\,x(t)$$

where everything is as before but now $B': X \to U$ and B, C may be unbounded. This class is of particular interst to us since distributed networks give rise to this kind of dynamical equations.

For many interesting systems which are described by models like (2), we can produce equivalent models like (1). For more details on that see [12]. We denote by e^{At} the semigroup generated by A, and then the input-output relation of the system (1) is given via the convolution

$$y = T * u \qquad (3)$$

where $T(t) = Ce^{At}B$ is the weighting pattern. The realization problem consists of finding a Hilbert space X and operators A, B, C, so that (3) holds, where T(t) is a given function. Of course T completely characterizes the input-output behaviour of the system. To simplify the discussion we consider here scalar inputs and scalar outputs only. The results have been extended to finite dimensional inputs and outputs in [13]. The triple (A, B, C) is a realization of T. The transfer function (the Laplace transform of T) is denoted by $\hat{T}$ and we have

$$\hat{T}(s) = C(Is - A)^{-1}B \qquad (4)$$

for complex s in some right half plane. For details on these see [6]. To characterize the transfer functions which admit such realizations, and construct realizations with additional properties based on engineering requirements, we use heavily the theory of invariant subspaces [14, 15] and properties of the Hardy spaces [11].

If we denote by Π^+ the right half plane $\text{Re}\, s > 0$, then $H^\infty(\Pi^+)$ is the space of all functions analytic and bounded in Π^+. The space $H^2(\Pi^+)$ consists of the Laplace tranforms of functions in $L_2(0,\infty)$. These are also analytic in Π^+. Usually we will write H^∞ and H^2. Functions in these spaces have nontangential limits as $\text{Re}\, s \to 0$ almost everywhere on the imaginary axis, and we denote also by H^∞ and H^2 the spaces of boundary values (which space we refer to will be clear from the context). The space H^2 of boundary values is the Fourier transform of $L_2(0,\infty)$. Without loss of generality (see [5]) we restrict the discussion in this section to weighting patterns that are elements of $L_2(0,\infty)$.

Theorem 1: Let T be continuous and in $L_2(0,\infty)$. If $\hat{T}(i\omega) = F_1(i\omega)\,\overline{F_2(i\omega)}$ where F_1, F_2 are in H^2, then T is realizable.

For a proof of this theorem and more details see [5]. These sufficient conditions for realizability include a large class of weighting patterns. The realization which we construct uses the Hilbert space $L_2(0,\infty)$ and the semigroup of left translations on that space. However we are not only interested in providing just any model for a system, but mainly to construct models that reflect in the best possible way the natural properties of the system that are inherent in the description (or performance specifications) given to us through the weighting pattern T. In finite dimensional linear systems this is accomplished by constructing minimal realizations, which have been proven to inherit the maximum amount of information about the system, that can be inferred from T. In infinite dimensional systems such a concept is not available in general. Nevertheless engineering considerations permit us to single out a class of transfer functions, for which a complete realization theory can be developed. We outline here the major results that are relevant to network theory and we refer to [5, 6, 7, 8, 9] and the references there for the complete expositions.

The basic observation is that many practical problems and especially distributed networks will give us transfer functions which are mero morphic. For this type of systems a model like (1) where the resolvent set of the operator A is not connected is not natural. It follows then from (4) that the singularities of $\hat{T}$ are part of the spectrum of A in any realization. We thus arrive at the concept of an S-minimal realization (spectrally minimal) introduced in [5]. Definition: A realization (A, B, C) is S-minimal if the singularities of $\hat{T}$ concide with the spectrum of A, multiplicities counted.

To proceed then we reduce the realization constructed in theorem 1 to obtain a controllable and observable realization. This is the restricted translation realization. Weighting patterns which have the property that their left translations span $L_2(0,\infty)$ are termed cyclic. Those which do not have this property are noncyclic. For cyclic weighting patterns we can not obtain S-minimal realizations by reducing the translation realization. For noncyclic we can (see [6]).

Theorem 2: Every noncyclic realizable weighting pattern has an S-minimal realization.

Noncyclicity is equivalent to the transfer function $\hat{T}$ (which is analytic in π^+) being the boundary value of a function which is mero morphic in the left half plane and the ratio of two bounded functions there. For transfer functions of many networks this is satisfied (see [4] where mero morphic functions of bounded type are shown to correspond to fairly general lumped-distributed networks).

To proceed further, any noncyclic $\hat{T}$ can be written as $\hat{T}(i\omega) = \overline{H(i\omega)}\, \phi(i\omega)$ where $H \in H^2$ and ϕ is inner (that is $|\phi(s)| \leq 1$ in $\text{Re}s > 0$, $|\phi(i\omega)| = 1$ and ϕ is analytic in π^+; for more on these functions see section 4). Then $\hat{T}(i\omega)$ is the boundary value of the function $\frac{\overline{H(-\bar{s})}}{\phi(-\bar{s})}$ which is meromorphic in $\text{Re}s < 0$ and of bounded type. Moreover if $\hat{T}$ is meromorphic then it is equal to this function for $\text{Re}s < 0$. Clearly ϕ displays the singularities of $\hat{T}$ and these are the points where $\phi(-\bar{s}) = 0$ and the points on the imaginary axis where ϕ cannot be continued through. It turns out that these points exactly with the same multiplicity constitute the spectrum of A in the restricted translation realization of $\hat{T}$. Notice the similarity with rational functions.

Finally by strengthening either the controllability or observability notions one obtains a state space isomorphis m theorem between two exactly controllable (exactly observable) and observable (controllable) realizations of T, see Helton [9]. A consequence of this is that all such realizations of a noncyclic transfer function are S-minimal, and they differ from the restricted translation realization by a similarity. Certain considerations for the class of systems we studied here, similar to the invariant factors analysis of a rational transfer function have been developed in [13,16]. It would be interesting to investigate if the analysis can lead to an extension of the notion of the degree of a rational transfer function.

3. SYNTHESIS VIA REACTANCE EXTRACTION

In this section we consider the problem of synthesizing a lumped distributed network via reactance extraction [2, 3]. Although this method does not lead to easily implementable constructions it has theoretical importance and provides the link with realization theory. Again we make the simplifying assumption that we want to synthesize a nonrational scalar impedance Z(s). The elements we are allowed to use are resistors, gyrators, transformers, inductors, capacitors, and loss less transmission lines which are short circuited or open circuited. We assume that the network is regular, that is we can extract reactances and be left with an interconnecting wiring network. The approach is depicted in the figure below

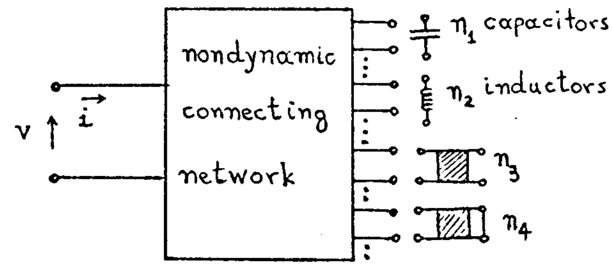

Fig. 1 Reactance Extraction

Let $i_1, v_1, i_2, v_2, i_3, v_3, i_4, v_4$ denote the vector currents and voltages at the ports to be loaded with capacitors, inductors, open circuited lossl ess transmission lines and short circuited lossl ess transmission lines respectively. Let C_i, $i = 1, \cdots n_1$ be the capacitors, L_i, $i = 1 \cdots n_2$ be the inductors, ℓ_i, c_i, d_i, $i = 1, \cdots n_3+n_4$ the specific inductances, capacitances and lengths of the transmission lines of the network. We assume (with no loss of generality) that the connecting network is described by the hybrid matrix

$$\begin{bmatrix} i \\ i_1 \\ v_2 \\ i_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} M_{00} & M_{01} & M_{02} & M_{03} & M_{04} \\ M_{10} & M_{11} & M_{12} & M_{13} & M_{14} \\ M_{20} & M_{21} & M_{22} & M_{23} & M_{24} \\ M_{30} & M_{31} & M_{32} & M_{33} & M_{34} \\ M_{40} & M_{41} & M_{42} & M_{43} & M_{44} \end{bmatrix} \begin{bmatrix} v \\ v_1 \\ i_2 \\ v_3 \\ i_4 \end{bmatrix} \qquad (5)$$

Assumptions on the nature of the interconnecting network induce properties on the matrix M. It is then straightforward to show that Z(s) must necessarily have the form of a ratio of exponential polynomials, see [4] [17]. After appropriate space variable normalizations we can describe the transmission lines by:

a) open circuit lines (capacitive behaviour)

$$\left.\begin{aligned} \frac{\partial V_{3i}(t,z)}{\partial z} &= -d_i \ell_i \frac{\partial I_{3i}(t,z)}{\partial t} \\ \frac{\partial I_{3i}(t,z)}{\partial z} &= -d_i c_i \frac{\partial V_{3i}(t,z)}{\partial t} \end{aligned}\right\} \qquad (6)$$

with the boundary conditions

$$I_{3i}(t,1) = 0\ ,\ I_{3i}(t,0) = -i_{3i}(t),\ V_{3i}(t,0) = v_{3i}(t) \qquad (7)$$

b) short circuit lines (inductive behaviour). Same equation as before, changing the subscripts from 3 to 4. The boundary conditions become

$$V_{4i}(t,1) = 0\ ,\ V_{4i}(t,0) = v_{4i}(t),\ I_{4i}(t,0) = -i_{4i}(t) \qquad (8)$$

On the other hand the lumped elements give us the constraints

$$i_1 = -C\frac{dv_1}{dt} \quad ; \quad v_2 = -L\frac{di_2}{dt} \tag{9}$$

where $C = \text{diag}(C_1, \ldots C_{n1})$, $L = \text{diag}(L_1 \ldots L_{n2})$

Equations (6) and the corresponding for I_4, V_4 constitute a system of hyperbolic partial differential equations with boundary conditions (5) (7) (8) (9) (all data are assumed C^1).

We will use a method similar to that employed initially by Brayton [18] to develop a completely equivalent differential-difference dynamical model for the network. To simplify matters and best illustrate the idea let us assume that $n_3 = n_4 = 1$. The general case proceeds similarly but one has to be carefull with the indices. Solutions of (6) have the form

$$\left.\begin{aligned} V_i(t,z) &= \frac{1}{2}[\phi_i(z-\frac{2t}{\tau_i}) + \psi_i(z+\frac{2t}{\tau_i})] \\ I_i(t,z) &= \frac{1}{2r_i}[\phi_i(z-\frac{2t}{\tau_i}) - \psi_i(z+\frac{2t}{\tau_i})] \end{aligned}\right\} \tag{10}$$

where $r_i = \sqrt{\ell_i/c_i}$ and $\tau_i = 2d_i\sqrt{\ell_i c_i}$, $i = 3,4$ are the characteristic impedances and delay times of the lines. It is then straight forward to use the boundary conditions (7) and (8) to establish that

$$\begin{array}{l|l} v_3(t) = \frac{1}{2}(\phi_{30}(t) + \phi_{30}(t-\tau_3)) & v_4(t) = \frac{1}{2}(\phi_{40}(t) - \phi_{40}(t-\tau_4)) \\ r_3 i_3(t) = \frac{1}{2}(\phi_{30}(t-\tau_3) - \phi_{30}(t)) & r_4 i_4(t) = -\frac{1}{2}(\phi_{40}(t-\tau_4) + \\ & \quad +\phi_{40}(t)) \end{array} \tag{11}$$

where $\phi_{i0}(t) = \phi_i(-\frac{2t}{\tau_i})$, $i = 3, 4$.

Using the relations between between i_3, v_3, i_4, v_4, provided by the last two rows of (5) and substituting (11) we get

$$\frac{1}{2}\begin{bmatrix} r_3^{-1} - M_{33} & r_4^{-1}M_{34} \\ -M_{43} & 1 + r_4^{-1}M_{44} \end{bmatrix}\begin{bmatrix} \phi_{30}(t) \\ \phi_{40}(t) \end{bmatrix} + \frac{1}{2}\begin{bmatrix} r_3^{-1} - M_{33} & 0 \\ -M_{43} & 0 \end{bmatrix}\begin{bmatrix} \phi_{30}(t-\tau_3) \\ \phi_{40}(t-\tau_3) \end{bmatrix} +$$
$$+\frac{1}{2}\begin{bmatrix} 0 & r_4^{-1}M_{34} \\ 0 & -1 + r_4^{-1}M_{44} \end{bmatrix}\begin{bmatrix} \phi_{30}(t-\tau_4) \\ \phi_{40}(t-\tau_4) \end{bmatrix} = \begin{bmatrix} M_{30} \\ M_{40} \end{bmatrix} v + \begin{bmatrix} M_{31} & M_{32} \\ M_{41} & M_{42} \end{bmatrix}\begin{bmatrix} v_1 \\ i_2 \end{bmatrix} \tag{12}$$

The first line of (5) gives us using (11)

$$i = M_{00}v + [M_{01}\ M_{02}]\begin{bmatrix} v_1 \\ i_2 \end{bmatrix} + \frac{1}{2}[M_{03}\ M_{04}r_4^{-1}]\begin{bmatrix} \phi_{30}(t) \\ \phi_{40}(t) \end{bmatrix} +$$
$$+\frac{1}{2}[M_{03}\ 0]\begin{bmatrix} \phi_{30}(t-\tau_3) \\ \phi_{40}(t-\tau_3) \end{bmatrix} + \frac{1}{2}[0 - r_4^{-1}M_{04}]\begin{bmatrix} \phi_{30}(t-\tau_4) \\ \phi_{40}(t-\tau_4) \end{bmatrix} \tag{13}$$

While (9) (12) and the second and third lines of (5) give us the following differential difference equation of neutral type:

$$\frac{d}{dt}\begin{bmatrix} v_1(t) \\ i_2(t) \\ \sum_{0,3,4} D_n \begin{bmatrix} \phi_{30}(t-\tau_n) \\ \phi_{40}(t-\tau_n) \end{bmatrix} \end{bmatrix} = \begin{bmatrix} I \\ \hline M_{31} \ M_{32} \\ M_{41} \ M_{42} \end{bmatrix}\begin{bmatrix} -C^{-1} & 0 \\ 0 & -L^{-1} \end{bmatrix}\left(\begin{bmatrix} v_1(t) \\ i_2(t) \end{bmatrix} + \right.$$
$$\left. + \frac{1}{2}\begin{bmatrix} M_{13} & M_{14} \\ M_{23} & M_{24} \end{bmatrix}\begin{bmatrix} \phi_{30}(t) + \phi_{30}(t-\tau_3) \\ -r_4^{-1}\phi_{40}(t-\tau_4) - r_4^{-1}\phi_{40}(t) \end{bmatrix} + \begin{bmatrix} M_{10} \\ M_{20} \end{bmatrix} v \right) + \begin{bmatrix} 0 \\ 0 \\ M_{30} \\ M_{40} \end{bmatrix}\dot{v} \tag{14}$$

The D_n represent the matrices on the ℓ-h-s-of (12). We can eliminate the derivative of the control by changing the state variable (see [2] p. 196). Then (13) and (14) constitute a realization of the transfer function Z by a neutral differential difference dynamical system. They are also a state space description of the network. Of course the system is infinite dimensional due to the delays. That we ended up with a neutral equation is no accident. It is due to the fact that we started with a hyperbolic partial differential equation. If instead we had lossy RC lines we would have obtained a differential difference equation of retarted type. Equation (14) can be put in the form of (1), (see [19]) the difference being that the standard description and study of these equations are in Banach spaces, not Hilbert spaces. It is easy to see that the transfer function of a system like (13) (14) will be a ratio of exponential polynomials. Solutions to the synthesis problem provide a solution to the problem of realization of differential difference systems of the neutral type. Notice that these realization problems are not covered by our theory in section 2, basically due to the continuity requirements for the weighting patterns there. Conversely solution to the realization problem for the delay system may provide a solution to the synthesis problem. This will involve determination of the matrix M from the matrices of the delay realization and then synthesis of the nondynamical network hybrid matrix M by standard techniques [2], and loading of the parts with reactances. We have not completed as yet this side of the problem. These questions are under investigation.

4. CASCADE SYNTHESIS

From the practical point of view a cascade synthesis is very desirable and provides easily implementable structures. We study here the synthesis of lossless scattering parameter S. Such a function is necessarily inner by losslessness, i.e. $S(i\omega)\overline{S(i\omega)} = 1$. Now any inner function has a factorization [11]

$$S = \text{Blaschke product} \times \text{Singular function} \tag{15}$$

where a <u>Blaschke product</u> has the form

$$\left(\frac{s-1}{s+1}\right)^k \prod_n \frac{|1-\beta_n^2|}{1-\beta_n^2} \cdot \frac{s-\beta_n}{s+\bar{\beta}_n} \qquad (16)$$

where $\beta_1, \beta_2, \ldots$ are complex numbers in π^+ different from 1 and such that $\sum_n \frac{R_e(\beta_n)}{1+|\beta_n|^2} < \infty$.

A singular function has the form

$$e^{-\rho s} \exp\left(-\int_{-\infty}^{\infty} \frac{s\omega + i}{\omega + is}\, d\mu(\omega)\right) \qquad (17)$$

where μ is a finite singular positive measure on the imaginary axis and $\rho \geq 0$. Since S is bounded real, the β_n's in (16) occur in complex conjugate pairs or are real, and the measure μ in (17) is concentrated symmetrically with respect to the origin.

The factors $\frac{s-1}{s+1}$ and $\frac{s-\beta_n}{s+\bar{\beta}_n}$ for real β_n can be easily realized in cascade by standard techniques and similarly for complex β_n, combining the conjugate pairs β_n and $\bar{\beta}_n$. For the network to be realizable by finite number of elements the measure must be concentrated on a finite number of points. Then considering the symmetry of the measure we will have factors of the form

$$\exp\left(-\frac{2s(\omega^2+1)}{s^2+\omega^2}\right), \quad \omega \text{ real}$$

which can be readily realized as loss less transmission lines and transformers in cascade (see [3] p. 262 and p. 236). These facts were observed initially by Dominquez [20] who also investigated the matrix case, using results of Potapov [21]. A complete synthesis theory along these lines is not available however.

Certain questions now arise quite naturally. What type of measures in (17) arise from practical networks? How can we synthesize measures which are not concentrated on a finite number of points? Perhaps using waveguides or nonuniform lines. On theother hand it is not true that a cascade of finite number of lossless lines will have a scattering parameter with a finite Blaschke product. This is immediately seen from the counterexample provided by a loss1 ess line loaded with an open circuit loss1 ess line, through a gyrator. What is true however is that such a cascade will have finite number of sequences of β_n's, (arithmetic sequences) each corresponding to some transmission line, which thus can be factored out. Finally let us consider the lossy case. Then the scattering parameter will have another factor of the form

$$\exp\left(\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{s\omega + i}{is + \omega} \log|S(i\omega)| \frac{d\omega}{1+\omega^2}\right) \qquad (18)$$

which is the outer part of S and as we see depends only on the magnitude of S. How such a factor can be synthesized ?

In conclusion we believe that the mathematical tools described here are effective and well equiped to study problems in lumped distributed network synth esis. Further research along these lines will result to more precise design techniques .

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