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ESTIMATION OF RANDOM SIGNALS BASED ON QUANTUM MECHANICAL MEASUREMENTS

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Abstract

We consider here the problem of estimating a scalar random signal based on quantum mechanical measurements, a situation frequently appearing in optical communication systems. We find the minimum variance estimator based on the combination of optimal observable selection at each time and optimal linear processing of past measurements. We compare this estimator with the one obtained by optimal selection of observables at each time independently. Under additional assumptions we demonstrate that the former estimator uses the same physical devices as the latter, followed by recursive classical filtering. The results are illustrated in the estimation of the real amplitude of a random coherent signal through quantum mechanical measurements in a single mode cavity with background thermal noise.

1. Introduction

With the recent advances in lasers and quantum electronics, and their unquestionable bearings to optical communication systems, problems related to the filtering of signals based on quantum mechanical measurements are no longer of academic interest only. In optical communication systems, where the signals are composed of optical frequencies, the various fields and interactions can be described accurately only by the laws of quantum mechanics. At these optical frequencies the intrinsic limitations on performance due to quantum mechanical laws (quantum mechanical uncertainty), become more important than other forms of random disturbance (as thermal noise). It is believed that the systematic study of optical communications systems in their natural quantum mechanical framework, will lead to guidelines for the design of more accurate components and in particular signal processing devices.

Although the problems of estimation and detection of parameters through quantum mechanical measurements have received considerable attention [1], [2], [3], the problem of filtering of a random signal via quantum mechanical measurements, has been considered only very recently [4], [6]. In this paper we consider the problem of estimating a discrete time random scalar signal x_k, $k=1,2,\dots$ with zero mean for all k, which is carried by an electromagnetic field at optical frequencies, for example a laser beam. The field is received in a cavity by opening an aperture at the appropriate time. This cavity is then closed and we make measurements on the received field, in order to generate information about the signal, that was carried by the field. Then, later, the cavity is cleansed and the process is repeated again at regular time intervals. Let us briefly describe the quantum mechanical description of the physical process described above [1], [5]. The field in the cavity at time k is described by a density operator $\rho(x_k)$, acting on a Hilbert space H, which is self-adjoint, positive definite, and has trace equal to 1. This density operator represents the quantum mechanical state of the field in the cavity. Each measurement we perform on the received field, is represented by a self-adjoint operator V_k on H, called an observable. If we let v_k be the outcome of the measurement represented by V_k, when the field is at the state represented by $\rho(x_k)$, then v_k is a random variable with probability distribution function given by

$$F_{v_k}(\xi)=\Pr\{v_k\le\xi\}=\operatorname{Tr}\rho(x_k)P((-\infty,\xi]) \qquad (1)$$

where $P(\cdot)$ is the spectral measure associated with the self-adjoint operator V_k, [5].

Care must be exercised in the sequel, in handling the two kinds of randomness appearing in our problem. Namely the randomness due to the stochastic signal x_k, and that due to the stochastic interpretation of quantum mechanical measurements. For that purpose we will denote by E expectation, by E_s expectation with respect to the signal process and by $E_{Q/s}$ quantum mechanical expectation given the values of the signal process (an operation necessary since our ρ's are parametrized by the signal, x_k). For example in (1) above, we display a conditional probability distribution function. We then have

$$E_{Q/s} v_k = \int v_k \, dF_{v_k} = \operatorname{Tr} \rho(x_k) \cdot V_k \tag{2}$$

It is by now well established that the new and interesting feature in quantum estimation problems, is that the observer (or experimenter) has the possibility of selecting what measurement to perform on the received field in order to extract more meaningful information about the signal carried by the field. This corresponds to an optimal selection of the corresponding observable V_k and generates the problem of subsequent physical realization of this measurement, [1], [2], [3], [4], [6].

The major problem studied in this paper is the optimal selection of observables at each time, together with optimal selection of a linear processing scheme of the outcomes of previous measurements, in order to obtain the minimum variance estimator of x_k. The paper is organized as follows: In section 2, we generate the general formulas for linear quantum filtering. In section 3, we illustrate the special form that these filters obtain under additional assumptions. In section 4, we give an application to a practical situation.

2. Quantum Linear Filtering

At first let us examine the following estimation scheme. At each time k, find the optimum observable V_k, to minimize

$$E(x_k - v_k)^2 = E_s E_{Q/s} (x_k - v_k)^2 = $$
$$= E_s \operatorname{Tr} \rho(x_k)(x_k I - V_k)^2 \tag{3}$$

We do this independently at each time, and the outcome of the selected measurement, is the estimator of x_k. By an easy application of the projection theorem [7] the solution for the optimal observable T_k is given by

$$T_k \eta_k + \eta_k T_k = 2\delta_{kk} \tag{4}$$

where $\eta_k = E_s \rho(x_k) = \int p(x_k) \rho(x_k) dx_k$ (5)

$$\delta_{kk} = E_s x_k \rho(x_k) = \int x_k p(x_k) \rho(x_k) dx_k \tag{6}$$

and $p(x_k)$ is the propability density function of x_k. This result is due to Personick [10], see also Yuen [3]. The optimal estimator in this case is

$$\hat{x}_k = \tau_k \tag{7}$$

Existence and uniqueness of solution to the problem defined by (3) are governed by existence and uniqueness of solutions to equation (4).

It may be possible, however, to improve the estimates by using outcomes of previous measurements. In this paper we consider only linear processing of previous measurements. Suppose that at time k we have already chosen optimal measurements $\hat{V}_0, \hat{V}_1, \hat{V}_2, \ldots \hat{V}_{k-1}$, which gave outcomes $\hat{v}_0, \hat{v}_1, \ldots \hat{v}_{k-1}$. The new estimator now is

$$\hat{x}_k = \hat{v}_k + \sum_{i=0}^{k-1} \hat{c}_i^{(k)} \hat{v}_i \tag{8}$$

where, the real numbers $\hat{c}_i^{(k)}$, $i = 0, 1, \ldots k-1$ and the optimal observable $\hat{V}_k$, with outcome $\hat{v}_k$, are to be chosen so that we minimize

$$\epsilon^2 = E(x_k - \hat{x}_k)^2 = E(x_k - \hat{v}_k - \sum_{i=0}^{k-1} \hat{c}_i^{(k)} \hat{v}_i)^2 \tag{9}$$

This problem has been previously considered by Park in [4]. Our first result is an improvement of the result appearing in [4]. The derivation however is entirely new, and is based on an elegant application of the projection theorem [7]. It has the advantage that demonstrates the geometric interpretation of the result and in addition settles quite readily the problem of existence and uniqueness of solution.

<u>Theorem 1:</u> There exist optimal observable $\hat{V}_k$, and optimal processing coefficients $\hat{c}_i^{(k)}$, $i = 0, 1, \ldots k-1$, if and only if there exist solution to the equations

$$\eta_k \hat{V}_k + \hat{V}_k \eta_k = 2\delta_{kk} - 2\sum_{i=0}^{k-1} \hat{c}_i^{(k)} \zeta_{ki} \tag{10}$$

$$\begin{bmatrix} \operatorname{Tr} \zeta_{00} \hat{V}_0 & \cdots & \operatorname{Tr} \zeta_{k-1\,0} \hat{V}_{k-1} \\ \vdots & & \\ \operatorname{Tr} \zeta_{0\,k-1} \hat{V}_0 & \cdots & \operatorname{Tr} \zeta_{k-1\,k-1} \hat{V}_{k-1} \end{bmatrix} \begin{bmatrix} \hat{c}_0^{(k)} \\ \hat{c}_1^{(k)} \\ \vdots \\ \hat{c}_{k-1}^{(k)} \end{bmatrix} = \begin{bmatrix} \operatorname{Tr} \delta_{k0} \hat{V}_0 \\ \operatorname{Tr} \delta_{k1} \hat{V}_1 \\ \vdots \\ \operatorname{Tr} \delta_{kk-1} \hat{V}_{k-1} \end{bmatrix} - \begin{bmatrix} \operatorname{Tr} \zeta_{k0} \hat{V}_k \\ \operatorname{Tr} \zeta_{k1} \hat{V}_k \\ \vdots \\ \operatorname{Tr} \zeta_{kk-1} \hat{V}_k \end{bmatrix} \tag{11}$$

where $\delta_{ij}=E_s x_i\rho(x_j)$

$$\eta_i = E_s\rho(x_i)$$

$$\zeta_{ij}=E_s\rho(x_i)\mathrm{Tr}\rho(x_j)\hat{V}_j \quad \text{for } i\neq j \tag{12}$$

and $\zeta_{ii}=\eta_i\hat{V}_i$

and $\hat{V}_i$, $i=0,1,\ldots k-1$ the previously chosen optimal observables. Moreover existence and uniqueness of solutions of the optimization problem is determined by existence and uniqueness of solutions to these equations.

<u>Proof:</u> We have

$$\begin{aligned}\epsilon^2 &= E\Big(x_k-\hat{v}_k-\sum_{i=0}^{k-1}c_i^{(k)}\hat{v}_i\Big)^2 = E_s E_{Q/s}\Big(x_k^2-2x_k\hat{v}_k + \\ &+\hat{v}_k^2-2\sum_{i=0}^{k-1}\hat{c}_i^{(k)}x_k\hat{v}_i+\sum_{\substack{i=0\\ i\neq j}}^{k-1}\sum_{j=0}^{k-1}\hat{c}_i^{(k)}\hat{c}_j^{(k)}\hat{v}_i\hat{v}_j+\sum_{i=0}^{k-1}\hat{c}_i^{(k)2}\hat{v}_i^2+ \\ &+2\sum_{i=0}^{k-1}\hat{c}_i^{(k)}\hat{v}_k\hat{v}_i\Big)=E_s\{\mathrm{Tr}\,x_k^2\rho(x_k)-2\mathrm{Tr}\,x_k\rho(x_k)\hat{V}_k+ \\ &+\mathrm{Tr}\rho(x_k)\hat{V}_k^2-2\sum_{i=0}^{k-1}\hat{c}_i^{(k)}\mathrm{Tr}x_k\rho(x_i)\hat{V}_i+ \\ &+\sum_{\substack{i=0\\ i\neq j}}^{k-1}\sum_{j=0}^{k-1}\hat{c}_i^{(k)}\hat{c}_j^{(k)}\mathrm{Tr}\rho(x_i)\hat{V}_i\,\mathrm{Tr}\rho(x_j)\hat{V}_j+ \\ &+\sum_{i=0}^{k-1}\hat{c}_i^{(k)2}\mathrm{Tr}\rho(x_i)\hat{V}_i^2+2\sum_{i=0}^{k-1}\hat{c}_i^{(k)}\mathrm{Tr}\rho(x_k)\hat{V}_k\mathrm{Tr}\rho(x_i)\hat{V}_i\}\end{aligned}\tag{13}$$

We recast (13) in the following setting. We let

$$H^{(k)}=\underbrace{H\otimes H\otimes\ldots\otimes H}_{k\text{ times}}$$

$$\hat{V}_i^{(k)}=\overbrace{I\otimes I\otimes\ldots\otimes \hat{V}_i\otimes I\ldots\otimes I}^{k\text{ times}}$$

$\uparrow$ i^{th} position

$$I^{(k)}=\underbrace{I\otimes I\otimes\ldots\otimes I}_{k\text{ times}}$$

Then (13) can be readily rewritten as

$$\epsilon^2=E_s\mathrm{Tr}\rho(x_0)\otimes\rho(x_1)\otimes\ldots\otimes\rho(x_k)\Big(x_kI^{(k)}-\hat{V}_k^{(k)}-\sum_{i=0}^{k-1}\hat{c}_i^{(k)}\hat{V}_i^{(k)}\Big)\tag{14}$$

Let now $\mathcal{L}$ be the set of operator valued functions of the form

$$f(x)=\beta xI^{(k)}+\sum_{i=0}^{k-1}\alpha_i\hat{V}_i^{(k)}+A_k^{(k)}\tag{15}$$

where β, x, α_i, $i=0,\ldots k-1$ are real numbers and $A_k^{(k)}=\underbrace{I\otimes\ldots\otimes I\otimes A_k}_{k\text{ times}}$, with A_k a self-adjoint operator on H, for which

$E_s\mathrm{Tr}\,\rho(x_0)\otimes\rho(x_1)\otimes\ldots\otimes\rho(x_k)\,f(x_k)^2$ is finite. Clearly $\mathcal{L}$ is a real vector space. Now for f, $g\in\mathcal{L}$ let us define

$$\langle f,g\rangle=E_s\mathrm{Tr}\rho(x_0)\otimes\ldots\otimes\rho(x_k)\{f(x_k)g(x_k)+g(x_k)f(x_k)\}\tag{16}$$

It is readily checked that $\langle\cdot,\cdot\rangle$ satisfies on $\mathcal{L}$ the properties of an inner product, except from the fact that $\langle f,f\rangle=0$, does not necessarily imply $f\equiv0$. So (16) defines on $\mathcal{L}$ a <u>degenerate inner product.</u> Let $\mathcal{H}$ be the linear manifold in $\mathcal{L}$ of functions of the form

$$h(x)=\sum_{i=0}^{k-1}\alpha_i\hat{V}_i^{(k)}+A_k^{(k)}\tag{17}$$

So from (14), it follows that we are looking for the element of $\mathcal{H}$ which has minimum distance defined by (16), from the element $x_kI^{(k)}$ of $\mathcal{L}$. This is precisely the setting for the projection theorem. The fact that our inner product is degenerate prohibits the proof of existence and uniqueness of solution, but nevertheless, the projection theorem can still be used to provide necessary and sufficient conditions characterizing the optimal solution. These are

$$\begin{aligned}&E_s\mathrm{Tr}\rho(x_0)\otimes\ldots\otimes\rho(x_k)\Big\{\Big(x_kI^{(k)}-\hat{V}_k^{(k)}-\sum_{i=0}^{k-1}\hat{c}_i^{(k)}\hat{V}_i^{(k)}\Big)\\ &\Big(\sum_{j=0}^{k-1}\alpha_j\hat{V}_j^{(k)}+D_k^{(k)}\Big)+\Big(\sum_{j=0}^{k-1}\alpha_j\hat{V}_j^{(k)}+D_k^{(k)}\Big)\Big(x_kI^{(k)}-\\ &-\hat{V}_k^{(k)}-\sum_{i=0}^{k-1}\hat{c}_i^{(k)}\hat{V}_i^{(k)}\Big)\Big\}=0\end{aligned}\tag{18}$$

for all real numbers α_i, $i=0,\ldots k-1$, and all self-adjoint operators D_k, on H. This is just the expression of the fact that the error must be orthogonal to $\mathcal{H}$. Carrying out the trace operation in (18) we get

$$\begin{aligned}&E_s\Big\{\sum_{j=0}^{k-1}2\alpha_jx_k\mathrm{Tr}\rho(x_j)\hat{V}_j-2\sum_{\substack{i=0\\ i\neq j}}^{k-1}\sum_{j=0}^{k-1}\hat{c}_i^{(k)}\alpha_j\mathrm{Tr}\rho(x_i)\hat{V}_i\,\cdot\\ &\mathrm{Tr}\rho(x_j)\hat{V}_j-\sum_{i=0}^{k-1}2\hat{c}_i^{(k)}\alpha_i\mathrm{Tr}\rho(x_i)\hat{V}_i^2-\\ &-\sum_{j=0}^{k-1}2\alpha_j\mathrm{Tr}\,\rho(x_k)\hat{V}_k\mathrm{Tr}\rho(x_j)\hat{V}_j+2x_k\mathrm{Tr}\rho(x_k)D_k-\\ &-\sum_{i=0}^{k-1}2\hat{c}_i^{(k)}\mathrm{Tr}\rho(x_i)\hat{V}_i\,\mathrm{Tr}\rho(x_k)D_k-\mathrm{Tr}\rho(x_k)\\ &(\hat{V}_kD_k+D_k\hat{V}_k)\Big\}=0\end{aligned}\tag{19}$$

for all real numbers α_i, $i=0,\ldots k-1$ and for all self-adjoint operators D_k, on H.

Letting $\alpha_i = 0$, $i=0,1,\dots k-1$ in (19) we get

$$2\mathrm{Tr}E_s x_k \rho(x_k) D_k - 2\sum_{i=0}^{k-1} \hat{c}_i^{(k)} E_s \mathrm{Tr}\rho(x_i)\hat{V}_i \mathrm{Tr}\rho(x_k) D_k -$$

$$- \mathrm{Tr}(E_s \rho(x_k)\hat{V}_k + \hat{V}_k E_s \rho(x_k)) D_k = 0 \tag{20}$$

for all self-adjoint D_k.

Letting $D_k = 0$ in (19) we get

$$\sum_{j=0}^{k-1} 2\alpha_j E_s x_k \mathrm{Tr}\rho(x_j)\hat{V}_j - 2\sum_{i=0}^{k-1} \sum_{\substack{j=0 \\ i \neq j}}^{k-1} \hat{c}_i^{(k)} \alpha_j E_s \mathrm{Tr}\rho(x_i)\hat{V}_i \cdot$$

$$\cdot \mathrm{Tr}\rho(x_j)\hat{V}_j - 2\sum_{i=0}^{k-1} \hat{c}_i^{(k)} \alpha_i E_s \mathrm{Tr}\rho(x_i)\hat{V}_i^2 -$$

$$- \sum_{j=0}^{k-1} 2\alpha_j E_s \mathrm{Tr}\rho(x_k)\hat{V}_k \mathrm{Tr}\rho(x_j)\hat{V}_j = 0 \tag{21}$$

for all α_j, $j=0,\dots k-1$. Therefore our original problem has a solution if and only if (20), (21) have a solution. Using the notation of (12) we get from (20)

$$\mathrm{Tr}\Big(2\delta_{kk} - \sum_{i=0}^{k-1} \hat{c}_i^{(k)} 2\zeta_{ki} - \eta_k \hat{V}_k - \hat{V}_k \eta_k\Big) D_k = 0 \tag{22}$$

for all self-adjoint D_k. But clearly (bv letting $D_k = 2\delta_{kk} - \sum_{i=0}^{k-1} \hat{c}_i^{(k)} 2\zeta_{ki} - \eta_k \hat{V}_k - \hat{V}_k \eta_k$) (22) can hold if and only if

$$\eta_k \hat{V}_k + \hat{V}_k \eta_k = 2\delta_{kk} - \sum_{i=0}^{k-1} \hat{c}_i^{(k)} 2\zeta_{ki} \tag{23}$$

Similarly from (21)

$$\sum_{j=0}^{k-1} (\mathrm{Tr}\delta_{kj}\hat{V}_j - \sum_{i=0}^{k-1} \hat{c}_i^{(k)} \mathrm{Tr}\zeta_{ij}\hat{V}_i - \mathrm{Tr}\zeta_{kj}\hat{V}_k)\alpha_j = 0 \tag{24}$$

for all α_j, $j=0,1,\dots k-1$. Clearly (24) holds if and only if

$$\mathrm{Tr}\delta_{kj}\hat{V}_j - \mathrm{Tr}\zeta_{kj}\hat{V}_k = \sum_{i=0}^{k-1} \hat{c}_i^{(k)} \mathrm{Tr}\zeta_{ij}\hat{V}_i \tag{25}$$

for $j=0,1,\dots k-1$.
and this completes the proof of the theorem.

The coupling between the equations determining the optimal observable $\hat{V}_k$ and the optimal processing vector $\hat{c}^{(k)}$, in the result of theorem 1, is only apparent and can indeed be disposed of, as the following corollary indicates.

<u>Corollary 1</u>: The optimal observables $\hat{V}_k$ and the optimal coefficients $\hat{c}^{(k)}$ satisfy the equations

$$\hat{V}_k = T_k - \sum_{i=0}^{k-1} \hat{c}_i^{(k)} \sigma_{ki} \tag{26}$$

$$\left(\begin{bmatrix} \mathrm{Tr}\zeta_{00}\hat{V}_0 & \dots & \mathrm{Tr}\zeta_{k-1\,0}\hat{V}_{k-1} \\ \vdots & & \\ \mathrm{Tr}\zeta_{0\,k-1}\hat{V}_0 & \dots & \mathrm{Tr}\zeta_{k-1\,k-1}\hat{V}_{k-1} \end{bmatrix} - \begin{bmatrix} \mathrm{Tr}\zeta_{k0}\sigma_{k0} & \dots & \mathrm{Tr}\zeta_{k0}\sigma_{k\,k-1} \\ \vdots & & \\ \mathrm{Tr}\zeta_{k\,k-1}\sigma_{k0} & \dots & \mathrm{Tr}\zeta_{k\,k-1}\sigma_{k\,k-1} \end{bmatrix} \right) \begin{bmatrix} \hat{c}_0^{(k)} \\ \vdots \\ \hat{c}_{k-1}^{(k)} \end{bmatrix} =$$

$$= \begin{bmatrix} \mathrm{Tr}\delta_{k0}\hat{V}_0 \\ \vdots \\ \mathrm{Tr}\delta_{k\,k-1}\hat{V}_{k-1} \end{bmatrix} - \begin{bmatrix} \mathrm{Tr}\zeta_{k0}T_k \\ \vdots \\ \mathrm{Tr}\zeta_{k\,k-1}T_k \end{bmatrix} \tag{27}$$

where T_k is as in eqution (4) and

$$\eta_k \sigma_{ki} + \sigma_{ki}\eta_k = 2\zeta_{ki},\ i=0,\dots k-1 \tag{28}$$

<u>Proof</u>: T_i satisfies $\eta_i T_i + T_i \eta_i = 2\delta_{ii}$ and we let σ_{ij} be the solution of $\eta_i \sigma_{ij} + \sigma_{ij}\eta_i = 2\zeta_{ij}$ then from (10) above $\hat{V}_k = T_k - \sum_{i=0}^{k-1} \hat{c}_i^{(k)} \sigma_{ki}$ which is (26). But then

$$\mathrm{Tr}\zeta_{ki}\hat{V}_k = \mathrm{Tr}\zeta_{ki}T_k - \sum_{j=0}^{k-1} \hat{c}_j^{(k)} \mathrm{Tr}\zeta_{ki}\sigma_{kj} \tag{29}$$

for $i=0,1,\dots k-1$. Substituting (29) into (11) completes the proof.

We note that in (26), (27) the equations for $\hat{V}_k$ and $\hat{c}^{(k)}$ are decoupled in the following sense. First we solve for $\hat{V}_0$ and we find $\hat{V}_0 = T_0$. Having this we can calculate all the coefficients in equation (27) and therefore solve for $c_0^{(1)}$. Also we can solve (28) for σ_{10}. Then (26) gives us $\hat{V}_1$. This in turn allows us to calculate all coefficients in (27) and solve for $(c_0^{(2)}, c_1^{(2)})$. We can also solve (28) for σ_{21}, σ_{20}. Then (26) gives $\hat{V}_2$, and so on repeating the process. That is the calculation of $\hat{c}^{(k)}$ each time requires only past data.

We would like to emphasize that the results of theorem 1 and corolary 1 are quite general; no assumption was made about the stochastic process x_k, $k=0,1,2,\dots$. We shall see in the next section, how various assumptions on the process x_k, modify these results.

Let us note also that the estimator described by equation (7) is an <u>unbiased estimator</u>. Indeed

$$E\tau_k = E_s \mathrm{Tr}\rho(x_k)T_k = \mathrm{Tr}\eta_k T_k = \mathrm{Tr}\delta_{kk} = E_s x_k \tag{30}$$

Moreover the estimator described by equation (8) (when we allow processing of previous measurements) is also unbiased. Indeed

$$E\hat{x}_k = E\hat{v}_k + \sum_{i=0}^{k-1} \hat{c}_i^{(k)} E\hat{v}_i = E\tau_k - \sum_{i=0}^{k-1} \hat{c}_i^{(k)} E_s \mathrm{Tr}\rho(x_k)\sigma_{ki} +$$

$$+ \sum_{i=0}^{k-1} \hat{c}_i^{(k)} E\hat{v}_i = E_s x_k - \sum_{i=0}^{k-1} \hat{c}_i^{(k)} \mathrm{Tr}\zeta_{ki} + \sum_{i=0}^{k-1} \hat{c}_i^{(k)} E\hat{v}_i = E_s x_k -$$

$$- \sum_{i=0}^{k-1} \hat{c}_i^{(k)} E_s \mathrm{Tr}\rho(x_i)\hat{V}_i + \sum_{i=0}^{k-1} \hat{c}_i^{(k)} E\hat{v}_i = E_s x_k - \sum_{i=0}^{k-1} \hat{c}_i^{(k)} E\hat{v}_i +$$

$$+ \sum_{i=0}^{k-1} \hat{c}_i^{(k)} E\hat{v}_i = E_s x_k \tag{31}$$

Recently [2], [3], [8] has been observed, that one can obtain better estimators, in some quantum estimation problems, by considering generalized measurements. A generalized measurement is visualized as a measurement (corresponding to a self-ajoint operator) on the composite system which consists of the original system (cavity) and an adjoined to the cavity auxiliary apparatus. The composite system is represented by a density operator $\rho(x_k)\otimes\rho_A$ (ρ_A is the same for all k) on the extended Hilbert space $H\otimes H_A$. However since we are dealing with a scalar random process, considering generalized measurements (i.e., minimizing (3) or (9) over all generalized measurements as well) does not change the results of theorem 1 and corollary 1. This follows from a modification of results in [3]. So we have

Theorem 2: The optimal measurements, and optimal processing coefficients, derived in theorem 1 or corollary 1, remain the same even if we consider generalized measurements. That is no additional (auxiliary) apparatus is necessary for the filtering schemes presented here.

3. Recursive Quantum Linear Filtering

The results described in section 2, solve completely the linear filtering problem in quantum estimation. They are not recursive however. This can lead to serious difficulties when we try to physically interpret the results, by means of physical realizations of the optimal measurements. In this section we will see how these filtering schemes become truly recursive under appropriate additional assumptions.

For the rest of the paper, we assume that the signal x_k, has the property that x_k, x_ℓ are jointly Gaussian for any pair of indices k, ℓ. A special case of this is when x_k satisfies the recursion

$$x_{k+1} = \phi_k x_k + w_k \tag{32}$$

where ϕ_k is a real number, w_k is a white noise Gaussian process with zero mean, x_0 is a Gaussian zero mean random variable and w_k is indepented of past signals x_j, $j<k$. So x_k is a Gaussian, zero mean random variable for each k, and moreover any finite collection of x_k's are jointly Gaussian.

The observables T_k, are intrinsic variables in any filtering scheme. In general the function $Tr\rho(x_k)T_k$ will be a nonlinear function of x_k. If it is linear however, we have a considerable reduction in the complexity of the result of the previous section, as the following theorem indicates.

Theorem 3: If $Tr\rho(x_k)T_k = \Gamma_k x_k$ for $k=0,1,\ldots,$ where Γ_k are scalar constants, and x_k, x_ℓ are jointly Gaussian for any pair of indices k, ℓ, then there exist scalars B_k, $k=0,1,\ldots$ such that the optimal observables of theorem 1, can be expressed as $\hat{V}_k = B_k T_k$.

Proof: Since x_i, x_j are jointly Gaussian there exist scalars A_{ij} such that the conditional expectations $E(x_j|x_i)=A_{ij}x_i$ for any pair i, j.

Now $\hat{V}_0 = T_0$ and

$$\zeta_{10} = E_s\rho(x_1)Tr\rho(x_0)\hat{V}_0 = E_s\rho(x_1)Tr\rho(x_0)T_0 =$$

$$= E_s\rho(x_1)\Gamma_0 x_0 = E_s(\rho(x_1)\Gamma_0 E(x_0|x_1)) = E_s(\rho(x_1)\Gamma_0 A_{10}x_1) =$$

$$= \Gamma_0 A_{10}\,\delta_{11}$$

So from (28) $\sigma_{10} = \Gamma_0 A_{10} T_1$ and from (26) $\hat{V}_1 = T_1 - \hat{c}_0^{(1)}\sigma_{10} = (1-\hat{c}_0^{(1)}\Gamma_0 A_{10})T_1 = B_1 T_1$; we complete the proof by induction. So let $\hat{V}_i = B_i T_i$, $i=0,\ldots k-1$, then $\zeta_{ki} = E_s\rho(x_k)Tr\rho(x_i)\hat{V}_i = B_i E_s\rho(x_k)Tr\rho(x_i)T_i =$

$$= B_i E_s\rho(x_k)\Gamma_i x_i = B_i\Gamma_i E_s(\rho(x_k)E(x_i|x_k)) = B_i\Gamma_i E_s(\rho(x_k)A_{ki}x_k) =$$

$= B_i\Gamma_i A_{ki}\delta_{kk}$. So from (28) $\sigma_{ki} = B_i\Gamma_i A_{ki}T_k$, $i=0,\ldots k-1$ and from (26)

$$\hat{V}_k = T_k - \sum_{i=0}^{k-1}\hat{c}_i^{(k)}B_i\Gamma_i A_{ki}T_k = (1-\sum_0^{k-1}\hat{c}_i^{(k)}B_i\Gamma_i A_{ki})T_k =$$

$$= B_k T_k$$

We note in particular that

$$B_k = 1 - \sum_{i=0}^{k-1}\hat{c}_i^{(k)}B_i\Gamma_i A_{ki} \tag{33}$$

Remark 1: If the random variables τ_i, x_i are jointly Gaussian for each i, then clearly there exist scalars Γ_i such that $Tr\rho(x_i)T_i = E(\tau_i|x_i) = \Gamma_i x_i$. So this represents a special case of theorem 3. Moreover if τ_i, x_i for each i, and x_k, x_ℓ for each pair of indices k, ℓ are jointly spherically invariant [12], the result of theorem 3 holds.

We would like to emphasize the physical significance of the result of theorem 3. It states that under these conditions the estimator that utilizes past measurements, uses the same measuring devices as the other estimator where we use only current measurements. These are represented by the T_k's. Indeed the quantum linear filter in this case displays an interesting separation property. At the first stage one optimally selects the measurements at each time independently, and then proceeds with the selection of an optimal post processing linear scheme which is a classical estimation problem. The following two figures, display the estimators, and should be helpful in the interpretation of the result.

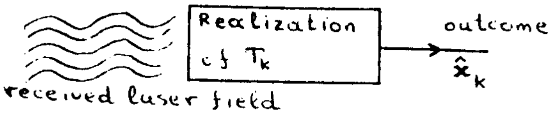

Figure 1. Estimator without past measurements.

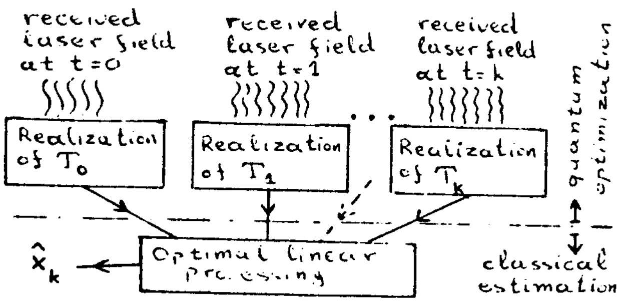

Figure 2. Estimator using past measurements.

We would also like to note, that in many practical applications the observables T_k, for all k, will represent the same physical measuring process.

The following theorem makes these ideas precise, and represents the main result of this paper.

Theorem 4: Let τ_k be the outcome of the measurements represented by the observables T_k, defined in equation (4) and let the conditions of theorem 3 be satisfied. Then the minimum variance estimator of x_k, which combines optimal observable selection and optimal linear postprocessing of past measurements (i.e., the one described by equation (8) and theorem 1), is precisely the minimum variance linear estimator of x_k based on the random variables τ_i, $i=0,1,\dots,k$.

Proof: Using the result of theorem 3 we can write (8) in the form

$$\hat{x}_k = B_k\tau_k + \sum_{i=0}^{k-1}\hat{c}_i^{(k)}B_i\tau_i \tag{34}$$

Moreover we have from (12)

$$Tr\delta_{kj}\hat{V}_j = B_j Tr\delta_{kj}T_j = B_j TrE_s x_k\rho(x_j)T_j = B_j Ex_k\tau_j$$

$$Tr\zeta_{ij}\hat{V}_i = B_iB_j TrE_s\rho(x_i)(Tr\rho(x_j)T_j)T_i = B_iB_jE\tau_i\tau_j, \; i\neq j$$

$$Tr\zeta_{ii}\hat{V}_i = B_i^2 Tr\eta_i T_i^2 = B_i^2 TrE_s\rho(x_i)T_i^2 = B_i^2E\tau_i^2$$

$$Tr\zeta_{ki}\hat{V}_k = B_kB_iE_s(Tr\rho(x_k)T_k)(Tr\rho(x_i)T_i) = B_k\Gamma_kB_iEx_k\tau_i$$

Then we can rewrite (11) as

$$\begin{bmatrix} E\tau_0^2 & \cdots & E\tau_{k-1}\tau_0 \\ \vdots & & \vdots \\ E\tau_{k-1}\tau_0 & & E\tau_{k-1}^2 \end{bmatrix}\begin{bmatrix} B_0\hat{c}_0^{(k)} \\ \vdots \\ B_{k-1}\hat{c}_{k-1}^{(k)} \end{bmatrix} = (1-B_k\Gamma_k)\begin{bmatrix} Ex_k\tau_0 \\ \vdots \\ Ex_k\tau_{k-1} \end{bmatrix} \tag{35}$$

The j^{th} row of (35), for $j=0,1,\dots k-1$ gives

$$\sum_{i=0}^{k-1}\hat{c}_i^{(k)}B_iE\tau_i\tau_j = (1-B_k\Gamma_k)Ex_k\tau_j \tag{36}$$

$$E\hat{x}_k\tau_j = (1-B_k\Gamma_k)Ex_k\tau_j + B_kE\tau_k\tau_j = Ex_k\tau_j - B_kE(\tau_jE(\tau_k|x_k)) + B_kE\tau_k\tau_j = Ex_k\tau_j \tag{37}$$

since $E(\tau_jE(\tau_k|x_k)) = E_sE(\tau_jE(\tau_k|s)|s) = E_s(E(\tau_j|s)E(\tau_k|s)) = E\tau_k\tau_j$, since $\tau_k\tau_j$ are independent conditioned on the signal process.

We rewrite (37) as

$$E(\hat{x}_k - x_k)\tau_j = 0;\; j=0,1,\dots k-1 \tag{38}$$

Now from (33) we have

$$B_k + \sum_{i=0}^{k-1}\hat{c}_i^{(k)}B_i\Gamma_iA_{ki} = 1 \tag{39}$$

and since $A_{ki} = \dfrac{Ex_kx_i}{Ex_k^2}$ by definition we get

$$\Gamma_kB_kE_sx_k^2 + \sum_{i=0}^{k-1}\hat{c}_i^{(k)}B_i\Gamma_k\Gamma_iE_sx_kx_i = \Gamma_kE_sx_k^2 \tag{40}$$

$$\Gamma_kE_sx_k^2 = E_sx_k\Gamma_kx_k = E_s(x_kE(\tau_k|x_k)) = E\tau_kx_k \tag{41}$$

$$\Gamma_k\Gamma_iE_sx_kx_i = E_s(E(\tau_k|x_k)E(\tau_i|x_i)) = E\tau_k\tau_i \tag{42}$$

Notice from (4) that $Tr\eta_kT_k^2 = Tr\delta_{kk}T_k$ and so

$$E\tau_k^2 = E_sTr\rho(x_k)T_k^2 = TrE_sx_k\rho(x_k)T_k = E_s(x_kE(\tau_k|x_k)) = Ex_k\tau_k \tag{43}$$

Using (41), (42), (43), we can rewrite (40) as

$$B_kE\tau_k^2 + \sum_{i=0}^{k-1}\hat{c}_i^{(k)}B_iE\tau_k\tau_i = Ex_k\tau_k \tag{44}$$

which using (34) reads

$$E(\hat{x}_k - x_k)\tau_k = 0 \tag{45}$$

With (38) and (45) the proof is completed.

Remark 2: From the result of theorem 4, it is obvious that the minimum variance achieved by the estimator which used previous measurements is in general (with the exception of trivial cases) smaller than the minimum variance achieved by the estimator which does not use previous measurements. So with the same equipment, employing a simple signal processing scheme we obtain a better filter.

If we let $k^{(i)}$ be the vector of coefficients in the expression of $\hat{x}_i$ as a linear combination of the τ_i's, then any recursion on the $k^{(i)}$'s (which will depend on the nature of the signal process and of the τ_i's) will produce under the assumptions of this section a truly recursive filter.

4. An Example

In this section we illustrate the results of this paper with the example of estimation of the real amplitude of a random coherent signal through quantum mechanical measurements in a single mode cavity with backgound thermal radiation. Then the receiver field has the density (with appropriate normalizations for simplicity [11])

$$\rho(x_k)=\int \frac{1}{\pi N}\exp\left(-\frac{|a-x_k|^2}{N}\right)|a\rangle\langle a|\, d^2a$$

in the P-representation, [1], [2], [3]. Here the coherent states $|a\rangle$ are eigenstates of the photon annihilation operator a. Let us suppose in addition that the amplitude x_k, $k=0,1,\ldots$, is a Gaussian stochastic process, generated by a dynamical model driven by white noise, as in equation (32) of the previous section (e.g., amplitude modulation). Then if we denote by $\lambda_k=Ex_k^2$ and by $Q_k=Ew_k^2$, the observables T_k take the form

$$T_k=\frac{\lambda_k}{N+2\lambda_k+\frac{1}{2}}\,(a_k+a_k^*) \qquad (46)$$

Personick [10], where a_k and a_k^* are the photon annihilation and creation operators. If we let ψ_k denote the outcome of optical homodyning, then this corresponds to the measurement represented by the operator $\frac{a_k+a_k^*}{2}$, see [10] p. 78. Then of course $\tau_k=\Gamma_k\psi_k$ where Γ_k is as in (47) below. One computes, [4], [10]

$\mathrm{Tr}\rho(x_k)T_k=\frac{2\lambda_k}{N+2\lambda_k+1/2}x_k$, so the conditions of theorem 3 are satisfied, in this example, with

$$\Gamma_k=\frac{2\lambda_k}{N+2\lambda_k+1/2} \qquad (47)$$

We see from (46), that all T_k's correspond to the same physical measurement (apart from a scalar scaling), so we need only consider one device which produces measurements at each time k. From Personick [10] we also know that if we let ψ_k denote the outcome of the measurement represented by $\frac{a_k+a_k^*}{2}$, then ψ_k conditioned on x_k is Gaussian with mean x_k and variance $\frac{N}{2}+\frac{1}{4}$. So ψ_k, x_k are jointly Gaussian. Therefore by introducing a white noise Gaussian process ϵ_k, with zero mean and covariance $E\epsilon_k\epsilon_\ell=\left(\frac{N}{2}+\frac{1}{4}\right)\delta_{k\ell}$ which is independent from x_k we can represent τ_k as (see [10], p. 77)

$$\tau_k=\Gamma_k(x_k+\epsilon_k)=\Gamma_k\psi_k \qquad (48)$$

where Γ_k is given by (47). But then from theorem 4 it follows that the minimum variance estimator using past measurements, is just the Kalman filter estimator of x_k, satisfying (32), using observations τ_i, $i=0,\ldots k$, described by (48). From [7] we have the following recursive form for $\hat{x}_k$.

$$\hat{x}_k=\phi_{k-1}\hat{x}_{k-1}+(P_{k|k-1}/\Gamma_k)\left(P_{k|k-1}+\frac{N}{2}+\frac{1}{4}\right)^{-1}(\tau_k-\Gamma_k\phi_{k-1}\hat{x}_{k-1}) \qquad (49)$$

or using (48),

$$\hat{x}_k=\phi_{k-1}\hat{x}_{k-1}+P_{k|k-1}\left(P_{k|k-1}+\frac{N}{2}+\frac{1}{4}\right)^{-1}(\psi_k-\phi_{k-1}\hat{x}_{k-1}) \qquad (50)$$

$$\left.\begin{aligned} &P_{k+1|k}=\phi_k^2P_{k|k}+Q_k\\ &P_{k|k}=P_{k|k-1}-P_{k|k-1}^2\left(P_{k|k-1}+\frac{N}{2}+\frac{1}{4}\right)^{-1}\end{aligned}\right\} \qquad (51)$$

for $k=1,2,\ldots$, $\hat{x}_0=x_0$, $P_{0|0}=Ex_0^2$ and $P_{k|k}=E(x_k-\hat{x}_k)^2$

The following picture illustrates the filter derived in this example, where $K(k)=P_{k|k-1}(P_{k|k-1}+\frac{N}{2}+\frac{1}{4})^{-1}$ is the Kalman gain, computed via (51).

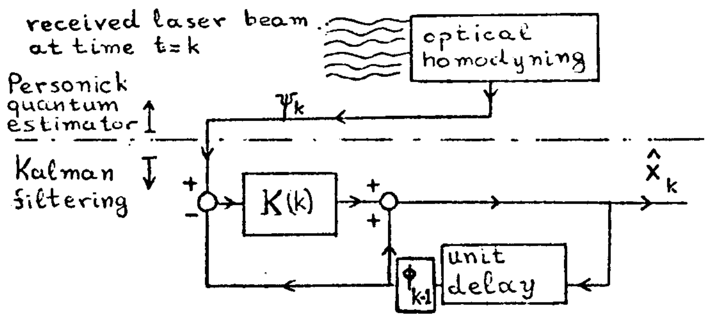

Figure 3. Illustrating the filter for the example.

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