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ON CANONICAL REALIZATIONS WITH UNBOUNDED INFINITESIMAL GENERATORS*

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Abstract

In this paper we study canonical (controllable and observable) realizations for infinite dimensional linear systems. In these realizations the infinitesimal state transition operator is unbounded but the generator of a C_0-semigroup in a Hilbert space. We present ways to reduce a realization to a canonical one. We also study the relation between the analytic properties of a transfer function and the spectral properties of the infinitesimal generators in its realizations. Finally we describe a class of transfer functions which can be realized by a system with infinitesimal generator having spectral properties closely related to the singularities of the transfer functions.

0. INTRODUCTION

The theory of infinite dimensional linear systems has progressed significantly in the last few years. This is mainly due to the application of certain fairly recent operator theoretic results in the study of infinite dimensional linear systems in Hilbert spaces. In particular the circle of ideas related to invariant subspace theory in H^2 spaces, provides a natural framework for a very successful study of the realization problem for non-rational transfer functions. Based on this approach several problems in infinite dimensional realization theory have been studied in our previous work [1] (with Roger W. Brockett), [2], by Paul A. Fuhrmann in [3], [4] and by J. W. Helton in [5].

The problem of realization, in our setting, is to express a given real function T defined on $[0,\infty)$ as $T(t) = c[e^{At}b]$, or to express its Laplace transform $\tilde{T}(s)$ as $c[(Is-A)^{-1}b]$ in some appropriately defined region of the complex plane. T is called the <u>weighting pattern</u> [6] and $\tilde{T}$ the <u>transfer function</u>. For further details, notation and relations to external and internal description of systems we refer to [1], [2]. In [1] we considered the following two cases: when A generates a C_0-semigroup on $\mathcal{H}$, $b \in \mathcal{H}$ and c is a bounded linear functional on $\mathcal{H}$, $[A, b, c]$ is a <u>regular realization</u>; in that case we write also $y(t) = \langle c, x(t)\rangle$ where c belongs to $\mathcal{H}$; $[A, b, c]$ is a <u>balanced realization</u> when A is as before, b belongs to the domain of A (denoted $\mathcal{D}_0(A)$), and c is a linear functional, defined on $\mathcal{D}_0(A)$ and such that $|c[x]| \le k(\|Ax\| + \|x\|)$ for $x \in \mathcal{D}_0(A)$ and some constant k. For a comparison between these two concepts we refer to

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[1]. Our motivation for introducing the notion of a balanced realization comes from systems governed by partial differential equations with observations on the boundary of the domain of definition (see[2]p. 34, and [8] p. 200). For general motivation and examples we refer to [1], [2].

The following theorem ([1], Theorem 3) proves that the class of weighting patterns which admit balanced realizations is identical with the class of weighting patterns which admit regular realizations.

Theorem 1: A weighting pattern T has a balanced realization if and only if it has a regular one. Moreover the infinitesimal generators in both realizations can be taken to be the same.

We denote by Π^+ the half-plane Res>0. $H^2(\Pi^+)$ consists of functions analytic in Π^+ and square integrable along vertical lines in Π^+ such that

$$\sup_{x>0} \int_{-\infty}^{\infty} |\psi(x+iy)|^2 dy \le M < \infty$$

We denote by $\mathbb{I}$ the imaginary axis in the complex plane. The Fourier transform

$$g(t) \xrightarrow{\mathfrak{F}} \int_{-\infty}^{\infty} e^{-i\omega t} g(t)dt = G(i\omega) \tag{1}$$

is a unitary map between $L_2(-\infty,\infty)$ and $L_2(\mathbb{I}, d\omega/2\pi)$. $L_2(o,\infty)$ is considered as the subspace of $L_2(-\infty,\infty)$, of functions which vanish on $(-\infty, o)$. $H^2(\mathbb{I}) = \mathfrak{F} L_2(o,\infty)$ and $\tilde{H}^2(\mathbb{I}) = \mathfrak{F} L_2(-\infty, o)$. We have also that $H^2(\mathbb{I})^{\perp} = \tilde{H}^2(\mathbb{I}) = \overline{H^2(\mathbb{I})}$. $H^2(\mathbb{I})$ consists exactly of the boundary values of the elements of $H^2(\Pi^+)$. The Paley-Wiener Theorem establishes that $H^2(\Pi^+) = \mathcal{L}\, L_2(o,\infty)$, where $\mathcal{L}$ denotes the Laplace transform. $\mathbb{D}$ denotes the open unit disk and $\mathbb{T}$ the unit circle. $H^2(\mathbb{T})$ denotes the subspace of $L^2(\mathbb{T})$ of functions with vanishing negative Fourier coefficients. $H^2(\mathbb{D})$ denotes the space of functions analytic in $\mathbb{D}$, with Taylor series around zero having square summable coefficients. $H^2(\Pi^+)$, $H^2(\mathbb{I})$, $\tilde{H}^2(\mathbb{I})$, $H^2(\mathbb{D})$ and $H^2(\mathbb{T})$ are called Hardy spaces.

In [1] we derived also criteria for a transfer function to admit a regular (and hence a balanced) realization. The most general conditions for realizability derived in [1] are given by the following theorem and corollary ([1] Theorem 6, Corollary 6.1).

Theorem 2: Let $T \in L_2(o,\infty)$ and continuous. If $\tilde{T}(i\omega) = \overline{F_c(i\omega)} F_o(i\omega)$, where F_c, F_o belong to $H^2(\mathbb{I})$, then T has a regular realization.

Corollary 2.1: Let T be continuous and of exponential order. If for some α the function $T_1(t) = e^{-\alpha t} T(t)$ satisfies the conditions of Theorem 2, then T has a regular realization.

We denote, as usual, by e^{At} the C_0-semigroup of bounded operators generated by A. An element g of $\mathcal{K}$ is called a cyclic vector of the semigroup e^{At}, whenever the linear span of the vectors $e^{At}g, t \ge o$ is dense in $\mathcal{K}$. A realization [A, b, c] (balanced or regular) is controllable if and only if b is a cyclic vector of the semigroup e^{At} (Fattorini [9], p. 393). A regular realization [A, b, c] is observable if and only if c is a cyclic vector of the semigroup $e^{A^* t}$. A balanced realization is observable if and only if $x \in \mathcal{D}_o(A)$ and $c[e^{At}x] = 0$ for $t \ge 0$ imply $x = 0$. This implies that any two states that can be reached exactly are distinguishable. (For a summary of Controllability and Observability theory for infinite dimensional linear systems see [2]p. 59-65). A realization is canonical whenever is controllable and observable. In section 1 of this paper we show how to construct a canonical regular (or balanced) realization starting from a given regular (or balanced) realization.

It is often desirable that the internal model for a given input-output map satisfy certain requirements (besides simplicity) which are due to engineering considerations. The importance of the connectedness of the resolvent set of the infinitesimal generator in relation to frequency response methods for system identification is explained in [1]. The requirement that the spectral properties of the infinitesimal generator of a model should reflect closely the properties of the singularities of $\tilde{T}$ is essential from the engineering point of view. It is used as a guide in many ad hoc modeling or synthesis methods in electrical engineering [11], [12]. In section 3 of this paper we construct canonical realizations satisfying both these requirements. We characterize also a class of transfer functions which admit such realizations. It turns out that this class includes many of the transfer functions appearing in practical problems.

1. REDUCTION OF REALIZATIONS TO CANONICAL ONES

The following theorem describes a way to construct a canonical regular realization for a weighting pattern T, starting from any regular realization of T.

Note: When L is a closed subspace of $\mathcal{H}$ and B a linear operator, $P_L B|_L$ denotes the operator 'B restricted on L,' provided it is well defined.

Theorem 3: Let $[A, b, c]$ be a regular realization of a weighting pattern T on the Hilbert space $\mathcal{H}$. Let M be the closure of the linear span of the vectors $e^{At}b$ with $t \geq 0$, and P_M the associated orthogonal projection. Then i) $[P_M A|_M, b, P_M c]$ is a regular realization of T, with state space M.

Let now N be the closed linear span in M of the vectors $P_M e^{A^*t} P_M c$ with $t \geq 0$, and let P_N be the associated orthogonal projection ($P_N: \mathcal{H} \to N$). Then ii) $[P_N A|_N, P_N b, P_M c]$ is a canonical regular realization of T, with state space N.

Proof: Obviously M is invariant under e^{At} for $t \geq 0$. Hence

$$P_M e^{At} P_M = e^{At} P_M \tag{2}$$

Let $S(t) = P_M e^{At}|_M$. Then $S(t_1)S(t_2) = S(t_1 + t_2)$ for $t_1, t_2 \geq 0$, $S(0)$=identity on M and S(t) is strongly continuous on M. Hence S(t), $t \geq 0$, is a C_o-semigroup on M. It is clear that the infinitesimal generator of S(t) is the operator $P_M A|M = A|_M$, which has domain dense in M. Then we write in our usual notation $S(t) = \exp(P_M A|_M t)$. Now

$$<P_M c, e^{(P_M A|_M)t} b> = <c, P_M e^{At} b> = <c, e^{At} b> = T(t)$$

Since M, $\mathcal{H}$ are Hilbert spaces we have that

$$(S(t))^* = (P_M e^{At}|_M)^* = P_M e^{A^*t}|_M = e^{(P_M A^*|_M)t}$$

is also a C_o-semigroup on M with infinitesimal generator $P_M A^*|_M$. Obviously N is invariant under $P_M e^{A^*t}|_M$ for $t \geq 0$ and so

$$P_N e^{A^*t}|_N = P_M e^{A^*t}|_N \overset{def}{=} S_1(t) \tag{3}$$

Based on (3) we prove as above that $S_1(t)$ is a C_o-semigroup on N, with infinitesimal generator the operator $P_N A^*|_N$. We write again in standard notation $P_N e^{A^*t}|_N = \exp(P_N A^*|_N t)$. We have also that

$$P_N e^{At} P_N = P_N e^{At} P_M \tag{4}$$

Hence $P_N e^{At}|_N = (S_1(t))^*$ is a C_o-semigroup on N with inf. gen. $P_N A|_N$. Now

$$<P_M c, e^{(P_N A|_N)t} P_N b> = <P_M c, P_N e^{At} P_M b> = <c, e^{At} b> = T(t)$$

Moreover if $x \in N$ and $<P_M c, \exp(P_N A|_N t)x> = 0$ for all $t \geq 0$, then it follows using (3) that $<P_M e^{A^* t} P_M c, x> = 0$ for $t \geq 0$, and so $x=0$ by the definition of N. If now $x \in N$ and $<\exp(P_N A|_N t) P_N b, x> = 0$ for all $t \geq 0$, then it follows from (4) that $<P_N e^{At} P_M b, x> = <e^{At} b, x> = 0$ for $t \geq 0$, and so $x=0$ by the definition of M. This proves (ii).

We can now use this Theorem and Theorem 1 to produce a canonical balanced realization starting from a given balanced realization. First we show that if the regular realization in Theorem 1 is canonical, then the balanced one is canonical too. The relations connecting the balanced realization $[A, b, c]$ to the regular realization $[A, b_1, c_1]$ ([1] Theorem 3) are given by $b = (\lambda I - A)^{-1} b_1$, $c[x] = <c_1, (\lambda I - A)x>$ with $\lambda > 1$ and in $\rho(A)$. Suppose $x \in \mathcal{D}_o(A)$ and $c[e^{At}x] = 0$ for $t \geq 0$. Then $<c_1, e^{At}(\lambda I - A)x> = 0$ for $t \geq 0$ and hence $(\lambda I - A)x = 0$. This implies $x = 0$ since $\lambda \in \rho(A)$. Suppose now $<e^{At} b, x> = 0$ for $t \geq 0$. Then $<e^{At} b_1, (\lambda I - A^*)^{-1} x> = 0$ and so $(\lambda I - A^*)^{-1} x = 0$. This implies $x = 0$ since $\lambda \in \rho(A^*)$. Therefore $[A, b, c]$ is canonical.

Hence we have the following:

Corollary 3.1: Given the balanced realization $[A, b, c]$ of T, we construct the associated regular one $[A, b_1, c_1]$ (using Theorem 1), which we reduce according to Theorem 3 to a canonical regular realization $[F, g_1, h_1]$. Finally we construct (using Theorem 1) the associated balanced realization $[F, g, h]$. Then $[F, g, h]$ is a canonical balanced realization of T.

Based on the above we can restrict our study to canonical regular realizations.

2. SPECTRAL CONSIDERATIONS AND THE HARDY CLASS ON HALF-PLANES

In the rest of this paper we investigate the problem of constructing realizations with infinitesimal generators having spectral properties closely related to the singularities of $\tilde{T}$. Following [1] we denote by $\sigma(\tilde{T})$ the set of nonanayticity of $\tilde{T}$ and by $\rho_o(A)$ the connected component of $\rho(A)$ which contains the half-plane $\text{Re} s > \beta$. ($\text{Re} s > \beta$ is the half-plane contained in $\rho(A)$, described in the Hille-Yosida Theorem [7]). We showed in [1] that if $[A, b, c]$ is a realization of T (balanced or regular) and if $\sigma_o(A)$ is the complement of $\rho_o(A)$ we must have the spectral inclusion property

$$\sigma(\tilde{T}) \subseteq \sigma_o(A)$$

Questions about the connectedness of the resolvent set of the infinit. generators, the simplicity of spectrum, and the relations to analytic properties of the transfer functions were for the first time raised and answered in [1], for bounded realizations (i.e. when the infinitesimal generators are bounded). In contrast with the finite dimensional theory the spectrum of the infinitesimal generator in a canonical regular realization is not uniquely determined by the transfer function (see [1], [2], [3], [4] for counterexamples with bounded realizations, which are a special class of regular realizations).

Here we present a similar investigation for transfer functions which admit regular realizations. In the rest of this paper we restrict to the study of the class described in Theorem 2. Extensions of our results to include the larger class described in Corollary 2.1 are easy.

To proceed we need some facts from the theory of $H^2(\Pi^+)$ functions. For details we refer to Hoffman [12], Duren [14]. A function Φ is inner if it is analytic in Π^+, with $|\Phi(s)| \le 1$ for $s \in \Pi^+$, while $|\Phi(i\omega)| = 1$ a.e. A function $H \in H^2(\Pi^+)$ is outer if its boundary value on the imaginary axis is a cyclic vector for the semigroup 'multiplication by $e^{-i\omega t}$' on $H^2(\mathbb{I})$. Equivalently H is outer if its inverse Laplace transform h, has the property that its right translations form a dense set in $L_2(0,\infty)$. A subpsace of $L_2(0,\infty)$ which is invariant under right translations is mapped by the Fourier transform to a subspace of $H^2(\mathbb{I})$ which is invariant under 'multiplication by $e^{-i\omega t}$', $t \ge 0$. Following Lax [15], we call such a subspace of $H^2(\mathbb{I})$ a right translation invariant subspace. The orthogonal complement of a right translation invariant subspace is called a left translation invariant subspace and is invariant under 'multiplication by $e^{i\omega t}$, $t \ge 0$, followed by projection on $H^2(\mathbb{I})$'. The inverse Fourier transform of a left translation invariant subspace is a subspace of $L_2(0,\infty)$ which is invariant under left translations followed by restriction to $(0,\infty)$. Clearly we have corresponding facts for such subspaces in $H^2(\Pi^+)$.

For clarity we use the variable z for complex numbers in $\mathbb{D}$, while the variable s is used for complex numbers in Π^+. The map

$$z \xrightarrow{\eta} s = \frac{1+z}{1-z} \tag{5}$$

maps $\mathbb{D}$ onto Π^+. It is well known [12], [17], that the map $\mathcal{V}$ defined by

$$[\mathcal{V} F](z) = g(z) = \frac{2}{1-z} F\left(\frac{1+z}{1-z}\right) \tag{6}$$

is a unitary map from $H^2(\Pi^+)$ onto $H^2(\mathbb{D})$. $\mathcal{V}$ restricted to boundary values is a unitary map of $H^2(\mathbb{I})$ onto $H^2(\mathbb{T})$. The inverse of $\mathcal{V}$ is given by

$$[\mathcal{V}^{-1} g](s) = F(s) = \frac{1}{s+1} g\left(\frac{s-1}{s+1}\right). \tag{7}$$

Let as usual $\mathcal{H}$ denote a Hilbert space and e^{At} a C_o-semigroup of contractions on $\mathcal{H}$, with generator A. The operator $B = (A+I)(A-I)^{-1}$ is a contraction, called the cogenerator of e^{At} (see [16] p. 141). The relation between the semigroup and its cogenerator is the following:

$$\left.\begin{aligned} e^{At} &= e_t(B), \quad t \ge 0 \\ B &= \lim_{t \to 0+} \phi_t(e^{At}) \end{aligned}\right\} \tag{8}$$

where $e_t(\lambda) = \exp(t\,\lambda + t/\lambda - 1)$, $t \ge 0$, $\phi_t(\lambda) = \lambda - 1 + t/\lambda - 1 - t$. Using these relations it is easy to show that a vector $b \in \mathcal{H}$ is a cyclic vector for B if and only if it is a cyclic vector for e^{At}. Moreover B^* is the cogenerator of the adjoint semigroup $(e^{At})^*$ and the relations described above are valid if we replace B with B^* and e^{At} by $(e^{At})^*$ (see [16] p. 143).

The semigroup of contractions 'multiplication by $e^{-i\omega t}$' on $H^2(\mathbb{I})$ has as its cogenerator B, the operator 'multiplication by $i\omega - 1/i\omega + 1$'. Then for $g \in H^2(\mathbb{D})$ we have

$$[\mathcal{V} B \mathcal{V}^{-1} g](z) = z g(z) = [Ug](z) \tag{9}$$

Hence B is unitarily equivalent to the forward shift on $H^2(\mathbb{T})$ (denoted as usual by U). The adjoint semigroup is 'multiplication by $e^{i\omega t}$ followed by projection on $H^2(\mathbb{I})'$, and its cogenerator is B^* and hence it is unitarily equivalent to the backward shift on $H^2(\mathbb{T})$. Clearly $F \in H^2(\mathbb{I})$ is a cyclic vector for B (resp. B^*) on $H^2(\mathbb{I})$ if and only if $\mathcal{V} F$ is a cyclic vector for the forward (resp. backward shift) on $H^2(\mathbb{T})$. Thus we have proved the following:

Lemma 1: A function $F \in H^2(\Pi^+)$ is outer if and only if $\mathcal{V} F \in H^2(\mathbb{D})$ is outer. A function $F \in H^2(\mathbb{I})$ is a cyclic vector for the semigroup "multiplication by $e^{i\omega t}$ followed by projection on $H^2(\mathbb{I})'$ if and only if $\mathcal{V} F \in H^2(\mathbb{T})$ is a cyclic vector for the backward shift on $H^2(\mathbb{T})$.

The following theorem is due to Lax and characterizes right translation invariant subspaces in $H^2(\Pi^+)$.

Theorem 4 ([15]): Every closed subspace R of $H^2(\Pi^+)$ invariant under 'multiplication by e^{-st}', $t \geq 0$, is of the form $\Phi H^2(\Pi^+)$ where Φ is inner. Φ is unique modulo a constant of modulus one. If ϕ is the inner function associated to $\mathcal{V} R$ by Beurling's theorem then $\Phi(s) = \phi(\frac{s-1}{s+1})$.

An inner function Φ is normalized whenever the corresponding ϕ on the disk is normalized. Any element $F \in H^2(\Pi^+)$ has a factorization $F = \Phi \cdot H$ where Φ is inner and H is outer. Every inner function has a factorization $\Phi = cBS$ where c is a constant of modulus 1, B a Blaschke product and S a singular function (see [12] for details).

Using Lemma 1 we can obtain many properties of cyclic or non-cyclic vectors for the semigroup 'multiplication by $e^{i\omega t}$ followed by projection on $H^2(\mathbb{I})'$, from properties of cyclic or noncyclic vectors for the backward shift on $H^2(\mathbb{T})$ (see [13]). For example any $F \in H^2(\Pi^+)$ with isolated branch points on the imaginary axis is a cyclic vector.

In this paper cyclic or noncyclic is understood with respect to the semigroup 'multiplication by $e^{i\omega t}$ followed by projection on $H^2(\mathbb{I})'$.

Definition: Let $F \in H^2(\mathbb{I})$, e^{At} denote the semigroup 'multiplication by $e^{i\omega t}$ followed by projection on $H^2(\mathbb{I})'$. The left translation invariant subspace generated by F is the closure of the linear span of the vectors $e^{At}F$, $t \geq 0$, in $H^2(\mathbb{I})$.

Theorem 5: An element F of $H^2(\mathbb{I})$ is a noncyclic vector if and only if there exist an inner function Φ and a function H in $H^2(\Pi^+)$ such that $F(i\omega) = \overline{H(i\omega)}\, \Phi(i\omega)$ a.e. on $\mathbb{I}$. Moreover if we choose Φ to be normalized and relatively prime to the inner factor of H this factorization is unique. In this case the left translation invariant subspace generated by F is $(\Phi H^2(\mathbb{I}))^{\perp}$.

Proof: This is a direct consequence of the corresponding theorem on the disk ([13] p. 56) and of the properties of the map $\mathcal{V}$ defined by (6), (7).

Definition: The inner function uniquely associated to every non-cyclic vector $F \in H^2(\Pi^+)$ by Theorem 5 is called the associated inner function of F.

We are ready now to proceed with the study of canonical realizations for the class of weighting patterns described in Theorem 2.

For any such transfer function we have the 'right translation realization' which is constructed by considering as Hilbert space $\mathcal{H}$ the space $H^2(\mathbb{I})$, the semigroup 'multiplication by $e^{-i\omega t}$' as e^{At}, F_c as b and F_o as c. Using inverse Fourier transform this realization is described via the semigroup of right translations on $L_2(0,\infty)$. Our plan is to apply Theorem 3 to this realization, obtain a canonical one, and then discuss the spectral properties of the latter. We assume with no loss of generality that the inner factors of F_c and F_o are relatively prime. We start with the following preliminary result.

Lemma 2: Suppose $\tilde{T} \in H^2(\Pi^+)$ and has a factorization $\tilde{T}(i\omega) = \overline{F_c(i\omega)}$ $F_o(i\omega)$ a.e. on the imaginary axis, where $F_c, F_o \in H^2(\Pi^+)$ and have no common inner factor. Then $\tilde{T}$ is noncyclic if and only if F_o is.

Proof: Suppose F_o is noncyclic. Then by Theorem 5, $F_o(i\omega) = \overline{H_o(i\omega)}$ $\Phi_o(i\omega)$ a.e. on $\mathbb{I}$, where $H_o \in H^2(\Pi^+)$ and Φ_o is inner. Hence $\tilde{T} = \overline{F_c(i\omega) H_o(i\omega)}\, \Phi_o(i\omega)$ a.e. on $\mathbb{I}$. Since $\tilde{T} \in H^2(\Pi^+)$ we have that $F_c H_o \in H^2(\Pi^+)$ and therefore by Theorem 5, $\tilde{T}$ is noncyclic.

Suppose now that $\tilde{T}$ is noncyclic. Then by (6) we have

$$[\mathcal{V}\tilde{T}](e^{i\theta}) = \frac{2}{1-e^{i\theta}} \overline{F_c\left(\frac{1+e^{i\theta}}{1-e^{i\theta}}\right)}\, F_o\left(\frac{1+e^{i\theta}}{1-e^{i\theta}}\right) = \overline{\frac{1}{2}e^{i\theta}[\mathcal{V}F_c](e^{i\theta})}(e^{i\theta}-1)[\mathcal{V}F_o](e^{i\theta}) \tag{10}$$

Let now

$$\mathcal{V}\tilde{T} = g;\ \frac{1}{2}e^{i\theta}[\mathcal{V}F_c](e^{i\theta}) = f(e^{i\theta});\ \ (e^{i\theta}-1)[\mathcal{V}F_o](e^{i\theta}) = h(e^{i\theta}) \tag{11}$$

Then $g \in H^2(\mathbb{D})$ and by Lemma 1 is a noncyclic vector for the backward shift; $f \in H^2(\mathbb{D})$ and $f(0) = 0$; $h \in H^2(\mathbb{D})$. By Theorem 3.1.5 in [13], there exists an element $g_1 \in H^2(\mathbb{D})$ with $g_1(0)=0$ and an inner function ϕ (on the disk) such that $g = \overline{g_1}\phi$ a.e. on $\mathbb{T}$. Therefore

$$g = \overline{f}h = \overline{g_1}\phi \tag{12}$$

Now $\phi f \in H^2(\mathbb{D})$. Let U denote, as usual, the forward shift and U^* the backward shift on $H^2(\mathbb{D})$. Then

$$\langle \phi f, U^{*n}h\rangle = \langle U^n \phi f, h\rangle = \int \overline{z^n \phi f}\, h\, d\mu(\theta) = \int \overline{z^n}\, \overline{g_1}\, d\mu(\theta) = 0\ ;\ n=0,1,2,\ldots$$

because $\overline{g_1} \perp H^2(\mathbb{D})$. Since $\phi f \perp U^{*n}h$, for $n=0,1,2,\ldots$ it follows that h is noncyclic. But $v \in H^2(\mathbb{T})$ is a noncyclic vector for the backward shift if and only if $e^{i\theta}v$ is one (see [13] Theorem 2.2.8). We conclude therefore from (11) that $\mathcal{V}F_o$ is noncyclic. Then by Lemma 1, F_o is noncyclic and this completes the proof.

Since the properties of the 'right translation realization' depend heavily on $\tilde{T}$ being cyclic or noncyclic, we study these two cases separately, in the sequel.

3. NONCYCLIC TRANSFER FUNCTIONS

Since $\tilde{T}$ is noncyclic it follows from Lemma 2 that F_o is noncyclic

We study first transfer functions, for which F_c is outer. Then the 'right translation realization' is controllable. Hence applying Theorem 3 we obtain a canonical realization with state space N = left translation invariant subspace generated by F_o in $H^2(\mathbb{I})$, with $P_N e^{-i\omega t}|_N$ as e^{At}, with F_o as c and $P_N F_c$ as b. Moreover by Theorem 5 since F_o is noncyclic we have that $N = (\Phi_o H^2(\mathbb{I}))^{\perp}$, where Φ_o is the associated inner function of F_o.

The following theorem describes the spectrum of the infinitesimal generator of the semigroup $P_N e^{-i\omega t}|_N$ (where $N = (\Phi_o H^2(\mathbb{I}))^{\perp}$) in terms of the inner function Φ_o.

Theorem 6 ([8] p. 70): Let N be a left translation invariant subspace of $H^2(\mathbb{I})$, i.e. $N = (\Phi H^2(\mathbb{I}))^{\perp}$ for some inner function Φ. Consider the semigroup 'multiplication by $e^{-i\omega t}$', restricted on N (i.e. $P_N e^{-i\omega t}|_N$). The spectrum of its infinitesimal generator is the set $\mathcal{S}_\Phi$ which consists of

i) all complex numbers μ with Re $\mu < 0$, such that $\Phi(-\mu)=0$

ii) all complex numbers μ with Re $\mu=0$ such that Φ cannot be continued analytically across the imaginary axis at $-\mu$.

Combining this result with the previous discussion we have

Theorem 7: Let $\tilde{T}$ be a transfer function which belongs to $H^2(\Pi^+)$, is noncyclic and has a factorization $\tilde{T}=\overline{F_c}F_o$ a. e. on the imaginary axis with F_c, F_o in $H^2(\Pi^+)$ and F_c outer. Then $\tilde{T}$ has a canonical realization with the spectrum of the infinitesimal generator being exactly $\mathfrak{S}_{\tilde{\Phi}_o}$, where Φ_o is the associated inner function of F_o. This realization is constructed by taking, as state space the subspace of $H^2(\mathbb{I})$, $N=(\Phi_o H^2(\mathbb{I}))^{\perp}$, as c the function F_o, as b the projection of F_c on N and as semigroup the restriction of the semigroup 'multiplication by $e^{-i\omega t}$' on N (i. e. $P_N e^{-i\omega t}|_N$).

Suppose now that $\tilde{T}$ has a meromorphic continuation across the imaginary axis in Π^- (i. e. it has a finite number of poles in any finite region of Π^-, which accumulate only on $\mathbb{I}$). Now

$$\tilde{T}(i\omega) = \overline{F_c(i\omega)}\, F_o(i\omega) = \overline{F_c(i\omega) H_o(i\omega)}\, \Phi_o(i\omega) \tag{13}$$

where H_o, Φ_o are the factors of F_o according to Theorem 5. Now since T is real valued we have

$$\tilde{T}(i\omega) = \overline{\tilde{T}}(-i\omega) = \frac{F_c(-i\omega)H_o(-i\omega)}{\Phi_o(-i\omega)} \tag{14}$$

The right hand side of (14) is the boundary value of the function

$G(s)=\dfrac{F_c(-s)H_o(-s)}{\Phi_o(-s)}$, which is meromorphic in Π^-. Since F_c does not have any inner factor and Φ_o is relatively prime to the inner factor of H_o we see that G is analytic in Π^- except at points $\mu \in \Pi^-$ where $\Phi_o(-\mu)=0$. Since $\tilde{T}$ has a mermorphic continuation in Π^- we must have that $\tilde{T}(s)=G(s)$ in Π^-. Hence the singularities of $\tilde{T}$ in Π^- are the points μ with $\Phi_o(-\mu)=0$.

Since T is real valued we have $\overline{\tilde{T}}(-\bar{s}) = \tilde{T}(-s)$. Therefore $\tilde{T}$ has an analytic continuation through $i\omega$ in Π^-, if and only if it has one through $-i\omega$. From (13) $\tilde{T}(i\omega) = \overline{F_c(i\omega)H_o(i\omega)}\,\Phi_o(i\omega)$, where Φ_o and the inner factor of $F_cH_o \in H^2(\Pi^+)$ are relatively prime. Hence by a theorem in [8]p. 66, $\tilde{T}$ has an analytic continuation in Π^- through $-i\omega$, if and only if Φ_o has one through $-i\omega$. Comparing $\sigma(\tilde{T})$ as described above with the set $\mathfrak{S}_{\tilde{\Phi}}$ of Theorem 6, we see that for the canonical realization given by Theorem 7 the spectral inclusion property becomes again equality

$$\sigma(\tilde{T}) = \sigma(A)$$

This motivates the following definition.

Definition: A canonical regular (resp. balanced) realization $[A, b, c]$ of a weighting pattern T is called S-minimal regular (resp. balanced) (S from spectrum) if and only if $\sigma(\tilde{T}) = \sigma(A)$, multiplicities counted whenever possible.

We have thus proved the following.

Corollary 7.1: Any transfer function which satisfies the conditions of Theorem 7 above and is meromorphic in the left half plane, has an S-minimal regular realization with infinitesimal generator having connected resolvent set. The construction of this realization is given by Theorem 7 above.

We discuss briefly an application of these results to a particularly interesting class of weighting patterns which belong to the class we are considering here. Recall (see [1]) that if T belongs to $L_2(0,\infty)$, is locally absolutely continuous, its derivative belongs to $L_2(0,\infty)$ and $T(0)=0$, then T belongs to the class we are studying. In this case we have $F_c(i\omega)=1/(1+i\omega)$, $F_o(i\omega)=(1-i\omega)\tilde{T}(i\omega)$. Now $[\mathscr{S}F_c](z)=1$ and by Lemma 1, F_c is outer. We thus have

Corollary 7.2: Let T be a weighting pattern which belongs to $L_2(0,\infty)$, is locally absolutely continuous, its derivative belongs to $L_2(0,\infty)$ and $T(0)=0$. If $\tilde{T}$ is noncyclic and meromorphic in the left half-plane, T has an S-minimal realization with infinitesimal generator having connected resolvent set. The construction is given in Theorem 7.

Remark: The meromorphic assumption is satisfied by many systems governed by several forms of the wave equation (e.g. Schrödinger Equation, Maxwell's Equations) (see [8] Ch. VI and appendix 4, [10]).

If F_c is not outer the 'right translation realization' is neither controllable nor observable. By Th. 3 we obtain first a realization with state space $M=\Phi_c H^2(\mathbb{I})$ where Φ_c is the normalized inner factor of F_c, with $e^{-i\omega t}|_M$ as e^{At}, $P_M F_o$ as c and F_c as b. Next we reduce the latter realization to obtain a canonical one. This has as state space N, the closure of the linear span of the vectors $P_M e^{i\omega t} P_M F_o$, $t\geq 0$, in M, $P_M F_o$ as c and $P_N F_c$ as b. The semigroup is $P_N e^{-i\omega t}|_N$. Since M is invariant under $e^{-i\omega t}$, $P_M e^{i\omega t} P_M F_o = P_M e^{i\omega t} F_o$ for $t\geq 0$. Since F_o is noncyclic, the left translation invariant subspace generated by F_o is $(\Phi_o H^2(\mathbb{I}))^\perp$ where Φ_o is the assoc. inner function of F_o. The situation here looks similar to the previous case. However the existence of the projection P_M complicates the discussion of the spectral properties of this realization.

4. CYCLIC TRANSFER FUNCTIONS

Since $\tilde{T}$ is cyclic, it follows from Lemma 2 that F_o is cyclic also. This class is very interesting because it contains transfer functions with branch points, such as those usually appearing in systems governed by partial differential equations. By the spectral inclusion property all the points on branch cuts of $\tilde{T}$ are included in the spectrum of any infinitesimal generator with connected resolvent set which realizes $\tilde{T}$. Hence there is no unique "minimal" spectrum, due to the nonuniqueness of the branch cuts. When F_c is outer, the 'right translation realization' is canonical. The spectrum of the infinitesimal generator in this realization is however the whole closed left half-plane and hence generally it is far from being equal to $\sigma(\tilde{T})$. So again (see [1]) canonical by no means implies S-minimal. When F_c is not outer the 'right translation realization' is observable. Reducing this realization by Theorem 3 we obtain a canonical realization with state space $M=\Phi_c H^2(\mathbb{I})$ where Φ_c is the normalized inner factor of F_c, with $P_M e^{-i\omega t}|_M$ as e^{At}, with $P_M F_o$ as c and F_c as b. Since M is invariant under 'multiplication by $e^{-i\omega t}$' it is easy to see that the spectrum of the infinitesimal generator in this realization is the whole closed left half-plane, and so this realization is generally far from being S-minimal.

Therefore independently of F_c being outer or not, whenever $\tilde{T}$ is cyclic the 'right translation realization' does not reduce to an S-minimal realization in general. Only when $\tilde{T}$ is noncyclic, we can reduce the 'right translation realization' to obtain an S-minimal one.

It may be possible to construct S-minimal realizations for $\tilde{T}$ cyclic, by other means however (compare with [1], section 6).

We conclude this paper with the following.

Remark: We do not have a complete picture for the relations between canonical (resp. S-minimal) realizations of the same weighting pattern T, for the class considered in this paper. It is apparent however that there is no analogue of the state space isomorphism theorem of the finite dimensional theory. Counterexamples similar to the ones presented for the bounded case (see [1],[2],[3],[4]), can be constructed easily. To obtain a state space isomorphism theorem we need more assumptions. J. W. Helton [5] introduced the notions of exact controllability and observability and obtained a state space isomorphism theorem for bounded realizations. This result can be extended in a straightforward manner to regular and balanced realizations.

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