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Optimal State Estimation for Discrete-Time Markovian Jump Linear Systems, in the Presence of Delayed Output Observations

Ion Matei and John S. Baras

***Abstract*—We investigate the design of optimal state estimators for Markovian Jump Linear Systems. We consider that the output observations and the mode observations are affected by delays not necessarily identical. Our objective is to design optimal estimators for the current state, given current and past observations. We provide a solution to this paradigm by giving an optimal recursive estimator for the state, in the minimum mean square sense, and a finitely parameterized recursive scheme for computing the probability mass function of the current mode conditioned on the observed output. We also show that if the output delay is less then the one in observing the mode, then the optimal state estimation becomes nonlinear in the output observations.**

***Index Terms*—Markovian jump linear systems (MJLS), minimum mean square error (MMSE).**

I. Introduction

This technical note deals with the problem of designing optimal state estimators in the mean square sense. More specifically, we consider a plant modeled by a linear system, with parameters varying in time according to a Markovian process that takes values in a finite alphabet; this class of systems is called Markovian jump linear systems (MJLS). In the following we present the definition of a discrete-time MJLS.

Definition 1.1: (MJLS) Consider n, m, q and s to be given positive integers together with a $s \times s$ probability transition matrix $P = (p_{ij})$ (rows sum up to one). Consider the set $\mathcal{S} = \{1, \dots, s\}$ and consider a set of matrices $\{A_i\}_{i\in\mathcal{S}}$, $\{C_i\}_{i\in\mathcal{S}}$ with $A_i \in \mathbb{R}^{n\times n}$ and $C_i \in \mathbb{R}^{q\times n}$. In addition consider two independent random variables X_0 and M_0 taking values in $\mathbb{R}^n$ and $\mathcal{S}$, respectively. Given the vector valued random processes W_t and V_t taking values in $\mathbb{R}^n$ and $\mathbb{R}^q$ respectively, the following stochastic dynamic equations describe a (noise-driven) discrete-time MJLS:

$$X_{t+1} = A_{M_t} X_t + W_t \tag{1}$$
$$Y_t = C_{M_t} X_t + V_t. \tag{2}$$

The process M_t is a homogeneous Markovian jump process taking values in $\mathcal{S}$, with conditional probabilities given by $pr(M_{t+1} = j | M_t = i) = p_{ij}$. Throughout this technical note, we will consider W_t and V_t to be independent identically distributed (i.i.d.) Gaussian noises with zero mean and covariance matrices Σ_W and Σ_V, respectively. The initial condition vector X_0 has a Gaussian distribution with mean μ_{X_0} and covariance matrix Σ_{X_0}. The Markovian process $\{M_t\}_{t=0}^\infty$, X_0 and the noises $\{W_t\}_{t=0}^\infty$, $\{V_t\}_{t=0}^\infty$, are assumed independent.

Notice that the MJLS defined by (1), (2) has a *hybrid state* composed of X_t, the continuous component, and of M_t, the discrete part of the state. Making the assumption that the state of the Markovian process M_t can be directly observed, the system has a hybrid output, with Y_t representing the continuous component and M_t representing the discrete part of the output.

For simplicity, we will adopt an abuse of notation by referring to X_t as *state vector* and to M_t as *mode* (since it determines the mode of operation, by selecting the matrices (A_{M_t}, C_{M_t}) in (1), (2)). With respect to the hybrid output observations, we will call Y_t output observation and M_t mode observation.

A. Motivation and Survey of Related Results

This section introduces the motivation for the problem addressed in this technical note and a short survey of the state of the art in the MJLS theory in general and in the design of optimal state estimators for a MJLS in particular.

MJLSs represent an important class of stochastic time-variant systems because they can be used to model random abrupt changes in structure. Linear plants with random time-delays [19] or more general networked control applications [14], where communication networks/channels are used to interconnect remote sensors, actuators and processors, have been shown to be amenable to MJLS modeling. Motivated by a wide spectrum of applications, there has been active research in the analysis [6], [7], [9], and in the design of controllers and estimators [8], [9], [11], [12] for MJLSs.

In this technical note, we address the problem of optimal state estimation for discrete-time MJLS with Gaussian noise and arbitrary delays on the observation output components. Relevant results concerning the filtering problem for linear systems with time-delays can be found in [5], [15], [20]. Existing results solve the problem of state estimation for MJLS in the case of Gaussian noise for two main cases. In the first case, both the output and mode can be observed and the minimum mean square error (MMSE) estimator is derived from the Kalman filter for time varying systems [9], [12]. Off-line computation of the filter is inadvisable due to the dependence of the filter's gain on the mode path. An alternative estimator (filter), whose gain depends

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only on the current mode and for which off-line computations are feasible, is given in [10]. In the second case, only the output is observed, without any observation of the mode. The optimal nonlinear filter consists of a bank of Kalman filters, whose memory requirements and computation complexity increase exponentially with time [3]. To limit the computational requirements sub-optimal estimators have been proposed in the literature [1], [2], [4], [13]. A linear MMSE estimator, for which the gain matrices can be calculated off-line, is described in [11]. In this note we close the gap between the two cases mentioned above. We have partially addressed the problem described in this note in [18], where only the case with delays in the mode observations was considered. Also, in the current note we have changed and improved the proofs introduced in [18].

The framework proposed in this work is useful in addressing state estimation problems for systems affected by failures, which are detectable only after a delay. For instance, we can consider a set of sensors that measure the output of a plant. These sensors can fail, the number of fully functional sensors inducing a mode of operation. If we make the assumption that the time while the sensor is fully functional and the time it takes for a sensor to be repaired are exponentially distributed, then the time evolution of the mode of operation can be modeled by a Markov chain. However, the failure of a sensor is not necessarily immediately detected, but typically is detected after a few time units, depending on the fault detection techniques used. Hence the mode of operation is not known instantly but with some delay, which is one of the application domains that motivate the interest in pursuing the problem addressed here. Detection of sensor faults can be done for instance using techniques introduced in [16], [17].

Notations and Abbreviations: Consider a general random process Z_t. We denote by Z_0^t the history of the process from time 0 up to time time t, i.e. $Z_0^t = \{Z_0, Z_1, \ldots, Z_t\}$. A realization (sample path) of Z_0^t is denoted by $z_0^t = \{z_0, z_1, \ldots, z_t\}$. Let $\{X_t|Y_0^{t-h_1} = y_0^{t-h_1}, M_0^{t-h_2} = m_0^{t-h_2}\}$ be the vector valued random process representing the continuous part of the system state given the past history of the observations. We denote by $f_{X_t|Y_0^{t-h_1}M_0^{t-h_2}}$ its probability density function (p.d.f.) while $\mu^X_{t|(t-h_1,t-h_2)}$ and $\Sigma^X_{t|(t-h_1,t-h_2)}$ signify its mean and covariance matrix, respectively. We will compactly write the sum $\sum_{m_0=1}^{s}\sum_{m_1=1}^{s}\cdots\sum_{m_t=1}^{s}$ as $\sum_{m_0^t}$. Assuming that x is a vector in $\mathbb{R}^n$, by the integral $\int f(x)dx$ we understand $\int\ldots\int f(x_1,\ldots,x_n)dx_1\ldots dx_n$, for some function f defined on $\mathbb{R}^n$ with values in $\mathbb{R}$.

Paper Organization: This technical note has four more sections besides the introduction. After the formulation of the problem in Section II, in Section III we introduce the main results. Two corollaries will present the formulas for the optimal state estimator (discrete and continuous components) in the mean square sense. In Section IV we provide the proofs of these corollaries together with a number of supporting results. The technical note ends with conclusions in Section V.

II. Problem Formulation

In this Section, we formulate the problem for the MMSE state estimation for MJLS in the case of delayed output and mode observations.

Problem 2.1: (*MMSE state estimation for MJLS with delayed output and mode observations*) Consider a MJLS as in *Definition 1.1*. Let h_1 and h_2 be two positive integers representing the delays affecting the output and mode observations, respectively. Assuming that the state vector X_t and the mode M_t are not known, and that at the current time the data available consist of the output observations up to time $t-h_1$ and mode observations up to time $t-h_2$, i.e. the estimator has access to $Y_0^{t-h_1}$ and $M_0^{t-h_2}$, we want to derive the MMSE estimators for the state vector X_t and for the mode indicator function $\mathbb{1}_{\{M_t=i\}}$, $i\in\mathcal{S}$. More precisely, as is well known [21], we want to compute the following:

MMSE state estimator:

$$\hat{X}_t^{h_1,h_2} \stackrel{def}{=} E\left[X_t|Y_0^{t-h_1} = y_0^{t-h_1}, M_0^{t-h_2} = m_0^{t-h_2}\right]. \quad (3)$$

MMSE mode indicator function estimator:

$$\hat{\mathbb{1}}^{h_1,h_2}_{\{M_t=m_t\}} \stackrel{def}{=} E\left[\mathbb{1}_{\{M_t=m_t\}}|Y_0^{t-h_1} = y_0^{t-h_1}, M_0^{t-h_2} = m_0^{t-h_2}\right] \quad (4)$$

where the indicator function $\mathbb{1}_{\{M_t=m_t\}}$ is defined by

$$\mathbb{1}_{\{M_t=m_t\}} \stackrel{def}{=} \begin{cases} 1 & M_t = m_t \\ 0 & M_t \neq m_t. \end{cases}$$

Remark 2.1: We are interested in estimating the indicator function rather than the mode itself, because the MMSE estimator of the mode can produce real values, which may have limited usefulness. Also, obtaining an MMSE estimator of the mode indicator function, allows us to compute the estimator of any function of the mode. Indeed the following holds for any real function g defined on the set $\mathcal{S}$

$$\begin{aligned} \widehat{g(M_t)} &= E\left[g(M_t)|Y_0^{t-h_1} = y_0^{t-h_1}, M_0^{t-h_2} = m_0^{t-h_2}\right] \\ &= \sum_{m_t\in\mathcal{S}} g(m_t)\hat{\mathbb{1}}^{h_1,h_2}_{\{M_t=m_t\}} \end{aligned}$$

where by $\widehat{g(M_t)}$ we denote the MMSE estimator of the function $g(M_t)$.

Remark 2.2: Considering the definition of the indicator function, the MMSE mode indicator function estimator can be also written as: $\hat{\mathbb{1}}^{h_1,h_2}_{\{M_t=m_t\}} = pr(M_t = m_t|Y_0^{t-h_1} = y_0^{t-h_1}, M_0^{t-h_2} = m_0^{t-h_2})$. Then we can also produce a marginal maximal a posteriori mode estimator expressed in terms of the indicator function: $\tilde{M}_t^{h_1,h_2} = \arg\max_{m_t\in\mathcal{S}} pr(M_t = m_t|Y_0^{t-h_1} = y_0^{t-h_1}, M_0^{t-h_2} = m_0^{t-h_2}) = \arg\max_{m_t\in\mathcal{S}} \hat{\mathbb{1}}^{h_1,h_2}_{\{M_t=m_t\}}$.

III. Main Result

In this Section, we present the solution for *Problem 2.1*. We introduce here two Corollaries describing recursive formulas for computing the optimal estimators for the state and mode indicator function. The proofs of these Corollaries are deferred for the next section and they are a direct consequence of *Theorem 4.1* stated and proved in Section IV.

A. Standard Case

We begin by recapitulating some properties of the Kalman filter for MJLS in the standard case (i.e. the output and mode observations are available with no delay), summarized in the following Theorem.

Theorem 3.1: Consider a discrete MJLS as in *Definition 1.1*. The random processes $\{X_t|Y_0^t = y_0^t, M_0^t = m_0^t\}$, $\{X_t|Y_0^{t-1} = y_0^{t-1}, M_0^{t-1} = m_0^{t-1}\}$ and $\{Y_t|Y_0^{t-1} = y_0^{t-1}, M_0^t = m_0^t\}$ are Gaussian processes with the means and covariance matrices calculated by the following recursive equations:

$$\begin{aligned} \mu^X_{t|(t,t)} &= \mu^X_{t|(t-1,t-1)} + L_t\left(y_t - C_{m_t}\mu^X_{t|(t-1,t-1)}\right) \\ L_t &= \Sigma^X_{t|(t-1,t-1)}C^T_{m_t} \\ &\quad\times\left(C_{m_t}\Sigma^X_{t|(t-1,t-1)}C^T_{m_t} + \Sigma_V\right)^{-1} \\ \Sigma^X_{t|(t-1,t-1)} &= A_{m_{t-1}}\Sigma^X_{t-1|(t-1,t-1)}A^T_{m_{t-1}} + \Sigma_W \quad (5) \\ \mu^X_{t|(t-1,t-1)} &= A_{m_{t-1}}\mu^X_{t-1|(t-1,t-1)} \\ \Sigma^X_{t|(t,t)} &= \left(I - L_tC_{m_t}\right)\Sigma^X_{t|(t-1,t-1)} \\ \mu^Y_{t|(t-1,t)} &= C_{m_t}\mu^X_{t|(t-1,t-1)} \quad (6) \\ \Sigma^Y_{t|(t-1,t)} &= C_{m_t}\Sigma^X_{t|(t-1,t-1)}C^T_{m_t} + \Sigma_V \quad (7) \end{aligned}$$

with initial conditions $\mu^X_{0|(-1,-1)} = \mu_{X_0}$ and $\Sigma^X_{0|(-1,-1)} = \Sigma_{X_0}$.

Proof: Equations (5) represent the standard Kalman filter for the MJLS which is derived using Kalman filtering equations for time varying systems [9], [12], since the output and mode are known with no delay. Besides the Kalman filter equations, we added (6), (7) as they will be used throughout the technical note. These equations are an immediate consequence of the fact that the conditional random process $\{X_t|Y_0^{t-1}=y_0^{t-1}, M_0^{t-1}=m_0^{t-1}\}$ is Gaussian. ■

B. Delayed Output and Mode Observation Case

Our main result consists of *Corollaries 3.1 and 3.2*, which show the algorithmic steps necessary to compute the MMSE estimators for the state and mode indicator function for MJLS when the observations of the output and mode are affected by some arbitrary (but fixed) delays. These Corollaries are stated without proof. The proof of *Corollary 3.1* is given in Section IV-A and of *Corollary 3.2* is given in Section IV-B.

Corollary 3.1: Given a MJLS as in *Definition 2.1* and two positive integers h_1 and h_2, the MMSE state estimator from *Problem 2.1* is given by the following formulas:

(a) If $h_1 \geq h_2, h_2 > 1$

$$\hat{X}_t^{h_1,h_2} = \sum_{m_{t-h_2+1}^{t-1}} \prod_{k=1}^{h_2-1} p_{m_{t-k-1},m_{t-k}} \mu_{t|(t-h_1,t-1)}^X.$$

(b) If $(h_1 \geq h_2, h_{2=1})$ *or* $(h_1 > h_2, h_2 = 0)$

$$\hat{X}_t^{h_1,h_2} = \mu_{t|(t-h_1,t-1)}^X.$$

(c) If $h_1 < h_2, h_1 > 1$

$$\hat{X}_t^{h_1,h_2} = \sum_{m_{t-h_2+1}^{t-1}} \prod_{k=1}^{h_1-1} p_{m_{t-k-1},m_{t-k}} c_t\left(m_{t-h_2+1}^{t-h_1}\right) \mu_{t|(t-h_1,t-1)}^X.$$

(d) If $h_1 < h_2, h_1 \in \{0,1\}$

$$\hat{X}_t^{h_1,h_2} = \sum_{m_{t-h_2+1}^{t-h_1}} c_t\left(m_{t-h_2+1}^{t-h_1}\right) \mu_{t|(t-h_1,t-h_1)}^X$$

where $\mu_{t|(t-h_1,t-1)}^X$ (and $\mu_{t|(t-1,t-1)}^X$) is computed by the recurrence

$$\mu_{t|(t-h_1,t-1)}^X = \left(\prod_{k=1}^{h_1} A_{m_{t-k}}\right) \mu_{t-h_1|(t-h_1,t-h_1)}^X \tag{8}$$

for each of the unknown mode paths represented by a term in the above sums. The conditional means $\mu_{t-h_1|(t-h_1,t-h_1)}^X$ (or $\mu_{t|(t,t)}^X$) are calculated according to the Kalman filter (5), and the coefficients $c_t(m_{t-h_2+1}^{t-h_1})$ are given by

$$c_t\left(m_{t-h_2+1}^{t-h_1}\right) = \frac{\mathcal{N}}{\mathcal{D}} \tag{9}$$

with

$$\begin{aligned}\mathcal{N} = &\prod_{k=0}^{h_2-h_1-1} p_{m_{t-h_1-k-1},m_{t-h_1-k}} \\ &\times f_{Y_{t-h_1-k}|Y_0^{t-h_1-k-1},M_0^{t-h_1-k}} \\ &\left(y_{t-h_1-k}|y_0^{t-h_1-k-1}, m_0^{t-h_1-k}\right), \\ \mathcal{D} = &\sum_{m_{t-h_2+1}^{t-h_1}} \prod_{k=0}^{h_2-h_1-1} p_{m_{t-h_1-k-1},m_{t-h_1-k}} \\ &\times f_{Y_{t-h_1-k}|Y_0^{t-h_1-k-1},M_0^{t-h_1-k}} \\ &\left(y_{t-h_1-k}|y_0^{t-h_1-k-1}, m_0^{t-h_1-k}\right)\end{aligned}$$

where $f_{Y_{t-h_1-k}|Y_0^{t-h_1-k-1},M_0^{t-h_1-k}}(y_{t-h_1-k}|y_0^{t-h_1-k-1}, m_0^{t-h_1-k})$ is the Gaussian p.d.f. of the random process $\{Y_{t-h_1-k}|Y_0^{t-h_1-k-1}, M_0^{t-h_1-k}\}$, whose mean and covariance matrix are computed according to (6), (7) introduced in *Theorem 3.1*.

Corollary 3.2: Given a MJLS, as in *Definition 2.1*, and two positive integers h_1 and h_2, the MMSE mode indicator estimator from *Problem 2.1* is computed according to the following formulas:

(a) If $0 < h_2 \leq h_1$

$$\hat{\mathbb{1}}_{\{M_t=m_t\}}^{h_1,h_2} = \prod_{k=1}^{h_2} p_{m_{t-k},m_{t-k+1}}.$$

(b) If $0 < h_1 < h_2$

$$\hat{\mathbb{1}}_{\{M_t=m_t\}}^{h_1,h_2} = \sum_{m_{t-h_2+1}^{t-1}} \prod_{k=1}^{h_1} p_{m_{t-k},m_{t-k+1}} c_t\left(m_{t-h_2+1}^{t-h_1}\right).$$

(c) If $0 = h_1 < h_2$

$$\hat{\mathbb{1}}_{\{M_t=m_t\}}^{h_1,h_2} = \sum_{m_{t-h_2+1}^{t-1}} c_t\left(m_{t-h_2+1}^{t}\right).$$

where $c_t(m_{t-h_2+1}^{t-h_1})$ are computed according to (9).

Since the delays are fixed, the algorithms have time-independent computational complexity. We would like to note that the form of the optimal estimators will depend on whether $h_1 > h_2$ or $h_2 > h_1$. In the first case (i.e., $h_1 > h_2$), the optimal state estimator is linear with respect to the output observation Y_t, while the optimal estimation of the mode indicator function is independent of Y_t. In the second case (i.e. $h_2 > h_1$), the optimal state and mode indicator function estimators become nonlinear in the output observations due to the coefficients $c_t(m_{t-h_2+1}^{t-h_1})$. These coefficients reflect the fact that the modes from time $t-h_2+1$ up to t are indirectly observed through Y_t. In *Corollaries 3.1* and *3.2* we were not concerned by the numerical efficiency of the algorithms. However, we note that at the current time, the algorithm uses information computed at previous steps indicating that an economy in memory space and computation power can be achieved. *Corollaries 3.1* and *3.2* are consequences of a set of results (mainly *Lemmas 4.1* and *4.2* and *Theorem 4.1*) which will be detailed in the next section.

IV. Proof of the Main Result

In this Section, we introduce two Lemmas and a Theorem which will aid in the proof the main results presented in Section III. In particular, *Corollaries 3.1* and *3.2* are a direct consequence of *Theorem 4.1*, which is the main result of this section. This Theorem characterizes the p.d.f. of the conditional random process $\{X_t|Y_0^{t-h_1}=y_0^{t-h_1}, M_0^{t-h_2}=m_0^{t-h_2}\}$ where h_1 and h_2 are some known arbitrary positive integer values and is proved with the help of *Lemma 4.1* and *Lemma 4.2*. In *Lemma 4.1* we characterize the statistical properties of the conditional random process $\{X_t|Y_0^{t-h}=y_0^{t-h}, M_0^{t-1}=m_0^{t-1}\}$ where h is a positive integer. This result is related to the case of state estimation when the output observations are delayed but the modes are all known. In *Lemma 4.2* we analyze the statistical properties of the conditional random process $\{X_t|Y_0^t=y_0^t, M_0^{t-h}=m_0^{t-h}\}$. From *Lemma 4.2* we derive the MMSE state and mode indicator function estimators when all the output observations are known but the mode observations are delayed.

To simplify the exposition of *Lemmas 4.1* and *4.2* we introduce (without proof) the following Corollary presenting well known properties of the p.d.f of a linear combination of independent Gaussian random vectors.

Corollary 4.1: Consider two independent Gaussian random vectors V and X of dimension m and n respectively, with means $\mu_V = 0$ and μ_X, and covariance matrices Σ_V and Σ_X respectively. Let Y be a Gaussian random vector corresponding to a linear combination of X and V, $Y = CX + V$ where C is a matrix of appropriate dimensions. The following holds:

$$\int_{\mathbb{R}^n} f_V(y - Cx) f_X(x) dx = f_Y(y) \tag{10}$$

where $f_Y(y)$ is the multivariate Gaussian p.d.f. of Y with parameters $\mu_Y = C\mu_X$ and $\Sigma_Y = C\Sigma_X C^T + \Sigma_V$.

Lemma 4.1: Consider a discrete MJLS as in *Definition 1.1*. Let h be a known strictly positive integer value. Then the p.d.f. of the conditional random process $\{X_t | Y_0^{t-h} = y_0^{t-h}, M_0^{t-1} = m_0^{t-1}\}$ is Gaussian with mean computed by

$$\mu_{t|t-h,t-1} = \left(\prod_{k=1}^{h} A_{m_{t-k}}\right) \mu^X_{t-h|(t-h,t-h)} \tag{11}$$

and covariance matrix given by the recurrence

$$\Sigma^X_{t-k|(t-h,t-k)} = A_{m_{t-k-1}} \Sigma^X_{t-k-1|(t-h,t-k-1)} A^T_{m_{t-k-1}} + \Sigma_W \tag{12}$$

for $k \in \{h-1, h-2, \ldots, 1, 0\}$ and with initial covariance matrix $\Sigma^X_{t-h|t-h,t-h}$. In addition $\mu^X_{t-h|(t-h,t-h)}$ and $\Sigma^X_{t-h|t-h,t-h}$ are calculated according to the Kalman filter described in (5).

Proof: The Gaussianity is shown by induction. Assume that for a k from $\{0, 1, \ldots, h-1\}$, $f_{X_{t-k-1}|Y_0^{t-h}, M_0^{t-k-1}}$ is a Gaussian p.d.f. Then $f_{X_{t-k}|Y_0^{t-h}, M_0^{t-k}}$ can be expressed as

$$\begin{aligned}
&f_{X_{t-k}|Y_0^{t-h}, M_0^{t-k}}\left(x_{t-k}|y_0^{t-h} m_0^{t-k}\right)\\
&\quad= \int_{\mathbb{R}^n} f_{X_{t-k}, X_{t-k-1}|Y_0^{t-h}, M_0^{t-k}}\\
&\qquad\left(x_{t-k}, x_{t-k-1}|y_0^{t-h}, m_0^{t-k}\right) dx_{t-k-1}\\
&\quad= \int_{\mathbb{R}^n} f_{X_{t-k}|X_{t-k-1}, M_{t-k-1}}(x_{t-k}|x_{t-k-1}, m_{t-k-1})\\
&\qquad\times f_{X_{t-k-1}|Y_0^{t-h}, M_0^{t-k-1}}\\
&\qquad\left(x_{t-k-1}|y_0^{t-h}, m_0^{t-k-1}\right) dx_{t-k-1}.
\end{aligned}$$

Using (10) from *Corollary 4.1*, we conclude that $f_{X_{t-k}|Y_0^{t-h}, M_0^{t-k}}$ is a Gaussian p.d.f. with mean given by

$$\mu^X_{t-k|(t-h,t-k)} = A_{m_{t-k-1}} \mu^X_{t-k-1|(t-h,t-k-1)}$$

and covariance matrix

$$\Sigma^X_{t-k|(t-h,t-k)} = A_{m_{t-k-1}} \Sigma^X_{t-k-1|(t-h,t-k-1)} A^T_{m_{t-k-1}} + \Sigma_W.$$

Iterating over $k \in \{h-1, h-2, \ldots, 1, 0\}$, we obtain (11) and (12).■

Lemma 4.2: Consider a discrete MJLS as in *Definition 1.1* and let h be a known positive integer value. Then the p.d.f. of the conditional random process $\{X_t | Y_0^t = y_0^t, M_0^{t-h} = m_0^{t-h}\}$ is a mixture of Gaussian probability densities. More precisely

$$\begin{aligned}
&f_{X_t|Y_0^t, M_0^{t-h}}\left(x|y_0^t, m_0^{t-h}\right)\\
&\quad= \sum_{m_{t-h+1}^t} c_t\left(m_{t-h+1}^t\right) f_{X_t|Y_0^t, M_0^t}\left(x|y_0^t, m_0^t\right)
\end{aligned} \tag{13}$$

where $c_t(m_{t-h+1}^t) = f_{M_{t-h+1}^t|Y_0^t, M_0^{t-h}}(m_{t-h+1}^t|y_0^t, m_0^{t-h})$ are the mixture coefficients and $f_{X_t|Y_0^t, M_0^t}(x|y_0^t, m_0^t)$ is the Gaussian p.d.f. of the process $\{X_t|Y_0^t = y_0^t, M_0^t = m_0^t\}$, whose statistics are computed according to the recursions (5). The coefficients $c_t(m_{t-h+1}^t)$ are computed by (14), shown at the bottom of the page, where $f_{Y_{t-k}|Y_0^{t-k-1}, M_0^{t-k}}$ is the Gaussian p.d.f. of $\{Y_{t-k}|Y_0^{t-k-1} = y_0^{t-k-1}, M_0^{t-k} = m_0^{t-k}\}$ whose statistics are computed using (6) and (7).

Proof: Using the law of marginal probabilities we get

$$\begin{aligned}
&f_{X_t|Y_0^t, M_0^{t-h}}\left(x|y_0^t, m_0^{t-h}\right)\\
&\quad= \sum_{m_{t-h+1}^t} f_{X_t, M_{t-h+1}^t|Y_0^t, M_0^{t-h}}\left(x, m_{t-h+1}^t|y_0^t, m_0^{t-h}\right)\\
&\quad= \sum_{m_{t-h+1}^t} f_{X_t|Y_0^t, M_0^t}\left(x|y_0^t, m_0^t\right) f_{M_{t-h+1}^t|Y_0^t, M_0^{t-h}}\\
&\qquad\left(m_{t-h+1}^t|y_0^t, m_0^{t-h}\right)\\
&\quad= \sum_{m_{t-h+1}^t} c_t\left(m_{t-h+1}^t\right) f_{X_t|Y_0^t, M_0^t}\left(x|y_0^t, m_0^t\right).
\end{aligned}$$

Thus we obtained (13). We are left to compute the coefficients of this linear combination. Applying Bayes' rule we get

$$\begin{aligned}
&f_{M_{t-h+1}^t|Y_0^t, M_0^{t-h}}\left(m_{t-h+1}^t|y_0^t, m_0^{t-h}\right)\\
&\quad= \frac{f_{Y_0^t, M_0^t}\left(y_0^t, m_0^t\right)}{\sum_{m_{t-h+1}^t} f_{Y_0^t, M_0^t}\left(y_0^t, m_0^t\right)}.
\end{aligned} \tag{15}$$

The p.d.f. $f_{Y_0^t, M_0^t}$ can be expressed recursively as

$$\begin{aligned}
f_{Y_0^t M_0^t}\left(y_0^t, m_0^t\right) &= \int_{\mathbb{R}^n} f_{X_t, Y_0^t, M_0^t}\left(x_t, y_0^t, m_0^t\right) dx_t\\
&= p_{m_{t-1}, m_t} f_{Y_0^{t-1}, M_0^{t-1}}\left(y_0^{t-1}, m_0^{t-1}\right)\\
&\quad\times \int_{\mathbb{R}^n} f_{Y_t|X_t, M_t}(y_t|x_t, m_t) f_{X_t|Y_0^{t-1}, M_0^{t-1}}\\
&\qquad\left(x_t|y_0^{t-1}, m_0^{t-1}\right) dx_t.
\end{aligned}$$

Applying (10) we obtain

$$\begin{aligned}
&f_{Y_0^t, M_0^t}\left(y_0^t, m_0^t\right) = f_{Y_t|Y_0^{t-1}, M_0^t}\\
&\quad\left(y_t|y_0^{t-1}, m_0^t\right) p_{m_{t-1}, m_t} f_{Y_0^{t-1}, M_0^{t-1}}\left(y_0^{t-1}, m_0^{t-1}\right).
\end{aligned}$$

$$c_t\left(m_{t-h+1}^t\right) = \frac{\prod_{k=0}^{h-1} p_{m_{t-k-1}, m_{t-k}} f_{Y_{t-k}|Y_0^{t-k-1}, M_0^{t-k}}\left(y_{t-k}|y_0^{t-k-1}, m_0^{t-k}\right)}{\sum_{m_{t-h+1}^t} \prod_{k=0}^{h-1} p_{m_{t-k-1}, m_{t-k}} f_{Y_{t-k}|Y_0^{t-k-1}, M_0^{t-k}}\left(y_{t-k}|y_0^{t-k-1}, m_0^{t-k}\right)} \tag{14}$$

Using this recursive expression we get

$$f_{Y_0^t,M_0^t}\left(y_0^t,m_0^t\right)=\prod_{k=0}^{h-1}p_{m_{t-k-1},m_{t-k}}f_{Y_{t-k}|Y_0^{t-k-1},M_0^{t-k}}\left(y_{t-k}|y_0^{t-k-1},m_0^{t-k}\right)f_{Y_0^{t-h},M_0^{t-h}}\left(y_0^{t-h},m_0^{t-h}\right).$$

By substituting this last expression in (15), we obtain the coefficients $c_t(m_{t-h+1}^t)$ given in (14). We can conclude de proof by making the observation that the p.d.f. $f_{Y_{t-k}|Y_0^{t-k-1},M_0^{t-k}}$ is completely characterized in *Theorem 3.1*, (6) and (7). ■

The following Theorem is the main contribution of this Section.

Theorem 4.1: Consider a discrete MJLS as in *Definition 1.1* and let h_1 and h_2 be two non-negative integers. Then the p.d.f. of the random process $\{X_t|Y_0^{t-h_1}=y_0^{t-h_1},M_0^{t-h_2}=m_0^{t-h_2}\}$ is given by the following formulas

(a) If $h_1\geq h_2, h_2>1$

$$f_{X_t|Y_0^{t-h_1},M_0^{t-h_2}}\left(x_t|y_0^{t-h_1},m_0^{t-h_2}\right)=\sum_{m_{t-h_2+1}^{t-1}}\prod_{k=1}^{h_2-1}p_{m_{t-k-1},m_{t-k}}f_{X_t|Y_0^{t-h_1},M_0^{t-1}}\left(x_t|y_0^{t-h_1},m_0^{t-1}\right).$$

(b) If $(h_1\geq h_2, h_2=1)$ *or* $(h_1>h_2,h_2=0)$

$$f_{X_t|Y_0^{t-h_1},M_0^{t-h_2}}\left(x_t|y_0^{t-h_1},m_0^{t-h_2}\right)=f_{X_t|Y_0^{t-h_1},M_0^{t-1}}\left(x_t|y_0^{t-h_1},m_0^{t-1}\right).$$

(c) If $h_1<h_2, h_1>1$

$$f_{X_t|Y_0^{t-h_1},M_0^{t-h_2}}\left(x_t|y_0^{t-h_1},m_0^{t-h_2}\right)=\sum_{m_{t-h_2+1}^{t}}\prod_{k=1}^{h_1-1}p_{m_{t-k-1},m_{t-k}}c_t\left(m_{t-h_2+1}^{t-h_1}\right)\times f_{X_t|Y_0^{t-h_1},M_0^{t-1}}\left(x_t|y_0^{t-h_1},m_0^{t-1}\right).$$

(d) If $h_1<h_2, h_1\in\{0,1\}$

$$f_{X_t|Y_0^{t-h_1},M_0^{t-h_2}}\left(x_t|y_0^{t-h_1},m_0^{t-h_2}\right)=\sum_{m_{t-h_2+1}^{t-h_1}}c_t\left(m_{t-h_2+1}^{t-h_1}\right)f_{X_t|Y_0^{t-h_1},M_0^{t-h_1}}\left(x_t|y_0^{t-h_1},m_0^{t-h_1}\right)$$

where the p.d.f. $f_{X_t|Y_0^{t-h_1},M_0^{t-1}}$ is characterized in *Lemma 4.1* and the formulas for coefficients $c_t(m_{t-h_2+1}^{t-h_1})$ are given in (9).

Proof: Note that the most general cases are cases (a) and (c). The rest are particular cases of the aforementioned ones.

Proof of case (a) ($h_1\geq h_2$, $h_2>1$): By conditioning on the missing mode observation $M_{t-h_2+1}^{t-1}$ we can write

$$\begin{aligned}f_{X_t|Y_0^{t-h_1},M_0^{t-h_2}}&\left(x_t|y_0^{t-h_1},m_0^{t-h_2}\right)\\&=\sum_{m_{t-h_2+1}^{t-1}}f_{X_t,M_{t-h_2+1}^{t-1}|Y_0^{t-h_1},M_0^{t-h_2}}\left(x_tm_{t-h_2+1}^{t-1}|y_0^{t-h_1},m_0^{t-h_2}\right)\\&=\sum_{m_{t-h_2+1}^{t-1}}f_{X_t|Y_0^{t-h_1},M_0^{t-1}}\left(x_t|y_0^{t-h_1},m_0^{t-1}\right)f_{M_{t-h_2+1}^{t-1}|Y_0^{t-h_1},M_0^{t-h_2}}\left(m_{t-h_2+1}^{t-1}|y_0^{t-h_1},m_0^{t-h_2}\right).\end{aligned}$$

Observing that $\{M_{t-h_2+1}^{t-1}|Y_0^{t-h_1}, M_0^{t-h_2}\}=\{M_{t-h_2+1}^{t-1}|M_0^{t-h_2}\}$ and that

$$f_{M_{t-h_2+1}^{t-1}|Y_0^{t-h_1},M_0^{t-h_2}}\left(m_{t-h_2+1}^{t-1}|y_0^{t-h_1},m_0^{t-h_2}\right)=\prod_{k=1}^{h_2-1}p_{m_{t-k-1},m_{t-k}}$$

we conclude the proof of this case.

Proof of case (b) ($(h_1\geq h_2$, $h_2=1)$ *or* $(h_1>h_2$, $h_2=0)$): When $h_1\geq h_2$, $h_2=1$, the formula follows trivially. If $h_1>h_2$, $h_2=0$, the expression for $f_{X_t|Y_0^{t-h_1},M_0^{t-h_2}}(x_t|y_0^{t-h_1},m_0^{t-h_2})$ is obtained by noticing that $\{X_t|Y_0^{t-h_1},M_0^t\}=\{X_t|Y_0^{t-h_1},M_0^{t-1}\}$ since X_t does not depend on M_t.

Proof of case (c) ($h_1<h_2$, $h_1>1$): We start as in the case (a), by conditioning on $M_{t-h_2+1}^{t-1}$

$$\begin{aligned}f_{X_t|Y_0^{t-h_1},M_0^{t-h_2}}&\left(x_t|y_0^{t-h_1},m_0^{t-h_2}\right)\\&=\sum_{m_{t-h_2+1}^{t-1}}f_{X_t,M_{t-h_2+1}^{t-1}|Y_0^{t-h_1},M_0^{t-h_2}}\left(x_t,m_{t-h_2+1}^{t-1}|y_0^{t-h_1},m_0^{t-h_2}\right)\\&=\sum_{m_{t-h_2+1}^{t-1}}\underbrace{f_{X_t|Y_0^{t-h_1},M_0^{t-1}}\left(x_t|y_0^{t-h_1},m_0^{t-1}\right)}_{A}\\&\quad\times\underbrace{f_{M_{t-h_2+1}^{t-1}|Y_0^{t-h_1},M_0^{t-h_2}}\left(m_{t-h_2+1}^{t-1}|y_0^{t-h_1},m_0^{t-h_2}\right)}_{B}.\end{aligned}\tag{16}$$

In (16), the p.d.f. labeled by A is obtained from case (b) and for the p.d.f labeled by B we can further write

$$\begin{aligned}f_{M_{t-h_2+1}^{t-1}|Y_0^{t-h_1},M_0^{t-h_2}}&\left(m_{t-h_2+1}^{t-1}|y_0^{t-h_1},m_0^{t-h_2}\right)\\&=f_{M_{t-h_1+1}^{t-1}|Y_0^{t-h_1},M_0^{t-h_1}}\left(m_{t-h_1+1}^{t-1}|y_0^{t-h_1},m_0^{t-h_1}\right)\\&\quad\times f_{M_{t-h_2+1}^{t-h_1}|Y_0^{t-h_1},M_0^{t-h_2}}\left(m_{t-h_2+1}^{t-h_1}|y_0^{t-h_1},m_0^{t-h_2}\right).\end{aligned}$$

From the Markovian property of the process M_t the first term is

$$\begin{aligned}f_{M_{t-h_1+1}^{t-1}|Y_0^{t-h_1},M_0^{t-h_1}}&\left(m_{t-h_1+1}^{t-1}|y_0^{t-h_1},m_0^{t-h_1}\right)\\&=f_{M_{t-h_1+1}^{t-1}|M_0^{t-h_1}}\left(m_{t-h_1+1}^{t-1}|m_0^{t-h_1}\right)\\&=\prod_{k=1}^{h_1-1}p_{m_{t-k-1},m_{t-k}}.\end{aligned}$$

We can notice that $f_{M_{t-h_2+1}^{t-h_1}|Y_0^{t-h_1},M_0^{t-h_2}}$ is a shifted (by h_1) version of the coefficients c_t introduced in the *Lemma 4.2*. Thus

$$f_{M_{t-h_2+1}^{t-h_1}|Y_0^{t-h_1},M_0^{t-h_2}}\left(m_{t-h_2+1}^{t-h_1}|y_0^{t-h_1},m_0^{t-h_2}\right)=c_t\left(m_{t-h_2+1}^{t-h_1}\right)$$

where $c_t(m_{t-h_2+1}^{t-h_1})$ are given by (9). Thus we conclude the proof for the third case.

Proof of case (d) ($h_1<h_2$, $h_1\in\{0,1\}$): If $h_1<h_2$, $h_1=0$ we satisfy the conditions of *Lemma 4.2*. If $h_1<h_2$, $h_1=1$ we follow the same lines as in the case (c) with the difference that since $h_1=1$, there will be no products of the conditional probabilities p_{ij} multiplying the terms in the sum.

■

A. *Proof of Corollary 3.1*

The proof follows from the linearity of the expectation operator and by applying the results about the p.d.f. $f_{X_t|Y_0^{t-h_1},M_0^{t-h_2}}$ detailed in *Theorem 4.1* together with the properties of $f_{X_t|Y_0^{t-h},M_0^{t-1}}$ and $f_{X_t|Y_0^{t},M_0^{t-h}}$ shown in *Lemmas 4.1 and 4.2*.

B. *Proof of Corollary 3.2*

From the optimal estimator formula we have

$$\begin{aligned}\hat{\mathbb{1}}^{h_1,h_2}_{\{M_t=m_t\}} &= E\left[\mathbb{1}_{\{M_t=m_t\}}|Y_0^{t-h_1}=y_0^{t-h_1},M_0^{t-h_2}\right]\\ &= f_{M_t|Y_0^{t-h_1},M_0^{t-h_2}}\left(m_t|y_0^{t-h_1},m_0^{t-h_2}\right).\end{aligned}$$

In the case $h_1 \geq h_2$, from the Markovian property of the process M_t and from the fact that $\{M_t|Y_0^{t-h_1},M_0^{t-h_2}\}=\{M_t|M_{t-h_2}\}$ we obtain

$$\hat{\mathbb{1}}_{\left\{M_t=m_t|Y_0^{t-h_1}=y_0^{t-h_1},M_0^{t-h_2}=m_0^{t-h_2}\right\}} = \prod_{k=1}^{h_2-1} p_{m_{t-k-1},m_{t-k}}.$$

In the case when $h_1 < h_2$, $h_1 \geq 1$ we get

$$\begin{aligned}&\hat{\mathbb{1}}_{\left\{M_t=m_t|Y_0^{t-h_1}=y_0^{t-h_1},M_0^{t-h_2}=m_0^{t-h_2}\right\}}\\ &\quad= \sum_{m_{t-h_2+1}^{t-1}} f_{M_{t-h_2+1}^{t-1}|Y_0^{t-h_1},M_0^{t-h_2}}\\ &\qquad\left(m_{t-h_2+1}^{t}|y_0^{t-h_1},m_0^{t-h_2}\right)\\ &\quad= \sum_{m_{t-h_2+1}^{t-1}} f_{M_{t-h_1+1}^{t}|Y_0^{t-h_1},M_0^{t-h_1}}\\ &\qquad\left(m_{t-h_1+1}^{t}|y_0^{t-h_1},m_0^{t-h_1}\right) f_{M_{t-h_2+1}^{t-h_1}|Y_0^{t-h_1},M_0^{t-h_2}}\\ &\qquad\left(m_{t-h_2+1}^{t-h_1}|y_0^{t-h_1},m_0^{t-h_2}\right)\\ &\quad= \sum_{m_{t-h_2+1}^{t-h_1}} \left(\prod_{k=0}^{h_1-1} p_{m_{t-k-1},m_{t-k}}\right) c_t\left(m_{t-h_2+1}^{t-h_1}\right)\end{aligned}$$

where the last line was deduced from a similar analysis as in the proof of *Theorem 4.1*. When $h_1 < h_2$ and $h_1 = 0$ we obtain a formula similar to the one above, with the difference that there will be no longer any product of conditional probabilities.

V. Conclusion

In this technical note, we considered the problem of state estimation for MJLS, when the two components of the output observation are affected by delays. We gave formulas for the optimal estimators for both the continuous and discrete components of the state. These formulas admit recursive implementations and have time-independent complexity and therefore are feasible for practical implementation. However, the ordering relation between the delays affects the complexity of the estimators. An important observation is that when we have less mode observations than output observations the estimators become nonlinear in the outputs. Our problem setup can be viewed as a generalization of the state estimation problem for MJLS since it represents the link between the main cases addressed in the literature: all mode and output observations are known at the current time and only the output observations are known at the current time, respectively. Our framework can be used in monitoring applications where component failures are not instantly detected.

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