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NETWORK TOMOGRAPHY

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ABSTRACT. While conventional tomography is associated to the Radon transform in Euclidean spaces, electrical impedance tomography, or EIT, is associated to the Radon transform in the hyperbolic plane. We discuss some recent work on network tomography that can be associated to a problem similar to EIT on graphs and indicate how in some sense it may be also associated to the Radon transform on trees.

1. Introduction

As communication networks have become an essential part of everyday life, disruptions may have very serious consequences. Thus, the need to prevent or, at least, detect them early on, has become very important. In order to do that we discuss two models of the problem, one based on weighted graphs and the second based on trees. The first one is the discrete equivalent of the inverse conductivity problem, that is, of Electrical Impedance Tomography. The second model was mentioned recently by E. Jonckheere and his collaborators [**29**]. The reason we can think about this problem as a tomographic problem is that in both cases, the data we collect are obtained by monitoring traffic only at distinguished subsets of the network. We think about this subset as being the periphery of the network.

This paper is an expository version of ongoing work done by the authors in this subject and the proofs for results mentioned here can be found in [**9**] and [**11**].

2. The weighted graph model

In this case we model our network in the following way. We have a collection of nodes and edges between the nodes in a finite planar simple connected graph G. We denote by V the set of nodes of G and by E the set of edges of G. Usually, the graph G is denoted by $G(E, V)$. A particular subset of the vertices of this graph

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the "topology" of the graph has changed, and we refer to the important work of Fan Chung and her collaborators which offers crucial insights into this question. (See, for instance [**16**], [**17**] and [**18**]). In the other, the weights change because of "increase" of traffic, that is, the network configuration remains the same but the weights have either increased or remained the same. In this second situation, we can appeal to the following theorem, whose proof appears in [**11**]

THEOREM 1. *Let ω_1 and ω_2 be weights with $\omega_1 \leq \omega_2$ on $\overline{S} \times \overline{S}$, and f_1, $f_2 : S \to \mathbb{R}$ be functions satisfying for $j = 1, 2$,*

$$\begin{cases} \Delta_{\omega_j} f_j(x) = 0, \ x \in S \\ \frac{\partial f_j}{\partial n_{\omega_j}}(z) = \Phi(z), \ z \in \partial S \\ \int_S f_j d_{\omega_j} = K \end{cases}$$

for any given function $\Phi : \partial S \to \mathbb{R}$ with $\int_{\partial S} \Phi = 0$, and a given constant K with $K > m_0$, where $m_0 = \max_{j=1,2} |m_j| \cdot vol(S, w_j)$, $m_j = \min_{z \in \partial S} f_j(z), j = 1, 2$ and $vol(S, w_j) = \sum_{x \in S} d_{\omega_j} x$. If we assume that

$$(i)\, \omega_1(z, y) = \omega_2(z, y) \text{ on } \partial S \times \overset{\circ}{\partial S}$$
$$(ii)\, f_1|_{\partial S} = f_2|_{\partial S},$$

then we have

$$f_1 \equiv f_2$$

and

$$\omega_1(x, y) = \omega_2(x, y)$$

for all x and y in $\overline{S}$.

The condition that $\Delta_\omega f(x) = 0$ corresponds to the fact that the value $f(x)$ is the weighted average of the values of f at the adjacent nodes.

We conclude that the data distinguishes the two cases. That is, we can decide whether there is an increase of traffic somewhere in the network or not. While this is only a uniqueness theorem, nevertheless, we can effectively compute the actual weights from the knowledge of the Dirichlet data for convenient choices of the input Neumann data in a way similar to that done in [**21**] and [**23**] for lattices. Similarly, the Green function of this Neumann boundary value problem can be represented by an explicit matrix.

What we want to discuss now is the relationship between the above results to the problem of understanding a large network like the internet.

One way to make more concrete this problem was discussed by T. Munzner in [**32**] and [**33**] on visualizing the internet. It implies that the natural domain might be a hyperbolic space of dimension higher than 2. One can see that Munzner's suggestion leads to a question closely resembling EIT, and it is natural to consider it a problem in hyperbolic tomography [**7**], [**8**]. On the other hand, we have just obtained a significant result on the inversion of the Neumann-Dirichlet problem by studying it directly on "weighted" graphs [**11**]. Similarly, the Radon transform in the hyperbolic plane has been studied in [**7**], [**8**], and [**27**].

In addition, in a recent lecture E. Jonckheere [**29**] indicated that internet traffic, at least locally, could be modelled as being part of a tree and therefore it can be visualized using 2-dimensional hyperbolic geometry. As a consequence, a different way to locally study these kinds of networks would be by the use of the Radon

We give the following definitions. Let v, w be two vertices in T that are connected by a path $(v = v_0, ..., v_m = w)$, then the *distance* between v and w is the nonnegative integer $|v, w| = m$. Also, for $f \in L^1(T)$, let μ_n be the average operator defined by

$$\mu_n f(v) = \frac{1}{v_{(n)}} \sum_{|v,w|=n} f(w), \quad \text{for } v \in T$$

It can be seen that μ_n is basically a convolution with radial kernel

$$h_n(v, w) = \begin{cases} \frac{1}{v_{(n)}} & \text{if } |v, w| = n \\ 0 & \text{if } |v, w| \neq n \end{cases}$$

Let $\beta = q/(2(q+1))$ and R^* be the dual Radon transform defined for $\Phi \in L^\infty(\Gamma)$ by

$$R^*\Phi(v) = \int_{\Gamma_v} \Phi(\gamma) d\rho_v(\gamma) \text{ for each vertex } v \in T,$$

with respect to a suitable family $\{\rho_v : v \in T\}$ of measures on Γ where Γ_v is the set of all of the geodesics containing the vertex v.

In order to obtain the inversion of R we observe that R^*R acts as a convolution operator given by the radial kernel $h = \beta h_0 + \sum_{n=1}^{\infty} 2\beta h_v$.

PROPOSITION 1. [*Proposition* 3.2, 9] *The identity*

$$R^*R = \beta\mu_0 + \sum_{n=1}^{\infty} 2\beta\mu_n \quad \text{on } L^1(T),$$

holds in $L^1(T)$, where the series is absolutely convergent in the convolution operator norm on $L^2(T)$, thus providing a bounded extension of R^*R to $L^2(T)$.

THEOREM 2. [*Theorem* 3.4, 9] *The unique bounded extension to $L^2(T)$ of the operator R^*R is invertible on $L^2(T)$, and its inverse is the operator*

$$E = \frac{2(q+1)^3}{q(q-1)^2} \left[\mu_0 + \sum_{n=1}^{\infty} (-1)^n 2\mu_n \right]$$

which acts as the convolution with the radial kernel $\frac{2(q+1)^3}{q(q-1)^2} [h_0 + \sum_{n=1}^{\infty} (-1)^n 2h_n]$. As before, this series converges absolutely in the convolution operator norm on $L^2(T)$; in particular, E is bounded.

COROLLARY 1. [*Corollary* 3.5, 9] *The Radon transform $R: L^1(T) \to L^\infty(\Gamma)$ is inverted by*

$$ER^*Rf = f.$$

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