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CONTINUOUS AND DISCRETE INVERSE CONDUCTIVITY PROBLEMS

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ABSTRACT. Tomography using CT scans and MRI scans is now well-known as a medical diagnostic tool which allows for detection of tumors and other abnormalities in a noninvasive way, providing very detailed images of the inside of the body using low dosage X-rays and magnetic fields. They have both also been used for determination of material defects in moderate size objects. In medical and other applications they complement conventional tomography. There are many situations where one wants to monitor the electrical conductivity of different portions of an object, for instance, to find out whether a metal object, possibly large, has invisible cracks. This kind of tomography, usually called Electrical Impedance Tomography or EIT, has also medical applications like monitoring of blood flow. While CT and MRI are related to Euclidean geometry, EIT is closely related to hyperbolic geometry. A question that has arisen in the recent past is whether there is similar "tomographic" method to monitor the "health" of networks. Our objective is to explain how EIT ideas can in fact effectively be used in this context.

1. Introduction and preliminaries

Networks have become ubiquitous in present society and thus it has become important to avoid and detect disruptions. In particular, it is important to prevent malicious intruders from disrupting them. To achieve this sufficiently early, it is essential to count on a mathematical model that can allow early detection of attacks to the network. The mathematical tool that we consider to accomplish the early detection of disruptions is based on the use of tomographic ideas. One of the questions we are considering is how to find out whether an attack against the network by traffic overload is taking place by monitoring traffic only at the periphery of the network (input-output map), and hence, the use of a tomographic approach.

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5 with some new results on this last subject. The key ingredient is the attempt to understand what happens in a network from "boundary measurements", that is, to determine whether all the nodes and routers are working or not and also measure congestion in the links between nodes by means of introducing test packets (ICMP packets) in the "external" nodes, that is, the routers. The question of finding out whether there are nodes that are in working order is a classical question in graph theory. For networks, it is also interesting to try to predict future problems due to congestion. (Note that nodes could fail to work for other reasons than congestion on the links starting at a given node.) This requires to monitor also traffic intensity, also known as load, congestion, etc., in different contexts. There is another analogy to mathematical tomography that arose independently and maybe closer to the consideration of this question in the context of electrical networks. Curtis and Morrow have done very interesting work in this context, both theoretical and in simulations, see, for instance, [**20**] and [**19**]. Another analogy in the same direction arises when we consider very large networks, as the internet, which could be considered as the discretization of an underlying continuous model. In this way, we can see the analogy with the well-known inverse conductivity problem and we could try to profit from the large body of mathematical research in this area. The analogy with this particular inverse problem indicates that if one were to pursue this "abstract" approach the "correct" geometry is closer to be hyperbolic than to be Euclidean [**7**]. On the other hand, as of this moment, we have found that those tomographic analogies are more useful for providing directions of research and methods to consider these problems than providing an exact correspondence between the two phenomena. It is in this context that [**9**] modelled "internet tomography" as an inverse Dirichlet-to-Neumann problem for a graph with weights. In this situation, one can prove that characteristics of the graph, namely, its connectivity and the traffic along links can be uniquely determined by boundary-value measurements as shown in [**9**] which is the natural analogue of the continuous inverse conductivity problem.

Among the questions that arise naturally using the inverse conductivity problem as a guiding model there are a number of questions that have been previously addressed using other points of view. Namely, the problems already addressed in [**17**] for internet tomography are:

1. Link-level inference, in other words link-level parameter estimation based on end-to-end path-level traffic measurements. Examples of this are unicast inference of link loss rates, unicast inference of link delay distributions, topology identification, loss rates by using multicast probing and so on.

2. Path-level inference (origin-destination tomography OD) in other words sender-receiver path-level traffic intensity estimation based on link-level traffic measurements. One example of this is time-varying OD traffic matrix estimation.

We would like to conclude by thanking the editors and the referee for his useful comments.

2. The Radon transform in $\mathbb{R}^2$

Let $\omega \in S^1$, then $\omega = (\cos\theta, \sin\theta)$, and take $p \in \mathbb{R}$. The locus of equation $x \cdot \omega = p$ represents the line l that is perpendicular to the line r passing through the origin and forming an angle θ with the real line $\mathbb{R}$. If B is the intersection of l and r, the euclidean distance d (signed) from $B = p\omega$ to the origin is equal to p.

One can similarly define the Radon transform in $\mathbb{R}^n$ and verify that the properties (2.5) and (2.6) extend to this case. In particular for the Laplacian Δ in $\mathbb{R}^n$,

$$R(\Delta f) = \frac{\partial^2 Rf(\omega, p)}{\partial p^2}, \tag{2.7}$$

where, for each direction $\omega \in S^{n-1}$ the right hand side is the Laplace operator in dimension 1. Note that in general

$$R(\Delta f)(\omega, p) = (\omega_1^2 + \cdots + \omega_n^2)\frac{\partial^2 Rf(\omega, p)}{\partial p^2}$$

As a consequence, if the function f depends also on time, and $\Box_n$ represents the wave operator in n dimensions we conclude that

$$R\Box_n f = \Box_1 Rf.$$

therefore, the Radon transform in n dimensions is localizable if and only if the wave equation is localizable. Fixing $\omega \in S^{n-1}$, one can express this identity by saying that the Radon transform interwines the wave operator $\Box_n = \Delta - \frac{\partial^2}{\partial t^2}$ in $n-$dimensions with the wave operator $\Box_1 = \frac{\partial^2}{\partial p^2} - \frac{\partial^2}{\partial t^2}$ in $1-$space dimension. It follows that the Radon transform can not be localized in even dimensions [**10**]. In spite of this observation one can obtain an *almost* localization of the Radon transform in $\mathbb{R}^2$. The key elements is the use of wavelets as it will be described in the next section. Meanwhile, for the sake of completeness we remind the reader of the standard inversion formula for the Radon transform in $\mathbb{R}^2$. It depends on the following identity, usually called the Fourier slice theorem. Namely, writing the Fourier transform $F_2(f)$ of a nice function f in $\mathbb{R}^2$ in polar coordinates (s, ω) we have

$$\int_{\mathbb{R}^2} f(x)\, e^{-is\omega\cdot x}\, dx = \int_{-\infty}^{\infty} Rf(\omega, p)\, e^{-isp}\, dp, \quad x \in \mathbb{R}^2 \tag{2.8}$$

or, in a more concise form,

$$F_2(f) = F(Rf)$$

where, clearly, F_2 stands for the 2-dimensional Fourier transform and F stands for the 1-dimensional Fourier transform in the variable p which provides one standard inversion formula for the Radon transform

$$f = F_2^{-1} F(Rf) \tag{2.9}$$

There is another inversion formula that has a number of advantages for us, and we proceed to explain it now. To simplify we work in $X = \mathbb{S}(\mathbb{R}^2)$, the Schwartz space of functions f and $Y = \mathbb{S}(S^1 \times \mathbb{R})$ the Schwartz space of functions g. Let $f_1, f_2 \in X$ and $g_1, g_2 \in Y$, and $\langle f_1, f_2\rangle_X$, $\langle g_1, g_2\rangle_Y$ the inner products in X and Y respectively, then because of the linearity of the operator R, we write the equation that defines R^*, the adjoint operator of R

$$\langle Rf, g\rangle_Y = \langle f, R^* g\rangle_X \tag{2.10}$$

The explicit expression for $R^* g$ is given by

$$\int_{S^1} g(\omega, \omega \cdot x) d\omega = R^* g \tag{2.11}$$

following [8], given a "mother" wavelet $\Psi \in L_2(\mathbb{R}) \cap L_1(\mathbb{R})$ and $f \in L_2(\mathbb{R})$, we define the wavelet transform of f as

$$W_\Psi f(a,b) := \int_{-\infty}^{\infty} f(t)\ \overline{\Psi}(\frac{t-b}{a})\frac{dt}{\sqrt{a}} =< f, D_a\Psi_b(t) >_{L_2} \tag{3.3}$$

$b, a \in \mathbb{R}$, $a > 0$, where, for a function g and $b \in \mathbb{R}$ we let $g_b(t) = g(t-b)$.

One requires that the "mother" wavelet Ψ to be oscillatory, i.e. $\int_{-\infty}^{\infty} \Psi(x)dx = 0$. In fact, one assumes stronger condition

$$C_\Psi = \int_{-\infty}^{\infty} \frac{\left|\widetilde{\Psi}(\varsigma)\right|^2}{|\varsigma|} d\varsigma < \infty, \tag{3.4}$$

called the *admissibility condition.* The admissibility condition is satisfied when Ψ has several vanishing moments,i.e., for $0 \leq k < s$

$$\int_{-\infty}^{\infty} x^k\ \Psi(x)dx = 0$$

The functions $D_a\Psi_b$ are called the wavelets

The function f can be reconstructed from its wavelet transform by means of the "*resolution identity*" formula

$$f = C_\Psi^{-1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} < f, D_a\Psi_b(t) >_{L_2} D_a\Psi_b(t)dt$$

where $C_\Psi < \infty$ since $\Psi \in L_1(\mathbb{R})$.We refer to [27] for the general theory of wavelets.

Proposition 1 explains how to use wavelets to obtain (almost) localization.

PROPOSITION 1. [10] *Let n be an even integer, and $h \in L_2(\mathbb{R})$ a function with compact support such that for some integer $m \geq 0$ $\widetilde{h}$ is n+m-1 times differentiable and satisfies*

1. $\gamma^{\,j} \widetilde{h}^{(k)}(\gamma) \in L_1(\mathbb{R}) \cap L_2(\mathbb{R})$ for $0 \leq j \leq m$, $0 \leq k \leq m+n-1$
2. $\int_{-\infty}^{\infty} t^{\,j} h(t)dt = 0$ *for* $0 \leq j < m+1$, i.e., h *has* $m+1$ *vanishing moments*

Then

$$I^{\,1-n}h(t) = o(|t|^{-n-m+1}) as\ |t| \longmapsto \infty$$

and

$$t^{\,n+m-1} I^{\,1-n}h \in L_2(\mathbb{R})$$

The fact that $I^{\,1-n}h(t) = o(|t|^{-n-m+1})$ *as* $|t| \longmapsto \infty$ tells us that $I^{\,1-n}h$ decays as $|t|^{-(n+m-1)}$, and therefore, it does a good localization job.

For practical purposes, the continuos wavelet transform, CWT, is discretized and the discrete wavelet transform, DWT, is obtained. In order to discretize it, consider $m, n \in Z$ and the values a, b that appear in $W_\Psi f(a,b)$ are restricted to only discrete values $a = a_o^m$, $b = nb_oa_o^m$, $a_o > 1$, $b_o > 1$ fixed. (The fact that $a_o > 1$, $b_o > 1$ it really does not matter because m, n can be negative). The discrete wavelet transform DWT of f is defined as

$$W_{m,n}^{\Psi}(f) = a_0^{-m/2}\int_{-\infty}^{\infty} f(t)\overline{\Psi}(a_0^{-m}t - nb_0) \tag{3.5}$$

PROPOSITION 3. *Let Ψ be a separable 2-dimensional wavelet, i.e.,*

$$\Psi(x) = \Psi^1(x_1)\Psi^2(x_2), \quad x \equiv (x_1, x_2)$$

where for i=1,2 $\left|\tilde{\Psi}(\gamma)\right| \leq C_1(1+|\gamma|)^{-1}$ *for all* $\gamma \in \mathbb{R}$. *Defining the family of the one-dimensional functions* $\{\rho_\omega\}_{\omega \in S^1}$ *by*

$$\tilde{\rho_\omega}(\gamma) = \frac{1}{2}|\gamma|\,\tilde{\Psi}^1(\gamma\omega_1)\tilde{\Psi}^2(\gamma\omega_2), \quad \omega = (\omega_1, \omega_2)$$

i.e. $\rho_\omega = F_1^{-1}(\tilde{\rho_\omega})$. *Then for every* $f \in L_1(R^2) \cap L_2(R^2)$,

$$(W_\Psi f)(a,x) = a^{-1/2}\int_{S^1}(W_{\rho_\omega}\, R_\omega f)\,(a, x\cdot\omega)d\omega$$

The proposition shows that the wavelet transform of a function $f(x)$ given any mother wavelet and at any scale can be obtained by backprojecting the wavelets transform of the Radon transform of f using wavelets that vary with each angle, the argument of ω, but which are admissible for each angle, i.e. $C_\Psi < \infty$.

4. The hyperbolic Radon transform and EIT

In this section we discuss the Radon transform on the hyperbolic plane, state some formulae analogous to the ones that were given in section 2 to invert the Radon transform. The backprojection inversion formula is one of them, and later we will see how the hyperbolic Radon transform is related to electric impedance tomography (EIT).

In [6] and [7] it is shown that the hyperbolic Radon transform is involved in the problem of reconstructing the conductivity distribution on a plate by using electrical impedance tomography EIT.

4.1. The hyperbolic Radon transform.

Let D be the unit disk of the complex plane, i.e. $D = \{z \in \mathbf{C} / \ |z| < 1\}$. In D, a Riemannian structure is defined through the hyperbolic metric of arc-length ds given by

$$ds = \frac{2\,|dz|}{(1-|z|^2)} \tag{4.1}$$

with dz the Euclidean distance in $\mathbb{R}^2$, and the hyperbolic distance between two points $z, w \in D$ is given by

$$d(z,w) = \operatorname{arcsin} h\left(\frac{|z-w|}{(1-|z|^2)^{1/2}(1-|w|^2)^{1/2}}\right)$$

The set of lines that are diameters of D, and the set of intersections between the Euclidean circles and D such that the resultant lines (intersections) are perpendicular to the boundary ∂D of D are the geodesics or h-lines for the metric (4.1).

If $z \in D$ is expressed in polar coordinates by (w, r) where $w = z/|\,z|$, $r = d(z,0)$, then the metric (4.1) becomes

$$ds^2 = dr^2 + \sinh^2 r\; dw^2$$

where $dm(w)$ is the measure for the hyperbolic area which in polar coordinates is given by

$$dm = \sinh r dr dw$$

Recall the formula for R^*R can be written as

$$R^*Rf = \frac{2}{|x|} * f.$$

The analogous result for the hyperbolic Radon transform R_H is given by

$$R_H^* R_H f = k * f, \text{ where } k(t) = \frac{1}{\pi \sinh t} \tag{4.5}$$

In [7] and [8] it is shown that by letting f $S(t) = \coth t - 1$, we obtain

$$\frac{1}{4\pi}\Delta_H \; S *_H R_H^* R_H = I, \tag{4.6}$$

the analogue to the backprojection inversion formula given before.

The Fourier transform in the hyperbolic disk D for a radial function k is given by

$$\tilde{k}(\lambda) = 2\pi \int_0^\infty k(t) P_{i\lambda - 1/2}(\cosh t) \sinh t dt, \quad for \ \lambda \in \mathbb{R}$$

where $P_\nu(r)$ is the *Legendre* function if index ν. If m is another radial function then

$$\widetilde{(k * m)}(\lambda) = \tilde{k}(\lambda)\tilde{m}(\lambda)$$

as we know it, [8]. It follows that as $\tilde{k}(\lambda) \neq 0 \ \forall \lambda \in \mathbb{R}$ then the operator R_H, which takes f to $k *_H f$, is injective.

4.2. Electrical impedance tomography (EIT). EIT has a number of applications to medicine and non-destructive evaluation. For instance, to determine the existence and lengths of internal cracks in the wings of an airplane. These applications are related to the inverse problem which is formulated now.

Let D the unit disk in $\mathbb{R}^2$ and β an strictly positive function defined on $\overline{D}$ which is unknown and represents the conductivity distribution inside the disk. When currents are introduced at the boundary ∂D, let Ψ be a given integrable function representing such currents and such that the average of the values of Ψ on ∂D is zero

$$\int_{\partial D} \Psi ds = 0$$

and consider the boundary problem with Neumann conditions

$$\begin{cases} div(\beta grad \ u) = 0, & in \ \ D \\ \beta \frac{\partial u}{\partial n} = \Psi, & on \ \ \partial D \end{cases} \tag{4.7}$$

where Ψ is given and n is the outer unit normal vector on ∂D. This problem has a unique solution u where the uniqueness of u is up to an additive constant. The function u is the potential distribution on D so *grad* u is the electrical field. The variation of u on ∂D has to correspond to the known values of Ψ on ∂D, then, if s represents the tangent vector to ∂D, it follows that the tangential derivative of u, $\frac{\partial u}{\partial s}$, depends linearly on Ψ. So, for Ψ given and β, the unknown conductivity, there exists a solution u. This defines a mapping

δU satisfies

$$\begin{cases} \Delta(\delta U) = -\langle grad\ \delta\beta, grad\ U\rangle, & in\ \ D \\ \frac{\partial U}{\partial n} = -(\delta\beta)\Psi, & on\ \ \partial D \end{cases}$$

and since Ψ represents the input of the currents and they can be arbitrarily chosen with the only constraint

$$\int_{\partial D} \Psi ds = 0$$

then the input Ψ, can be well approximated by linear combination of dipoles where a dipole at a point $w \in \partial D$ is given by $-\pi\frac{\partial}{\partial s}\delta_w$, δ_w the Dirac delta at w. It follows that the problem (4.9) for the dipole (input) $-\pi\frac{\partial}{\partial s}\delta_w$ at w becomes

$$\begin{cases} \Delta U_w = 0, & in\ \ D \\ \frac{\partial U_w}{\partial n} = -\pi\frac{\partial}{\partial s}\delta_w, & on\ \ \partial D \end{cases} \tag{4.10}$$

and the solution U_w of (4.10) has level curves which are arcs of circles that pass through w and are perpendicular to ∂D. Therefore, the level curves of U_w are exactly the geodesics given by the hyperbolic metric. At this point, the hyperbolic Radon transform is involved in the problem and can be used to solve it.

In [**7**] is shown that the linearized problem can in fact be described explicitly in the context of hyperbolic geometry using R_H and a radial convolution operator with kernel k. Let k be given by

$$k(t) = \frac{\cosh^{-2}(t) - 3\cosh^{-4}(t)}{8\pi}$$

then, as the boundary data function $\mu = \frac{\partial(\partial U)}{\partial s}$ defined on the space of the geodesics in D, the relation between $\delta\beta$ and μ can be shown to be

$$R_H(k *_H \delta\beta) = \mu$$

and because of the backprojection operator, one obtains

$$R_H^* R_H(k *_H \delta\beta) = R_H^* \mu$$

hence

$$\frac{1}{4\pi}\Delta_H(S *_H (R_H^*\mu)) = k *_H \delta\beta \tag{4.11}$$

Computing the hyperbolic Fourier transform of k, $\widetilde{k}$, which can be done exactly, it can be seen that $\widetilde{k}(\lambda) \neq 0$, $\forall\lambda \in \mathbb{R}$, and consequently, the convolution operator with kernel or symbol k, $k*_H$ is invertible. Formula (4.11) requires to invert the convolution operator of symbol k to compute $\delta\beta$. Barber and Brown [**2**] proposed an approximate inversion and Santosa and Vogelius [**32**] shows that the inversion formula suggested by [**2**] is a generalized radon transform.

To numerically implement the reconstruction of $\delta\beta$ it is necessary to invert the geodesic Radon transform and perform a deconvolution. The difficulty of numerically implementing (4.11) lies in the fact that it is complicated to numerically implement a two-dimensional non-Euclidean convolution on the hyperbolic space. In [**26**], Lissianoi and Ponomarev focus on the problem of numerically inverting the geodesic Radon transform by developing an algorithm, and the problem regarding the deconvolution is also considered there. For this purpose, they consider the inversion formula (4.6) and use it to derive an inversion formula for the geodesic Radon transform that it is more suitable for computations. The interesting open problem here is to be able to define a class of "discrete hyperbolic wavelets" that only one neighboring node which is the interior node that has unit distance from p. If a line segment l connects a pair of neighboring nodes p and q in $intV$ or if it connects a boundary node p to its neighboring interior node q is called *edge* or *conductor* and denoted pq. In the case in which p is on the boundary, the edge is called a boundary edge. The set of edges is denoted by E, and usually the graph G is denoted by $G(V, E)$.

Let ω a non-negative real-valued function on E, the value $\omega(pq)$ is called the *conductance* of pq and $1/\omega(pq)$, the *resistance* of pq, and ω is the *conductivity* (ω is also called a *weight*). A function $u : V \to \mathbb{R}$ gives a current across each conductor pq by *Ohm's law*, $I = \omega(pq)(u(p) - u(q))$ (I the current). The function u is called *ω-harmonic* if for each interior node p,

$$\sum_{q \in N(p)} \omega(pq)(u(q) - u(p)) = 0$$

then the sum of the currents flowing out of each interior node is zero, and this is *Kirkhoff's law*. Let Φ a function defined at the boundary nodes, the network will acquire a unique *ω-harmonic* function u with $u(p) = \Phi(p)$ for each $p \in \partial G$ in other words, Φ induces u and u is called the *potential* induced by Φ. Considering a conductor pq then the potential drop across this conductor is $\Delta u(pq) = u(p) - u(q)$. The potential function u determines a current $I_\Phi(p)$ through each boundary node p, by $I_\Phi(p) = \omega(pq)(u(p) - u(q)$, q being the interior neighbor of p. As in the continuous case, for each conductivity ω on E, the linear map Λ_ω from boundary functions to boundary functions is defined by $\Lambda_\omega \Phi = I_\Phi$ where the boundary function Φ is called Dirichlet data, the boundary current I_Φ is called Neumann data, and the map Λ_ω is called the Dirichlet-to-Neumann map.

The problem to consider is to recover the conductivity ω from Λ_ω, which is analogous to the the inverse problem in the continuous case. The two basic problems are the connectivity and conductivity of the network. Note that the connectivity of the network or the situation where the network remains connected but some edges disappear is a topological problem, the *configuration* of the graph has changed. For detailed theory about electrical networks, planar graphs, recovering of a graph and harmonic functions, we refer to **[18]** and the work of Curtis and Morrow **[19]**.

The discrete or finite nature of graphs makes working on graphs basically easier than investigating these problems in the continuous case, although it gives rise to several disadvantages. For example, solutions of the Laplace equation for graphs have neither the local uniqueness property nor is their uniqueness guaranteed by the Cauchy data, contrary to the continuous case where they are the most important mathematical tools used to study the inverse conductivity problem and related problems **[9]**. The inverse problem that we study is to identify the connectivity of the nodes and the conductivity on the edges between each adjacent pair of nodes.

Given a network with a pattern of traffic measured as the "usual" load between adjacent nodes (e.g., number of messages) one can associate to it a Laplace operator denoted Δ_ω, where the weight ω is a sequence of values representing the usual loads between every pair of adjacent nodes in the network.

We define the degree $d_\omega x$ of a vertex in the weighted graph G with weight ω by

$$d_\omega x = \sum_{y \in V} \omega(x, y)$$

whenever $f_1(x) \neq f_1(y)$ *and* $f_2(x) \neq f_2(y)$.

Note that

$$\begin{cases} \Delta_\omega f(x) = 0, \ x \in S \\ \frac{\partial f}{\partial_\omega n}(z) = \Phi(z), \ z \in \partial S \end{cases}$$

is known as the Neumann boundary value problem *NBVP*. In [9] it is shown that the NBVP has a unique solution up to an additive constant.

The second conclusion of the theorem shows not only whether or not each pair of nodes is connected by a link, but also how nice the link is. Moreover, the proof gives an algorithm to detect if the weights change on the edges.

The conditions $\omega_1 \leq \omega_2$ (monotonicity condition) and $\int_G f_j d_{\omega_j} = K$ (the normalization condition) are essential for the uniqueness of the result. We know that the NBVP has a unique solution up to an additive constant; therefore, the Dirichlet data $f|_{\partial S}$, $z \in \partial S$ is well-defined up to an additive constant. Here we have discussed the inverse conductivity problem on the network (graph) S with nonempty boundary, which consists in recovering the conductivity (connectivity or weight) ω of the graph by using the Dirichlet-to-Neumann map with one boundary measurement. In order to deal with this inverse problem, we need at least to know or be given the boundary data such as $f(x)$, $\frac{\partial f}{\partial_\omega n}(z)$ for $z \in \partial S$ and ω near the boundary. So it is natural to assume that $f|_{\partial S}$, $\frac{\partial f}{\partial_\omega n}|_{\partial S}$ and $\omega|_{\partial S \times \overset{\circ}{S}}$ are known (given or measured). But even though we are given all these data on the boundary, we are not guaranteed, in general, to be able to identify the conductivity ω uniquely. For more details and counterexample, see [9].

There are many problems to be answered, for instance what happens if the number of nodes is not finite? What is the hyperbolic version of the discrete case?. If we allow to consider also $\omega = 0$ then the presence of zero weights tells us that the conductivity on the edge (a particular one) is either down or the nodes connected to that edge "disappear" in the sense that the edge length becomes infinite and this is because uniqueness is not true. We still need to get stronger results to determine the configuration of a network (connectivity). Let us add that very recently Bensoussan and Menaldi [3] have given a slightly different proof of theorem 4 relying on the fact that Δ_ω is a positive operator.

References

[1] J. Baras, C. A. Berenstein and F. Gavilánez, Local monitoring of the internet network. Available at http://techreports.isr.umd.edu/TechReports/ISR/2003/TR_2003-7/TR_2003-7.phtml

[2] D. C. Barber and B. H. Brown, Inverse problems in partial differential equations (Arcata, CA, 1989), 151-164, SIAM, Philadelphia, PA, 1990.

[3] A. Bensoussan and J. L. Menaldi, Difference Equations on weighted graphs, 2003

[4] R. M. Brown, and R. Torres, Uniqueness in the inverse conductivity problem for conductivities with $\frac{3}{2}$-derivatives in L^p, $p > 2n$, JFAA, 9 (2003), 563-574.

[5] R. W. Brown, E. Haacke, M. Thompson, R. Venkatesan, Magnetic Resonance Imaging: Physical Principles and Sequence Design. Wiley, May 1999.

[6] C. A. Berenstein and E. Casadio Tarabusi, The inverse conductivity problem and the hyperbolic Radon transform, "75 years of Radon Transform", S. Gindikin and P. Michor, editors. International Press, 1994.

[7] C. A. Berenstein and E. Casadio Tarabusi, Integral geometry in hyperbolic spaces and electrical impedance tomography, SIAM J. Appl. Math. 56 (1996), 755-764.

[8] C. A. Berenstein, Radon transforms, wavelets, an applications, Lecture notes in Mathematics. Berlin: Springer, Vol. 1684, 1998, pp.1-33.

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