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PARTIALLY OBSERVED DIFFERENTIAL GAMES, INFINITE-DIMENSIONAL HAMILTON–JACOBI–ISAACS EQUATIONS, AND NONLINEAR H_∞ CONTROL*

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Abstract. This paper presents new results for partially observed nonlinear differential games. Using the concept of *information state*, we solve this problem in terms of an infinite-dimensional partial differential equation, which turns out to be the Hamilton–Jacobi–Isaacs (HJI) equation for partially observed differential games. We give definitions of smooth and viscosity solutions and prove that the value function is a viscosity solution of the HJI equation. We prove a verification theorem, which implies that the optimal controls are separated in that they depend on the observations through the information state. This constitutes a separation principle for partially observed differential games. We also present some new results concerning the certainty equivalence principle under certain standard assumptions. Our results are applied to a nonlinear output feedback H_∞ robust control problem.

Key words. partially observed differential games, infinite-dimensional partial differential equations, viscosity solutions, nonlinear H_∞ robust control

AMS subject classifications. 90D25, 93B36, 93C10, 93C41, 49L25, 35R15

1. Introduction. The nonlinear H_∞ robust control problem has generated considerable activity in recent years, and important contributions have been made by a number of authors; see [1]–[3], [5], [8], [9], [12], [16], [20]–[24], [26], [29]–[35]. The state feedback problem is reasonably well understood, although the issue of controller synthesis for continuous-time systems remains outstanding. This is because the value functions solving the various partial differential equations (PDEs) that have been proposed need not be smooth—a standard difficulty even for simple deterministic optimal control problems. The output feedback problem is much more difficult, and various approaches have been suggested in the literature. Perhaps the most general of these approaches was initiated in [24], [25], where the concept of *information state* was used to solve a partially observed dynamic game, and applied in [23] to solve the output feedback H_∞ problem (see also the discussion in [6]). The results in [24], [23] are presented in the discrete-time context for technical simplicity, although the system-theoretic ideas are valid in continuous-time also; ideed, the key equations were presented in [25], [32] and later in [6]. The purpose of this paper is to commence the task of developing a mathematical theory for continuous-time partially observed differential games and output feedback H_∞ robust control.

The information state $p_t = p_t(x)$ is the solution of a first-order PDE and takes values in a suitable infinite-dimensional Banach space $p_t \in \mathcal{X}$ (here, $x \in \mathbf{R}^n$ is the state of the system being controlled, so $\mathcal{X}$ is a space of real-valued functions of x). The partially observed differential game that we consider can be transformed into an equivalent game with full state information, and this leads via dynamic programming to a value or optimal cost function $W(p, t)$ that "solves" a PDE on $\mathcal{X} \times [0, T]$.. This PDE is a nonlinear first-order equation and is the correct Hamilton–Jacobi–Isaacs (HJI) equation for partially observed differential games. This HJI equation appears to be new, and we are not aware of any results in the literature concerning this type of infinite-dimensional PDE. It is not clear what if anything the results in [7] have to say about this HJI equation. In the case of partially observed stochastic control,

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the idea of information state is familiar, a theory has been developed [14], [17], [27], and the dynamic programming equation is an infinite-dimensional nonlinear second-order PDE.

The particular class of problems that we consider in this paper is presented in §2. This class should be regarded as a prototype class, and the ideas and principles we develop are expected to apply in much more general contexts. The relevant information state is defined in §3, and some if its properties are analyzed for use in later sections. In particular, the key representation theorem is given. In §4, the value function and HJI equation are defined and studied. Definitions of smooth and viscosity solutions are given. We prove that the value function is a viscosity solution of the HJI equation. We do not know a proof of a uniqueness or comparison theorem for equations of this type, and consequently our definition of viscosity solution should be regarded as a provisional one. While in general it is not expected that smooth solutions will exist, a verification theorem is proven in §5 assuming a smooth solution exists, yielding that the optimal control is a separated control in the sense that it depends on the observations via the information state. The certainty equivalence principle proposed in [5] and [9] is considered in §6. We explain how this principle fits into the general information state framework and show that, under a generalization of the standard assumptions, the certainty equivalence controller can be optimal at certain values of the information state. The standard assumptions are very stringent and are unlikely to hold in general, and we explain what can happen in such an event. In §7, we apply our results to a relatively simple nonlinear H_∞ control problem, viz., finite horizon disturbance attenuation. The solution is expressed in terms of two PDEs, a finite-dimensional one for the information state and an infinite-dimensional equation for the value function. Infinite horizon H_∞ problems are closely related to the theory of dissipative systems [18], [36], and we present the relevant partial differential inequality (PDI) for the output feedback problem. Finally, we make some comments concerning more general cases.

2. Problem formulation. We consider the class of nonlinear partially observed deterministic systems described by the state space equations

$$\begin{cases} \dot{x}(t) = f(x(t), u(t)) + g(x(t), w(t)), \\ y(t) = h(x(t)) + w(t). \end{cases} \tag{2.1}$$

Here, $x(t) \in \mathbf{R}^n$ denotes the state of the system and is not directly measurable; instead, an output quantity $y(t) \in \mathbf{R}^p$ is observed. The control input is $u(t) \in U \subset \mathbf{R}^m$, and $w(t) \in \mathbf{R}^p$ is regarded as an opposing disturbance input. The functions $f : \mathbf{R}^n \times \mathbf{R}^m \to \mathbf{R}^n$, $g : \mathbf{R}^n \times \mathbf{R}^p \to \mathbf{R}^{n\times m}$, and $h : \mathbf{R}^n \to \mathbf{R}^n$ are assumed bounded and smooth with bounded derivatives of orders up to three, say. The set U is compact.

Most of this paper is concerned with a differential game problem on a finite-time horizon $[0, T]$, and we use the following type of admissible strategies. The admissible disturbances are the square integrable functions

$$w \in \mathcal{W}(t) = L_2([t, T], \mathbf{R}^p),$$

while the admissible controls are the nonanticipating (causal) maps

$$\mathbf{u} : \mathcal{Y}(t) \to \mathcal{U}(t),$$

where

$$\mathcal{U}(t) = L_2([t, T], U), \quad \mathcal{Y}(t) = L_2([t, T], \mathbf{R}^p).$$

The nonanticipating property means that if $y_1, y_2 \in \mathcal{Y}(t)$ and $y_1(r) = y_2(r)$ a.e. $r \in [t, s]$, then $\mathbf{u}[y_1](r) = \mathbf{u}[y_2](r)$ a.e. $r \in [t, s]$ (cf. the Elliott–Kalton notion of strategy [10], [11]).

We will denote by $\mathbf{U}(t)$ the class of such nonanticipating strategies for which (2.1) and (3.2) (for $u = \mathbf{u}[y]$) have unique solutions.

We next introduce several function spaces that will be used in the sequel. The Banach space of continuous functions with at most linear growth is denoted

$$\mathcal{X} = \{p \in C(\mathbf{R}^n) \ : \| p \| < \infty\},$$

where the norm is defined by

$$\| p \| = \sup_{x \in \mathbf{R}^n} \frac{|p(x)|}{1 + |x|}.$$

Denote by

$$\mathcal{X}^1 = \{p \in C^1(\mathbf{R}^n) \ : \| p \|_1 < \infty\}$$

the Banach space of continuously differentiable functions with bounded derivatives equipped with norm

$$\| p \|_1 = \sup_{x \in \mathbf{R}^n} \frac{|p(x)|}{1 + |x|} + \sup_{x \in \mathbf{R}^n} |\nabla_x p(x)|,$$

where $\nabla_x p$ is the gradient of p. Also, we need to define the function space

$$\mathcal{D} = \{p \in C(\mathbf{R}^n) \ : \ p(x) \leq -c_1|x| + c_2 \ \forall x \in \mathbf{R}^n, \text{ for some } c_1 > 0, c_2 \in \mathbf{R}\}.$$

Note that the subsets $\mathcal{D} \cap \mathcal{X} \subset \mathcal{X}$ and $\mathcal{D} \cap \mathcal{X}^1 \subset \mathcal{X}^1$ are open in their respective topologies. As sets, $\mathcal{X}^1 \subset \mathcal{X}$, but $\mathcal{X}^1$ is not a subspace of $\mathcal{X}$ as Banach spaces.

The minimax differential game is defined as follows. The payoff is

$$J(\mathbf{u}, w, x_0) = \alpha(x_0) + \int_0^T [L(x(t), \mathbf{u}[y](t)) - \gamma^2 \ell(w(t))]\, dt + \Phi(x(T)),$$

where the initial state $x(0) = x_0$ is in general unknown. The functions $L : \mathbf{R}^n \times \mathbf{R}^m \to \mathbf{R}$, $\ell : \mathbf{R}^p \to \mathbf{R}$, and $\Phi : \mathbf{R}^n \to \mathbf{R}$ are assumed bounded and smooth with bounded derivatives of orders up to three, and $\alpha \in \mathcal{D} \cap \mathcal{X}$. The assumptions imply that J is well defined and bounded uniformly in $w \in \mathcal{W}(0)$ and $\mathbf{u} \in \mathbf{U}(0)$. We will also assume that $L \geq 0$, $\ell \geq 0$, $\Phi \geq 0$. The controller's objective is to minimize J, while the disturbances attempt to maximize J. For $\mathbf{u} \in \mathbf{U}(0)$ define the functional

$$J(\mathbf{u}) = \sup_{w \in \mathcal{W}(0),\ x_0 \in \mathbf{R}^n} J(\mathbf{u}, w, x_0).$$

The problem addressed in this paper is that of minimizing $J(\mathbf{u})$ over $\mathbf{u} \in \mathbf{U}(0)$. This is a partially observed minimax differential game. Note that x_0 is regarded as an unknown opponent also.

3. Information state. The key to solving the partially observed game is to replace it by an equivalent one with full state information. The difficulty is that the new state is infinite dimensional in general.

To this end, for fixed output path $y \in \mathcal{Y}(0)$ and control $u \in \mathcal{U}(0)$, define the *information state* by

$$p_t(x) = \alpha(x_0) + \int_0^t [L(x(s), u(s)) - \gamma^2 \ell(y(s) - h(x(s)))]\, ds, \tag{3.1}$$

where $x(\cdot)$ is the solution of

$$\dot{x}(s) = f(x(s), u(s)) + g(x(s), y(s) - h(x(s))), \qquad 0 < s < t, \tag{3.2}$$

with terminal condition $x(t) = x$. This quantity describes the worst-case performance up to time t using the control u, which is consistent with the observed output and the constraint $x(t) = x$. It summarizes the observed information in a way that is suitable for fulfilling the control objective. The information state evolves according to the dynamics

$$\begin{cases} \dot{p}_t &= F(p_t, u(t), y(t)), \\ p_0 &= \alpha, \end{cases} \tag{3.3}$$

where F is the differential operator

$$F(p, u, y) = -\nabla_x p \cdot (f(\cdot, u) + g(\cdot, y - h)) + L(\cdot, u) - \gamma^2 \ell(y - h), \tag{3.4}$$

defined on a domain in $\mathcal{X} \times \mathbf{R}^m \times \mathbf{R}^p$ mapping into $\mathcal{X}$. (Note that F is not continuous on $\mathcal{X} \times \mathbf{R}^m \times \mathbf{R}^p$ but is continuous from $\mathcal{X}^1 \times \mathbf{R}^m \times \mathbf{R}^p$ to $\mathcal{X}$.)

The smoothness of $p_t(x)$ and consequently the sense in which (3.3) is to be understood depends on the smoothness of the initial data α (the other data are assumed smooth) and on the regularity of $u(\cdot)$ and $y(\cdot)$.

DEFINITION 3.1. *We say that a function $p_t(x)$ is a* smooth solution *of the dynamics* (3.3) *if $\alpha \in \mathcal{D} \cap \mathcal{X}^1$; and when $u \in \mathcal{U}(0)$ and $y \in \mathcal{Y}(0)$ are continuous,*

(i) *$p_t(x)$ is of class $C^1(\mathbf{R}^n \times [0, T])$ and*

(ii) *$p_t(x)$ satisfies* (3.3) *in $\mathbf{R}^n \times (0, T)$ in the usual sense.*

LEMMA 3.2. *If $\alpha \in \mathcal{D} \cap \mathcal{X}^1$, then $p_t(x)$ is the unique smooth solution of* (3.3), *and for any $u \in \mathcal{U}(0)$ and $y \in \mathcal{Y}(0)$, the information state p_t evolves in $\mathcal{D} \cap \mathcal{X}^1$ as*

$$p_t \in \mathcal{D} \cap \mathcal{X}^1, \qquad t \in [0, T], \tag{3.5}$$

whenever $\alpha \in \mathcal{D} \cap \mathcal{X}^1$. Moreover, the map $t \mapsto p_t$ from $[0, T]$ into $\mathcal{D} \cap \mathcal{X}^1$ is continuous, with modulus of continuity independent of $u \in \mathcal{U}(0)$, $y \in \mathcal{Y}(0)$.

Proof. 1. Let $\alpha \in \mathcal{D} \cap \mathcal{X}^1$, $u \in \mathcal{U}(0) \cap C([0, T], \mathbf{R}^m)$, and $y \in \mathcal{Y}(0) \cap C([0, T], \mathbf{R}^p)$. Then by the method of characteristics (see, e.g., [13]), we see that (3.3) has a unique solution given by (3.1) that is of class C^1, and moreover, the gradient has the representation

$$\begin{aligned} \nabla_x p_t(x) &= \nabla_x \alpha(x(0)) \\ &+ \int_0^t [\nabla_x L(x(s), u(s)) + \gamma^2 \nabla_w \ell(y(s) - h(x(s)))\, \nabla_x h(x(s))] \Xi(s)\, ds, \end{aligned} \tag{3.6}$$

where $x(\cdot)$ is the solution of (3.2) and

$$\begin{aligned} \dot{\Xi}(s) = [&\nabla_x f(x(s), u(s)) + \nabla_x g(x(s), y(s) - h(x(s))) \\ &- \nabla_w g(x(s), y(s) - h(x(s)) \nabla_x h(x(s))] \Xi(s), \end{aligned} \tag{3.7}$$

$0 \le s \le t$, where $\Xi(t) = I$.

2. These formulas are also valid for $u \in \mathcal{U}(0)$, $y \in \mathcal{Y}(0)$, by an approximation argument using continuous functions, as follows. Let $u^i \to u$ in $\mathcal{U}(0)$ and $y^i \to y$ in $\mathcal{Y}(0)$ as $i \to \infty$, where each u^i and y^i is continuous. We claim that

$$\lim_{i \to \infty} \sup_{0 \le t \le T} \| p_t^i - p_t \|_1 = 0, \tag{3.8}$$

where $p^i_\cdot$ and $p_\cdot$ denote the corresponding solutions of (3.3) with initial data α.

To prove this, let $x^i(\cdot)$ and $x(\cdot)$ denote the corresponding solutions of (3.2) with terminal data $x^i(t) = x, x(t) = x$. Then a standard estimate using the Gronwall and Holder inequalities gives

$$|x^i(s) - x(s)| \leq K(\| u^i - u \|_{L_2} + \| u^i - u \|_{L_2}),$$

for $s \in [0, t]$, where $K > 0$ is independent of $t \in [0, T]$, $x \in \mathbf{R}^n$. This implies

$$\begin{aligned}
|p_t^i(x) - p_t(x)| &\leq |\alpha(x^i(0)) - \alpha(x(0))| \\
&+ \int_0^t [|L(x^i(s), u^i(s)) - L(x(s), u(s))| + |\ell(y^i(s) - h(x^i(s))) - \ell(y(s) - h(x(s)))|]\, ds \\
&\leq \rho_\alpha(|x^i(0) - x(0)|) + K \int_0^t [|x^i(s) - x(s)| + |u^i(s) - u(s)| + |y^i(s) - y(s)|]\, ds \\
&\leq K(\| u^i - u \|_{L_2} + \| y^i - y \|_{L_2}),
\end{aligned}$$

uniformly, where ρ_α is a modulus of continuity function for α. A similar estimate for $|\nabla_x p_t^i(x) - \nabla_x p_t(x)|$ can be obtained using (3.6). This proves (3.8).

3. Note that by assumption and (3.6), $\nabla_x p_t(x)$ is bounded uniformly in $(x, t) \in \mathbf{R}^n \times [0, T]$. This implies $p_t \in \mathcal{X}^1$. The fact that $p_t \in \mathcal{D} \cap \mathcal{X}^1$ follows from the estimate (3.13).

4. Finally, we claim that there exists a modulus function ρ depending on α but independent of u and y such that

$$\| p_{t^2} - p_{t^1} \|_1 \leq \rho(|t^2 - t^1|). \tag{3.9}$$

To prove (3.9), assume $t^1 \leq t^2$. Let $x^i(\cdot)$ $(i = 1, 2)$ denote the solution of (3.2) with terminal data $x^i(t^i) = x$. Now

$$|x^2(t^1) - x| \leq K|t^2 - t^1|,$$

and for $0 \leq s \leq t^1$, using Gronwall's inequality,

$$|x^1(s) - x^2(s)| \leq K|x^1(t^1) - x^2(t^1)| \leq K|t^2 - t^1|.$$

Therefore

$$\begin{aligned}
|p_{t^2}(x) - p_{t^1}(x)| \leq |\alpha(x^2(0)) - \alpha(x^1(0))| &+ \int_0^{t^1} [|L(x^2(s), u(s)) - L(x^1(s), u(s))| \\
&+ K|\ell(y(s) - h(x^2(s))) - \ell(y(s) - h(x^1(s)))|]\, ds \\
&+ \int_{t^1}^{t^2} [|L(x^2(s), u(s))| + K|g(x^2(s), y(s) - h(x^2(s)))|]\, ds \\
\leq \rho_\alpha(|x^2(0) - x^1(0)|) + K \int_0^{t^1} &|x^2(s) - x^1(s)|\, ds + K|t^2 - t^1| \\
\leq \rho_\alpha(K|t^2 - t^1|) + K|t^2 - t^1|,&
\end{aligned}$$

uniformly in x, u, y. A similar estimate for $|\nabla_x p_{t^2}(x) - \nabla_x p_{t^1}(x)|$ can be obtained using (3.6). This completes the proof. □

If α is not differentiable, then (3.3) can be interpreted in the viscosity sense [13], [28].

DEFINITION 3.3. *We say that a function $p_t(x) \in C(\mathbf{R}^n \times [0, T])$ is a* viscosity subsolution *of the dynamics* (3.3) *if $\alpha \in \mathcal{D} \cap \mathcal{X}$, and if for all $\phi \in C^\infty(\mathbf{R}^n)$, $\psi \in L^1[0, T]$, whenever there* *exists* $(x', t') \in \mathbf{R}^n \times (0, T)$ *with* $p_{t'}(x') + \int_0^{t'} \psi(s)\,ds - \phi(x') = \max_{(x,t)\in\mathbf{R}^n\times[0,T]}(p_t(x) + \int_0^t \psi(s)\,ds - \phi(x))$, *then*

$$\lim_{\delta\to 0}\inf_{|t-t'|<\delta} \operatorname{ess\,inf}\{\psi(t) - \lambda\cdot(f(x,u(t)) + g(x,y(t)-h(x))) + L(x,u(t)) -\gamma^2\ell(y(t)-h(x)) \;:\; |x-x'|<\delta,\ |\lambda - \nabla_x\phi(x')| \le \delta\} \ge 0; \tag{3.10}$$

or a viscosity supersolution *of the dynamics* (3.3) *if* $\alpha \in \mathcal{D}\cap\mathcal{X}$, *and if for all* $\phi \in C^\infty(\mathbf{R}^n)$, $\psi \in L^1[0,T]$, *whenever there exists* $(x',t') \in \mathbf{R}^n\times(0,T)$ *with* $p_{t'}(x') + \int_0^{t'}\psi(s)\,ds - \phi(x') = \min_{(x,t)\in\mathbf{R}^n\times[0,T]}(p_t(x) + \int_0^t\psi(s)\,ds - \phi(x))$, *then*

$$\lim_{\delta\to 0}\sup_{|t-t'|<\delta} \operatorname{ess\,sup}\{\psi(t) - \lambda\cdot(f(x,u(t)) + g(x,y(t)-h(x))) + L(x,u(t)) -\gamma^2\ell(y(t)-h(x)) \;:\; |x-x'|<\delta,\ |\lambda - \nabla_x\phi(x')| \le \delta\} \le 0; \tag{3.11}$$

or a viscosity solution *if it is both a subsolution and a supersolution.*

LEMMA 3.4. *If* $\alpha \in \mathcal{D}\cap\mathcal{X}$, *then* $p_t(x)$ *is the unique viscosity solution of* (3.3). *Moreover, for any* $u \in \mathcal{U}(0)$ *and* $y \in \mathcal{Y}(0)$, *the information state* p_t *evolves in* $\mathcal{D}\cap\mathcal{X}$ *as*

$$p_t \in \mathcal{D}\cap\mathcal{X}, \qquad t \in [0,T], \tag{3.12}$$

whenever $\alpha \in \mathcal{D}\cap\mathcal{X}$.

Proof. Let $\alpha \in \mathcal{D}\cap\mathcal{X}$, $u \in \mathcal{U}(0)$, and $y \in \mathcal{Y}(0)$. From the formula (3.1) and from well-known continuity properties of ODEs, it is not hard to show that $p_t(x) \in C(\mathbf{R}^n \times [0,T])$, and we omit the details. The fact that $p_t(x)$ is the unique viscosity solution of (3.3) follows from the results in [28].

To show that $p_t \in \mathcal{D}\cap\mathcal{X}$, we must prove the estimate

$$-\underline{c}_1|x| - \underline{c}_2 \le p_t(x) \le -\bar{c}_1|x| + \bar{c}_2 \quad \text{for all } x \in \mathbf{R}^n,\ 0 \le t \le T, \tag{3.13}$$

where the constants are independent of $u \in \mathcal{U}(0)$, $y \in \mathcal{Y}(0)$ but may depend on α. To this end, let $x(\cdot)$ be the solution of (3.2). Then for $0 \le r \le s \le t \le T$, we have

$$|x(s)| \le |x(r)| + K|s-r| \tag{3.14}$$

for some constant $K > 0$. A similar inequality holds if $s \le r$. Therefore,

$$\begin{aligned} p_t(x) &\le \alpha(x(0)) + KT \\ &\le -c_1|x(0)| + c_2 + KT \\ &\le -c_1|x| + c_1KT + c_2 + KT. \end{aligned}$$

This proves the upper bound in (3.13). The lower bound is proven similarly. □

LEMMA 3.5. *The map* $(p,t) \mapsto p_T$ *from* $\mathcal{D}\cap\mathcal{X}\times[0,T]$ *into* $\mathcal{D}\cap\mathcal{X}$ *is continuous, where* p_T *denotes the solution of* (3.3) *at time T with initial data* $p_t = p$. *In addition, if* $u \in \mathcal{U}(0)$ *and* $y \in \mathcal{Y}(0)$ *are continuous, then the map* $t \mapsto p_t$ *from* $[0,T]$ *into* $\mathcal{X}$ *is* $\mathcal{X}$*-Frechet differentiable with continuous derivative* $t \mapsto F(p_t, u(t), y(t))$.

Proof. 1. Proof of first assertion. Fix $p^1 \in \mathcal{D}\cap\mathcal{X}$, and consider $p^2 \in \mathcal{D}\cap\mathcal{X}$ and $0 \le t^1 \le t^2 \le T$. If $u \in \mathcal{U}(t^1)$, $y \in \mathcal{Y}(t^1)$, then the natural truncations of u and y belong to $\mathcal{U}(t^2)$ and $\mathcal{Y}(t^2)$, respectively. Similarly, if $u \in \mathcal{U}(t^2)$, $y \in \mathcal{Y}(t^2)$ are given, then one can extend them to elements of $\mathcal{U}(t^1)$ and $\mathcal{Y}(t^1)$ by setting them equal to arbitrary but fixed elements of U and $\mathbf{R}^p$, respectively, on $[t^1, t^2]$. We use this convention to avoid any ambiguity in the sequel. We claim that there exist a constant $K > 0$ and modulus function ρ that may depend on p^1 such that

$$\| p_T^1 - p_T^2 \| \le K \| p^1 - p^2 \| + \rho(|t^2 - t^1|), \tag{3.15}$$

where p_T^i $(i = 1, 2)$ denotes the solution of (3.3) at time T with initial data $p_{t^i}^i = p^i$, and inputs $u(\cdot)$, $y(\cdot)$, with interpretations as explained above. This inequality implies that $(p, t) \mapsto p_T$ is continuous at $(p^1, t^1) \in \mathcal{D} \cap \mathcal{X} \times [0, T]$ (since (3.15) holds also if $t^2 \leq t^1$).

Fix x and let $x^i(\cdot)$ denote the corresponding solutions of (3.2) with terminal data $x^i(T) = x$. Then $x^1(s) = x^2(s)$ for $t^2 \leq s \leq T$ (in particular $x^1(t^2) = x^2(t^2)$),

$$|x^1(t^1) - x^1(t^2)| \leq K|t^2 - t^1|$$

and

$$|x^2(t^2)| \leq |x| + K.$$

Using these estimates,

$$\begin{aligned}
|p_T^1(x) - p_T^2(x)| &\leq |p^1(x^1(t^1)) - p^2(x^2(t^2))| \\
&\quad + \int_{t^1}^{t^2} |L(x^1(s), u(s)) - \gamma^2 \ell(y(s) - h(x^1(s)))|\, ds \\
&\leq |p_1(x^1(t^1)) - p_1(x^2(t^2))| + |p_1(x^2(t^2)) - p^2(x^2(t^2))| + K|t^2 - t^1| \\
&\leq \rho_{p^1}(|x^1(t^1) - x^2(t^2))|) + \| p^1 - p^2 \| (1 + |x^2(t^2)) + K|t^2 - t^1| \\
&\leq \rho_{p^1}(K|t^2 - t^1|) + K \| p^1 - p^2 \| (1 + |x|) + K|t^2 - t^1|.
\end{aligned}$$

This estimate implies (3.15) with $\rho(s) = \rho_{p^1}(K|s|) + K|s|$.

2. Proof of second assertion. Let $\alpha \in \mathcal{D} \cap \mathcal{X}^1$, and assume that u and y are continuous. We must prove that

$$\lim_{\delta \to 0} \frac{\| p_{t+\delta} - p_t - F(p_t, u(t), y(t))\delta \|}{\delta} = 0. \tag{3.16}$$

Since $p_t(x)$ is of class C^1, we have

$$\begin{aligned}
&\frac{|p_{t+\delta}(x) - p_t(x) - F(p_t, u(t), y(t))(x)\delta|}{\delta} \\
&= \left| \frac{1}{\delta} \int_t^{t+\delta} [F(p_s, u(s), y(s))(x) - F(p_t, u(t), y(t))(x)]\, ds \right| \\
&\leq \frac{1}{\delta} \int_t^{t+\delta} [|\nabla_x p_s(x) - \nabla_x p_t(x)||f(x, u(s)) + g(x, y(s) - h(x))| \\
&\quad + |\nabla_x p_t(x)||f(x, u(s)) - f(x, u(t)) + g(x)(y(s) - y(t))| \\
&\quad + |L(x, u(s)) - L(x, u(t))| + \gamma^2 K|y(s) - y(t)|]\, ds \\
&\leq \frac{1}{\delta} \int_t^{t+\delta} [K|\nabla_x p_s(x) - \nabla_x p_t(x)| \\
&\quad + K|f(x, u(s)) - f(x, u(t))| + K(1 + \gamma^2)|y(s) - y(t)| \\
&\quad + |L(x, u(s)) - L(x, u(t))|]\, ds \\
&\leq \frac{1}{\delta} \int_t^{t+\delta} C\rho(|s - t|)\, ds \to 0,
\end{aligned}$$

as $\delta \to 0$ uniformly, where ρ denotes a suitable modulus of continuity function. This follows because of our assumptions on the data and using (3.6), and is enough to prove (3.16). □

Using the definition (3.1), we have the following key representation theorem.

THEOREM 3.6. *For any* $\mathbf{u} \in \mathbf{U}(0)$ *we have*

$$J(\mathbf{u}) = \sup_{y \in \mathcal{Y}(0)} \{(p_T, \Phi) : p_0 = \alpha\}, \tag{3.17}$$

where $(p, \Phi) = \sup_{x \in \mathbf{R}^n}(p(x) + \Phi(x))$ *is the "sup-pairing"* [24].

Proof. For any $w \in \mathcal{W}(0)$, an output $y \in \mathcal{Y}(0)$ can be defined by solving the ODE (2.1) with $u(t) = \mathbf{u}[y](t)$, $0 \le t \le T$. Conversely, given any $y \in \mathcal{Y}(0)$, a disturbance $w \in \mathcal{W}(0)$ is defined by solving the ODE (3.2) with $u(t) = \mathbf{u}[y](t)$, $0 \le t \le T$, and setting $w(t) \stackrel{\Delta}{=} -h(x(t)) + y(t)$, $0 \le t \le T$. Therefore there is a natural bijection between $\mathcal{W}(0)$ and $\mathcal{Y}(0)$ (for each $\mathbf{u}$). Consequently,

$$\begin{aligned}
J(\mathbf{u}) &= \sup_{w \in \mathcal{W}(0),\, x_0 \in \mathbf{R}^n} \left\{\alpha(x_0) + \int_0^T [L(x(t), \mathbf{u}[y](t)) - \gamma^2 \ell(w(t))]\, dt + \Phi(x(T))\right\} \\
&= \sup_{y \in \mathcal{Y}(0),\, x_0 \in \mathbf{R}^n} \left\{\alpha(x_0) + \int_0^T [L(x(t), \mathbf{u}[y](t)) - \gamma^2 \ell(y(t) - h(x(t)))]\, dt + \Phi(x(T))\right\} \\
&= \sup_{y \in \mathcal{Y}(0),\, x \in \mathbf{R}^n} \left\{\alpha(x_0) + \int_0^T [L(x(t), \mathbf{u}[y](t)) - \gamma^2 \ell(y(t) - h(x(t)))]\, dt + \Phi(x(T)) \right. \\
&\qquad\qquad \left. : x(T) = x\right\} \\
&= \sup_{y \in \mathcal{Y}(0),\, x \in \mathbf{R}^n} \{p_T(x) + \Phi(x)\} \\
&= \sup_{y \in \mathcal{Y}(0)} \{(p_T, \Phi) : p_0 = \alpha\}. \qquad \square
\end{aligned}$$

The equivalent differential game with full state information is to minimize the right-hand side of (3.17) over $\mathbf{u} \in \mathbf{U}(0)$ subject to the infinite-dimensional dynamics (3.3).

We conclude this section with a brief discussion of an "adjoint" information state q_t, which runs backward in time and has the interesting property that the sup-pairing (p_t, q_t) is constant [24]. The *adjoint information state* is defined for fixed $u \in \mathcal{U}(t)$ and $y \in \mathcal{Y}(t)$ by

$$q_t(x) = \int_t^T [L(x(s), u(s)) - \gamma^2 \ell\,(y(s) - h(x(s)))]\, ds + \Phi(x(T)), \tag{3.18}$$

where $x(\cdot)$ is the solution of (3.2) on $[t, T]$ with initial data $x(t) = x$. The dynamics for the adjoint information state are

$$\begin{cases} \dot{q}_s = -F(-q_s, u(s), y(s)), & s \in [t, T], \\ q_T = \Phi. \end{cases} \tag{3.19}$$

THEOREM 3.7. *The sup-pairing of the information state and the adjoint information state is constant and expressed as*

$$(p_t, q_t) \text{ is independent of } t \in [0, T]. \tag{3.20}$$

Proof. The assertion can be verified easily by combining the definitions (3.1) and (3.18). Alternatively, suppose p_t and q_t are smooth solutions of (3.3) and (3.19), respectively. Define $\nu(t) = (p_t, q_t) = p_t(\bar{x}(t)) + q(\bar{x}(t))$. Then $\nabla_x p_t(\bar{x}(t)) = -\nabla_x q_t(\bar{x}(t))$ and

$$\begin{aligned}\dot{\nu}(t) &= \frac{\partial p_t}{\partial t}(\bar{x}(t)) + \frac{\partial q_t}{\partial t}(\bar{x}(t)) \\ &= -\nabla_x p_t(\bar{x}(t)) \cdot (f(\bar{x}(t), u(t)) + g(\bar{x}(t), y(t) - h(\bar{x}(t)))) \\ &\quad + L(\bar{x}(t), u(t)) - \gamma^2 \ell(y - h(\bar{x}(t))) \\ &\quad - \nabla_x q_t(\bar{x}(t)) \cdot (f(\bar{x}(t), u(t)) + g(\bar{x}(t), y(t) - h(\bar{x}(t)))) \\ &\quad - L(\bar{x}(t), u(t)) + \gamma^2 \ell(y - h(\bar{x}(t))) = 0.\end{aligned}$$

This shows that $\nu(t) = (\alpha, q_0) = (p_T, \Phi)$ is constant, as required. □

4. Value function and the HJI equation. Given Theorem 3.6, one can now apply dynamic programming methods to solve the equivalent problem and, hence, the original partially observed problem. The value function is defined for $(p, t) \in \mathcal{D} \cap \mathcal{X} \times [0, T]$ by

$$W(p, t) = \inf_{\mathbf{u} \in \mathbf{U}(t)} \sup_{y \in \mathcal{Y}(t)} \{(p_T, \Phi) \ : \ p_t = p\}. \tag{4.1}$$

This function is finite, as the following lemma shows.

LEMMA 4.1. *For all* $(p, t) \in \mathcal{D} \cap \mathcal{X} \times [0, T]$ *we have*

$$(p, 0) - K \leq W(p, t) \leq K + (p, 0) \tag{4.2}$$

for some constant $K > 0$.

Proof. For any $\mathbf{u} \in \mathbf{U}(t)$ and $y \in \mathcal{Y}(t)$ we have

$$\begin{aligned} p_T(x(T)) + \Phi(x(T)) &= p(x(t)) + \int_t^T [L(x(s), \mathbf{u}[y](s)) - \gamma^2 \ell(y(s) - h(x(s)))]\, ds + \Phi(x(T)) \\ &\leq p(x(t)) + K, \end{aligned}$$

where $K > 0$ does not depend on $\mathbf{u}$, y, and hence

$$(p_T, \Phi) = \sup_{x(T)} \{p(x(T)) + \Phi(x(T))\} \leq \sup_{x(t)} \{p(x(t)) + K\} = (p, 0) + K.$$

This proves the upper bound in (4.2).

To obtain the lower estimate in (4.2), select $x \in \operatorname{argmax} p$. Then

$$(p_T, \Phi) \geq p(x) + \int_t^T [L(x(s), \mathbf{u}[y](s)) - \gamma^2 \ell(y(s) - h(x(s)))]\, ds + \Phi(x(T)) \geq (p, 0) - K,$$

where $x(\cdot)$ is the solution of (2.1) with initial data $x(t) = x$. □

In the next lemma, $B(0, R)$ denotes the ball of radius R centered at 0 in $\mathbf{R}^n$.

LEMMA 4.2. *Fix* $p^1 \in \mathcal{D} \cap \mathcal{X}$. *Then there exist* $\delta^1 > 0$ *and* $R^1 > 0$ *such that* $\| p^2 - p^1 \| < \delta^1$ *implies that* $\operatorname{argmax}_{x \in \mathbf{R}^n} (p^2(x) + \Phi(x)) \subset B(0, R^1)$.

Proof. Since $p^1 \in \mathcal{D}$ and Φ is bounded, $p^1(x) + \Phi(x) \leq -c_1|x| + c_2 + K$. Then for $p^2 \in \mathcal{X}$,

$$p^2(x) + \Phi(x) = p^1(x) + \Phi(x) + (p^2(x) - p^1(x)) \leq -c_1|x| + c_2 + K + \| p^2 - p^1 \| (1 + |x|).$$

Set $\delta^1 = c_1/2$. Then for $\| p^2 - p^1 \| < \delta^1$,

$$p^2(x) + \Phi(x) \leq -c_1'|x| + c_2', \tag{4.3}$$

where $c_1' = c_1/2$, $c_2' = c_2 + K + \delta^1$.

Next, select a sequence x_i such that $\lim_{i\to\infty} p^2(x_i) = \sup_{x\in\mathbf{R}^n} p^2(x) = (p^2, 0) < +\infty$. Fix $\varepsilon > 0$. Then for all large i,

$$(p^2, \Phi) - \varepsilon \leq p^2(x_i) \leq -c_1'|x_i| + c_2',$$

and hence

$$|x_i| \leq R^1 \tag{4.4}$$

for some constant $R^1 > 0$ depending on p^1 and δ^1. Thus the sequence x_i is bounded, and any limit point x^2 satisfies $|x^2| \leq R^1$. Hence $\operatorname{argmax}_{x\in\mathbf{R}^n}(p^2(x) + \Phi(x)) \subset B(0, R^1)$. □

THEOREM 4.3. *The value function* $W(p, t)$ *defined by* (4.1) *is continuous, denoted*

$$W \in C(\mathcal{D} \cap \mathcal{X} \times [0, T]).$$

Proof. Fix $(p^1, t^1) \in \mathcal{D} \cap \mathcal{X} \times [0, T]$. Given $\varepsilon > 0$, we will show that there exists $\delta > 0$ (depending on p^1) such that $\| p^2 - p^1 \| < \delta$ and $|t^2 - t^1| < \delta$ imply

$$|W(p^1, t^1) - W(p^2, t^2)| \leq \varepsilon. \tag{4.5}$$

The proof of this assertion is based on the proof of [11, Thm. 3.2].

Assume that $0 \leq t^1 \leq t^2 \leq T$ and $0 < \delta < \delta^1$, where δ^1, R^1 are as in Lemma 4.2, and that $\| p^2 - p^1 \| < \delta$, $|t^2 - t^1| < \delta$.

Choose $\mathbf{u} \in \mathbf{U}(t^1)$ such that

$$W(p^1, t^1) \geq \sup_{y\in\mathcal{Y}(t^1)} \{(p_T^1, \Phi)\} - \varepsilon/3,$$

where p_s^1 is the solution of (3.3) with initial data $p_{t^1}^1 = p^1$ and using this $\mathbf{u}$ and any y. For any $y \in \mathcal{Y}(t^2)$ define $\tilde{y} \in \mathcal{Y}(t^1)$ by

$$\tilde{y}(s) = \begin{cases} 0, & t^1 \leq s < t^2, \\ y(s), & t^2 \leq s \leq T. \end{cases}$$

Define $\tilde{\mathbf{u}} \in \mathbf{U}(t^2)$ by

$$\tilde{\mathbf{u}}[y] = \mathbf{u}[\tilde{y}] \quad \text{for all } y \in \mathcal{Y}(t^2).$$

Select $y \in \mathcal{Y}(t^2)$ such that

$$W(p^2, t^2) \leq (p_T^2, \Phi) + \varepsilon/3,$$

where p_s^2 is the solution of (3.3) with initial data $p_{t^2}^2 = p^2$ and using $\tilde{\mathbf{u}}$ and y. Then

$$W(p^2, t^2) - W(p^1, t^1) \leq (p_T^2, \Phi) - (p_T^1, \Phi) + 2\varepsilon/3 = p_T^2(x^2) - p_T^1(x^2) + 2\varepsilon/3, \tag{4.6}$$

where $x^2 \in \operatorname{argmax}\{p^2 + \Phi\}$. By Lemma 4.2, since Φ is bounded, $|x^2| \leq R^1$. Then using $u(s) = \mathbf{u}[\tilde{y}](s)$ and $\tilde{y}(s)$, $s \in [t^1, T]$, and $s \in [t^2, T]$, inequality (3.15) of Lemma 3.5 implies that

$$p_T^2(x^2) - p_T^1(x^2) \leq (K\delta + \rho(|t^2 - t^1|))(1 + |x^2|) \leq (K\delta + \rho(|t^2 - t^1|))(1 + R^1).$$

Therefore there exists $\delta^2 < \delta^1$ such that $\delta < \delta^2$ implies, using (4.6),

$$W(p^2, t^2) - W(p^1, t^1) \leq \varepsilon. \tag{4.7}$$

The proof of the opposite inequality is similar. Choose $\mathbf{u} \in \mathbf{U}(t^2)$ such that

$$W(p^2, t^2) \geq \sup_{y \in \mathcal{Y}(t^2)} \{(p_T^2, \Phi)\} - \varepsilon/3,$$

where p_s^2 is the solution of (3.3) with initial data $p_{t^2}^2 = p^2$ and using $\mathbf{u}$ and any y. For all $y \in \mathcal{Y}(t^1)$, define $\tilde{y} \in \mathcal{Y}(t^2)$ by $\tilde{y} = y$ on $[t^2, T]$. Fix $u_0 \in U$. Define $\tilde{\mathbf{u}} \in \mathbf{U}(t^1)$ by

$$\tilde{\mathbf{u}}[y] = \begin{cases} u_0, & t^1 \leq s \leq t^2, \\ \mathbf{u}[\tilde{y}](s), & t^2 \leq s \leq T. \end{cases}$$

Now choose $y \in \mathcal{Y}(t^1)$ such that

$$W(p^1, t^1) \leq (p_T^1, \Phi) + \varepsilon/3,$$

where p_s^1 is the solution of (3.3) with initial data $p_{t^1}^1 = p^1$ and using $\tilde{\mathbf{u}}$ and y. Therefore,

$$W(p^1, t^1) - W(p^2, t^2) \leq (p_T^1, \Phi) - (p_T^2, \Phi) + 2\varepsilon/3,$$

and proceeding as above there exists $\delta^3 \leq \delta^2$ such that $\delta < \delta^3$ implies

$$W(p^1, t^1) - W(p^2, t^2) \leq \varepsilon. \tag{4.8}$$

Inequalities (4.7) and (4.8) are both valid for $\delta < \delta^3$, hence (4.5). □

The principle of optimality (dynamic programming principle) for this problem is as follows.

THEOREM 4.4. *For any* $0 \leq t \leq r \leq T$ *we have*

$$W(p, t) = \inf_{\mathbf{u} \in \mathbf{U}(t)} \sup_{y \in \mathcal{Y}(t)} \{W(p_r, r) \ : \ p_t = p\}. \tag{4.9}$$

Proof. The proof uses the same methods as in [10], [11].

Indeed, let $R(p, t)$ denote the right-hand side of (4.9), and fix $\varepsilon > 0$. Choose $\mathbf{u}^1 \in \mathbf{U}(p, t)$ such that

$$R(p, t) \geq \sup_{y \in \mathcal{Y}(t)} \{W(p_r, r)\} - \varepsilon.$$

For any $q \in \mathcal{D} \cap \mathcal{X}$ there exists $\mathbf{u}^2 \in \mathbf{U}(q, r)$ such that

$$W(q, r) \geq \sup_{y \in \mathcal{Y}(t)} \{(p_T, \Phi)\} - \varepsilon,$$

where $p_r = q$. Define $\mathbf{u}^3 \in \mathbf{U}(p, t)$ by

$$\mathbf{u}^3(y)(s) = \begin{cases} \mathbf{u}^1(p, y)(s), & t \leq s \leq r, \\ \mathbf{u}^2(p_r, y)(s), & r \leq s \leq T. \end{cases}$$

Then for any $y \in \mathcal{Y}(t)$ we have, using the control $\mathbf{u}^3$,

$$\begin{aligned} R(p, t) &\geq W(p_r, r) - \varepsilon \\ &\geq (p_T, \Phi) - 2\varepsilon; \end{aligned}$$

hence

$$(p_T, \Phi) \leq R(p, t) + 2\varepsilon \quad \text{for all } y \in \mathcal{Y}(t).$$

Therefore

$$\sup_{y\in\mathcal{Y}(t)} \{(p_T, \Phi)\} \le R(p,t) + 2\varepsilon \quad (\text{using } \mathbf{u}^3).$$

This implies

$$W(p,t) \le R(p,t) + 2\varepsilon. \tag{4.10}$$

To prove the opposite inequality, choose $\mathbf{u} \in \mathbf{U}(p,t)$ such that

$$W(p,t) \ge \sup_{y\in\mathcal{Y}(t)} \{(p_T, \Phi)\} - \varepsilon. \tag{4.11}$$

Then

$$R(p,t) \le \sup_{y\in\mathcal{Y}(t)} \{W(p_r, r)\},$$

and there exists $y^1 \in \mathcal{Y}(t)$ such that

$$R(p,t) \le W(p_r, r) + \varepsilon.$$

For each $y \in \mathcal{Y}(r)$, define $\tilde{y} \in \mathcal{Y}(t)$ by

$$\tilde{y}(s) = \begin{cases} y^1(s), & t \le s \le r, \\ y(s), & r \le s \le T. \end{cases}$$

Then define $\tilde{\mathbf{u}} \in \mathbf{U}(q, r)$ ($q = p_r$ results from $\mathbf{u}$, y^1, and $p_t = p$) by $\tilde{\mathbf{u}}(y)(s) = \mathbf{u}(\tilde{y})(s)$, $r \le s \le T$. Then

$$W(p_r, r) \le \sup_{y\in\mathcal{Y}(r)} \{(p_T, \Phi) \;:\; p_r = q\},$$

and there exists $y^2 \in \mathcal{Y}(r)$ such that

$$W(p_r, r) \le (p_T, \Phi) + \varepsilon.$$

Define $y^3 \in \mathcal{Y}(t)$ by

$$y^3(s) = \begin{cases} y^1(s), & t \le s \le r, \\ y^2(s), & r \le s \le T. \end{cases}$$

Therefore we have

$$R(p,t) \le W(p_r, r) + \varepsilon \le (p_T, \Phi) + 2\varepsilon,$$

which implies, by (4.11),

$$R(p,t) \le \sup_{y\in\mathcal{Y}(t)} \{(p_T, \Phi)\} + 2\varepsilon \le W(p,t) + 3\varepsilon. \tag{4.12}$$

Since $\varepsilon > 0$ was arbitrary, inequalities (4.10) and (4.12) imply (4.9). □

Equation (4.9) leads to the dynamic programming equation (DPE)

$$\begin{cases} \dfrac{\partial W}{\partial t} + \inf_{u\in U} \sup_{y\in \mathbf{R}^p} \langle \nabla_p W(p,t), F(p,u,y)\rangle = 0 & \text{in } \mathcal{D} \cap \mathcal{X}^1 \times (0,T), \\ W(p,T) = (p, \Phi) & \text{in } \mathcal{D} \cap \mathcal{X}. \end{cases} \tag{4.13}$$

In (4.13), $\nabla_p W(p,t)$ denotes the gradient of W with respect to p and, if it exists, belongs to the dual space $\mathcal{X}^*$ and $\langle \lambda, p\rangle$ denotes the value of $\lambda \in \mathcal{X}^*$ at $p \in \mathcal{X}$. In view of the structure of F (see (3.4)), the order of inf and sup in (4.13) is immaterial; i.e., the Isaacs condition holds.

The DPE (4.13) is the appropriate Hamilton–Jacobi–Isaacs (HJI) equation for the partially observed differential game.

We will make use of two classes $\mathcal{C}^1 \subset \mathcal{C} \subset C(\mathcal{D} \cap \mathcal{X} \times [0, T])$ of test functions. We take $\phi \in \mathcal{C}$ to mean that

(i) ϕ is $\mathcal{X}$-Frechet differentiable, with derivative denoted $(\nabla_p \phi, \frac{\partial \phi}{\partial t})$;

(ii) the Frechet derivative $(\nabla_p \phi, \frac{\partial \phi}{\partial t})$ is continuous on $\mathcal{D} \cap \mathcal{X} \times [0, T]$; and $\phi \in \mathcal{C}^1$ means that in addition

(iii) the Frechet derivative $(\nabla_p \phi, \frac{\partial \phi}{\partial t})$ is continuous on $\mathcal{D} \cap \mathcal{X}^1 \times [0, T]$.

These classes of functions will be used to define smooth and viscosity solutions of (4.13).

LEMMA 4.5. *Let $p \in \mathcal{D} \cap \mathcal{X}^1$ so that $p_r(x)$ is a smooth solution of* (3.3) *on* $[t, T]$ *with initial data $p_t = p$, and let $\phi \in \mathcal{C}$. Then we have the following version of the fundamental theorem of calculus:*

$$\phi(p_r, r) = \phi(p_t, t) + \int_t^r \left[\frac{\partial \phi}{\partial t}(p_s, s) + \langle \nabla_p \phi(p_s, s), F(p_s, u(s), y(s)) \rangle \right] ds. \tag{4.14}$$

Proof. Let u and y be continuous. Then the function $r \mapsto \phi(p_r, r)$ is continuously differentiable, and so by the usual fundamental theorem of calculus, (4.14) holds.

By an approximation argument, (4.14) holds for all $u \in \mathcal{U}(t)$, $y \in \mathcal{Y}(t)$ as follows. Let $u^i \to u$ in $\mathcal{U}(0)$ and $y^i \to y$ in $\mathcal{Y}(0)$ as $i \to \infty$, where each u^i and y^i is continuous. Then

$$\phi(p_r^i, r) = \phi(p_t^i, t) + \int_t^r \left[\frac{\partial \phi}{\partial t}(p_s^i, s) + \langle \nabla_p \phi(p_s^i, s), F(p_s^i, u^i(s), y^i(s)) \rangle \right] ds. \tag{4.15}$$

We claim that

$$\lim_{i \to \infty} \phi(p_r^i, r) = \phi(p_r, r), \qquad \lim_{i \to \infty} \phi(p_t^i, t) = \phi(p_t, t), \tag{4.16}$$

and

$$\begin{aligned} \lim\nolimits_{i \to \infty} & \int_t^r \left[\frac{\partial \phi}{\partial t}(p_s^i, s) + \langle \nabla_p \phi(p_s^i, s), F(p_s^i, u^i(s), y^i(s)) \rangle \right] ds \\ &= \int_t^r \left[\frac{\partial \phi}{\partial t}(p_s, s) + \langle \nabla_p \phi(p_s, s), F(p_s, u(s), y(s)) \rangle \right] ds, \end{aligned} \tag{4.17}$$

where p^i and $p_.$ denote the corresponding solutions of (3.3) with initial data $p \in \mathcal{D} \cap \mathcal{X}^1$ at time t. By (3.8) in Lemma 3.2, (4.16) follows directly by continuity. Thus it remains to prove (4.17). This can be done by showing that

$$\begin{aligned} &\left| \int_t^r \left[\frac{\partial \phi}{\partial t}(p_s^i, s) + \langle \nabla_p \phi(p_s^i, s), F(p_s^i, u^i(s), y^i(s)) \rangle \right. \right. \\ &\qquad \left. \left. - \frac{\partial \phi}{\partial t}(p_s, s) - \langle \nabla_p \phi(p_s, s), F(p_s, u(s), y(s)) \rangle \right] ds \right| \\ &\le \int_t^r \left| \frac{\partial \phi}{\partial t}(p_s^i, s) - \frac{\partial \phi}{\partial t}(p_s, s) \right| + |\langle \nabla_p \phi(p_s^i, s) - \nabla_p \phi(p_s, s), F(p_s, u(s), y(s)) \rangle| \\ &\qquad + |\langle \nabla_p \phi(p_s^i, s), F(p_s^i, u^i(s), y^i(s)) - F(p_s, u(s), y(s)) \rangle|] ds \\ &\le \int_t^r \left| \frac{\partial \phi}{\partial t}(p_s^i, s) - \frac{\partial \phi}{\partial t}(p_s, s) \right| + \| \nabla_p \phi(p_s^i, s) - \nabla_p \phi(p_s, s) \|_* \| F(p_s, u(s), y(s)) \| \\ &\qquad + \| \nabla_p \phi(p_s^i, s) \|_* \| F(p_s^i, u^i(s), y^i(s)) - F(p_s, u(s), y(s)) \|] ds \to 0 \end{aligned} \tag{4.18}$$

as $i \to \infty$. Here, $\|\cdot\|$ denotes the norm on $\mathcal{X}$ as in §2 and $\|\cdot\|_*$ indicates the norm on the dual space $\mathcal{X}^*$. We treat each term as follows.

Given $\varepsilon > 0$, the compactness of $[t, r]$, the assumed continuity of the partial derivatives, and the uniform convergence (3.8) imply that for i sufficiently large,

$$\tag{4.19} \begin{aligned} &\sup_{t\le s\le r} \| \nabla_p \phi(p_s^i, s) - \nabla_p \phi(p_s, s) \|_* \le \varepsilon, \\ &\sup_{t\le s\le r} \left| \frac{\partial \phi}{\partial t}(p_s^i, s) - \frac{\partial \phi}{\partial t}(p_s, s) \right| \le \varepsilon. \end{aligned}$$

This also implies

$$\tag{4.20} \sup_{t\le s\le r} \| \nabla_p \phi(p_s^i, s) \|_* \le K.$$

Next, because of (3.8) and the assumed bounds on the problem data, we have

$$\tag{4.21} \sup_{t\le s\le r} \| F(p_s, u(s), y(s)) \| \le K.$$

Finally, it is not hard to verify the estimate

$$\tag{4.22} \begin{aligned} &\int_r^t \| F(p_s^i, u^i(s), y^i(s)) - F(p_s, u(s), y(s)) \| \, ds \\ &\le K \sup_{t\le s\le r} \| p_s^i - p_s \|_1 + K \left(1 + \sup_{t\le s\le r} \| p_s^i \|_1 \right) \\ &\quad \cdot \int_t^r [|u^i(s) - u(s)| + |y^i(s) - y(s)|] \, ds \end{aligned}$$

Therefore using (4.19), (4.20), (4.21), (4.22) in (4.18), we have

$$(4.18) \le T\varepsilon + TK\varepsilon + K \int_t^r [|u^i(s) - u(s)| + |y^i(s) - y(s)|] \, ds \le K\varepsilon$$

for all i sufficiently large, for a suitable constant $K > 0$ not depending on i. This completes the proof. □

DEFINITION 4.6. *A function* $W : \mathcal{D} \cap \mathcal{X} \times [0, T] \to \mathbf{R}$ *is called a* smooth solution *of the DPE* (4.13) *if*

(i) $W \in C^1$;

(ii) W *satisfies* (4.13) *in* $\mathcal{D} \cap \mathcal{X}^1 \times (0, T)$ *in the usual sense.*

In general, it is too much to expect that the DPE will have smooth solutions, and so one must appeal to a weaker notion of solution. To this end, we will show that the value function W is a viscosity solution of (4.13) in a suitable sense. The definition we provide below is consistent with our definition of smooth solution and is a generalization of it. We do not know a proof of uniqueness, and it may be the case that the definition has to be modified to achieve this. Consequently, Definition 4.7 is a provisional one. An abstract formulation of the viscosity solution definition is given in [13]. It is not clear at present how our definition relates to those appearing in [7], [17], [27]; indeed, the PDE (4.13) does not appear to be covered by existing theory.

DEFINITION 4.7 (provisional viscosity solution). *We say that a function* $W \in C(\mathcal{D} \cap \mathcal{X} \times [0, T])$ *is a* viscosity subsolution *of the DPE* (4.13) *if for all* $\phi \in C$, *whenever there exists* $(p', t') \in \mathcal{D} \cap \mathcal{X}^1 \times (0, T)$ *with* $W(p', t') - \phi(p', t') = \max_{(p,t) \in \mathcal{D}\cap\mathcal{X}\times[0,T]} (W(p, t) - \phi(p, t)) = 0$, *then*

$$\tag{4.23} \frac{\partial \phi}{\partial t}(p', t') + \inf_{u\in U} \sup_{y\in \mathbf{R}^p} \langle \nabla_p \phi(p', t'), F(p', u, y) \rangle \ge 0;$$

a viscosity supersolution *of the DPE* (4.13) *if for all* $\phi \in \mathcal{C}$, *whenever there exists* $(p', t') \in \mathcal{D} \cap \mathcal{X}^1 \times (0, T)$ *with* $W(p', t') - \phi(p', t') = \min_{(p,t) \in \mathcal{D} \cap \mathcal{X} \times [0,T]} (W(p, t) - \phi(p, t)) = 0$, *then*

$$\frac{\partial \phi}{\partial t}(p', t') + \inf_{u \in U} \sup_{y \in \mathbf{R}^p} \langle \nabla_p \phi(p', t'), F(p', u, y) \rangle \leq 0; \tag{4.24}$$

and a viscosity solution *if it is both a subsolution and a supersolution.*

The proof that W is a viscosity solution of (4.13) requires the following result (cf. [11, Lem. 4.3]).

LEMMA 4.8. *Let* $\phi \in \mathcal{C}$. *Assume that* ϕ *satisfies*

$$\frac{\partial \phi}{\partial t}(p', t') + \inf_{u \in U} \sup_{y \in \mathbf{R}^p} \langle \nabla_p \phi(p', t'), F(p', u, y) \rangle \leq -\theta, \tag{4.25}$$

where $\theta > 0$, $p' \in \mathcal{D} \cap \mathcal{X}^1$, $t' \in [0, T]$. *Then there exists* $\delta_0 > 0$, $\mathbf{u} \in \mathbf{U}(t')$ *such that for all* $\delta < \delta_0$, $y \in \mathcal{Y}(t')$

$$\int_{t'}^{t'+\delta} \left[\frac{\partial \phi}{\partial t}(p_s, s) + \langle \nabla_p \phi(p_s, s), F(p_s, \mathbf{u}[y](s), y(s)) \rangle \right] ds \leq -\delta\theta/2. \tag{4.26}$$

Similarly, if ϕ *satisfies*

$$\frac{\partial \phi}{\partial t}(p', t') + \inf_{u \in U} \sup_{y \in \mathbf{R}^p} \langle \nabla_p \phi(p', t'), F(p', u, y) \rangle \geq \theta, \tag{4.27}$$

where $\theta > 0$, $p' \in \mathcal{D} \cap \mathcal{X}^1$, $t' \in [0, T]$, *then there exists* $\delta_1 > 0$, $y \in \mathcal{Y}(t')$ *such that for all* $\delta < \delta_1$, $\mathbf{u} \in \mathbf{U}(t')$

$$\int_{t'}^{t'+\delta} \left[\frac{\partial \phi}{\partial t}(p_s, s) + \langle \nabla_p \phi(p_s, s), F(p_s, \mathbf{u}[y](s), y(s)) \rangle \right] ds \geq \delta\theta/2. \tag{4.28}$$

Proof. Write

$$\Lambda(p, t, u, y) = \frac{\partial \phi}{\partial t}(p, t) + \langle \nabla_p \phi(p, t), F(p, u, y) \rangle.$$

Since $\phi \in \mathcal{C}$, $\Lambda : \mathcal{D} \cap \mathcal{X}^1 \times [0, T] \times U \times \mathbf{R}^p \to \mathbf{R}$ is continuous. In fact, for $\| p - p' \|_1 < \nu$, $|t - t'| < \nu$ ($\nu > 0$ small), we have the estimate

$$\begin{aligned} |\Lambda(p, t, u, y) - \Lambda(p', t', u', y')| \leq \left| \frac{\partial \phi}{\partial t}(p, t) - \frac{\partial \phi}{\partial t}(p', t') \right| \\ + K \| \nabla_p \phi(p, t) - \nabla_p(p', t') \|_* + K(\| p - p' \|_1 + |u - u'| + |y - y'|), \end{aligned} \tag{4.29}$$

where $K > 0$ depends on p', t', and ν.

By (4.25), $\inf_u \sup_y \Lambda(p', t', u, y) \leq -\theta$. Select $u_0 \in \operatorname{argmin}_u \sup_y \Lambda(p', t', u, y)$, which does not depend on y because the Isaacs condition holds. Therefore

$$\Lambda(p', t', u_0, y) \leq -\theta \quad \text{for all } y \in \mathbf{R}^p.$$

Define $\mathbf{u} \in \mathbf{U}(t')$ by $\mathbf{u}[y] \equiv u_0$. Let $y \in \mathcal{Y}(t')$. As in (3.9), Lemma 3.2, the map $s \mapsto p_s$ from $[t', T]$ into $\mathcal{D} \cap \mathcal{X}^1$ is continuous, with modulus of continuity independent of u, y. Thus there exists $\delta_0 > 0$ such that if $\delta < \delta_0$ and $t' \leq s \leq t' + \delta$, then

$$\Lambda(p_s, s, \mathbf{u}[y](s), y(s)) \leq -\theta/2.$$

Integrating from t' to $t' + \delta$ gives (4.26).

To prove (4.28), we note that (4.27) implies the existence of $y_0 \in \mathbf{R}^p$ (independent of u) such that

$$\Lambda(p', t', u, y_0) \geq \theta \quad \text{for all } u \in U.$$

Define $y \in \mathcal{Y}(t')$ by $y \equiv y_0$, and let $\mathbf{u} \in \mathbf{U}(t')$. Then by continuity, with $\delta < \delta_1$ (some $\delta_1 > 0$), $t' \leq s \leq t' + \delta$ implies

$$\Lambda(p_s, s, \mathbf{u}[y](s), y(s)) \geq \theta/2.$$

Integrating from t' to $t' + \delta$ gives (4.28). □

THEOREM 4.9. *The value function $W(p, t)$ defined by* (4.1) *is a continuous viscosity solution of the DPE* (4.13).

Proof. To show that $W(p, t)$ is a viscosity subsolution, assume there exist $\phi \in \mathcal{C}$, $(p', t') \in \mathcal{D} \cap \mathcal{X}^1 \times (0, T)$ with $W(p', t') - \phi(p', t') = \max_{(p,t) \in \mathcal{D} \cap \mathcal{X} \times [0,T]}(W(p, t) - \phi(p, t)) = 0$. We must show that ϕ satisfies (4.23). If not, then there exists $\theta > 0$ such that (4.25) holds. By Lemma 4.8, (4.26) holds, which implies

$$(4.30) \quad \inf_{\mathbf{u} \in \mathbf{U}(t')} \sup_{y \in \mathcal{Y}(t')} \left\{ \int_{t'}^{t'+\delta} \left[\frac{\partial \phi}{\partial t}(p_s, s) + \langle \nabla_p \phi(p_s, s), F(p_s, \mathbf{u}[y](s), y(s)) \rangle \right] ds \right\} \leq -\delta\theta/2.$$

Now

$$W(p', t') = \phi(p', t') \quad \text{and} \quad W(p, t) \leq \phi(p, t);$$

hence the dynamic programming principle (4.9) with $t = t'$, $r = t' + \delta$ implies

$$0 \leq \inf_{\mathbf{u} \in \mathbf{U}(t')} \sup_{y \in \mathcal{Y}(t')} \{\phi(p_{t'+\delta}, t' + \delta) - \phi(p', t')\}.$$

Since $\phi \in \mathcal{C}$, Lemma 4.5 and this inequality imply

$$(4.31) \quad 0 \leq \inf_{\mathbf{u} \in \mathbf{U}(t')} \sup_{y \in \mathcal{Y}(t')} \left\{ \int_{t'}^{t'+\delta} \left[\frac{\partial \phi}{\partial t}(p_s, s) + \langle \nabla_p \phi(p_s, s), F(p_s, \mathbf{u}[y](s), y(s)) \rangle \right] ds \right\}.$$

But (4.31) contradicts (4.30), hence (4.23) is valid. Therefore $W(p, t)$ is a viscosity subsolution of (4.13).

Now suppose there exists $\phi \in \mathcal{C}$, $(p', t') \in \mathcal{D} \cap \mathcal{X}^1 \times (0, T)$ with

$$W(p', t') - \phi(p', t') = \min_{(p,t) \in \mathcal{D} \cap \mathcal{X} \times [0,T]} (W(p, t) - \phi(p, t)) = 0.$$

If (4.24) does not hold, then there exists $\theta > 0$ such that (4.27) holds. Then by Lemma 4.8, (4.28) holds, implying

$$(4.32) \quad \inf_{\mathbf{u} \in \mathbf{U}(t')} \sup_{y \in \mathcal{Y}(t')} \left\{ \int_{t'}^{t'+\delta} \left[\frac{\partial \phi}{\partial t}(p_s, s) + \langle \nabla_p \phi(p_s, s), F(p_s, \mathbf{u}[y](s), y(s)) \rangle \right] ds \right\} \geq \delta\theta/2.$$

Now

$$W(p', t') = \phi(p', t') \quad \text{and} \quad W(p, t) \geq \phi(p, t),$$

and by dynamic programming,

$$0 \geq \inf_{\mathbf{u} \in \mathbf{U}(t')} \sup_{y \in \mathcal{Y}(t')} \{\phi(p_{t'+\delta}, t' + \delta) - \phi(p', t')\}.$$

This implies

$$(4.33) \quad 0 \geq \inf_{\mathbf{u} \in \mathbf{U}(t')} \sup_{y \in \mathcal{Y}(t')} \left\{ \int_{t'}^{t'+\delta} \left[\frac{\partial \phi}{\partial t}(p_s, s) + \langle \nabla_p \phi(p_s, s), F(p_s, \mathbf{u}[y](s), y(s)) \rangle \right] ds \right\},$$

contradicting (4.32). Therefore $W(p, t)$ is a viscosity supersolution of (4.13). □

5. Verification theorem. The main reason for defining value functions and using dynamic programming is to determine optimal controls. Typically, some type of smoothness is required. The following theorem says essentially that if both (3.3) and (4.13) have smooth solutions, then the optimal control is obtained by finding the control value $\mathbf{u}^*(p,t)$ that attains the minimum in (4.13) as

$$\mathbf{u}^*[y](t) = \mathbf{u}^*(p[y]_t, t). \tag{5.1}$$

This control is an *information state feedback* controller and depends on the output y via the information state. This is a type of *separation principle* for this partially observed differential game.

THEOREM 5.1. *Assume that there exists a smooth solution* $\tilde{W}$ *of the DPE* (4.13). *If there exist* $\mathbf{u}^* \in \mathbf{U}(t)$, $y^* \in \mathcal{Y}(t)$ *such that*

$$\mathbf{u}^*(s) \in \operatorname*{argmin}_{u\in U}\{\langle \nabla_p W(p_s, s), -\nabla_x p_s \cdot f(\cdot, u) + L(\cdot, u)\rangle\}, \tag{5.2}$$

$$y^*(s) \in \operatorname*{argmax}_{y\in \mathbf{R}^p}\{\langle \nabla_p W(p_s, s), -\nabla_x p_s \cdot g(\cdot, y-h) - \gamma^2 \ell(y-h)\rangle\} \tag{5.3}$$

for a.e. $s \in [t,T]$, *then* $\mathbf{u}^*$ *is optimal for the initial data* $(p,t) \in \mathcal{D}\cap\mathcal{X}^1 \times [0,T]$ *and* $\tilde{W}(p,t) = W(p,t)$, *where* $(p,t) \in \mathcal{D}\cap\mathcal{X}^1$. *In particular, for* $(p,t) = (\alpha, 0) \in \mathcal{D}\cap\mathcal{X}^1 \times [0,T]$ *the control* $\mathbf{u}^*$ *solves the partially observed minimax differential game.*

Proof. Since $p_t(x)$ and $\tilde{W}(p,t)$ are smooth solutions ($t \le r \le T$, $p_t = p \in \mathcal{D}\cap\mathcal{X}^1$), Lemma 4.5 implies

$$\tilde{W}(p_r, r) = \tilde{W}(p_t, t) + \int_t^r \left[\frac{\partial \tilde{W}}{\partial t}(p_s, s) + \langle \nabla_p \tilde{W}(p_s, s), F(p_s, u(s), y(s))\rangle\right] ds. \tag{5.4}$$

Fix $y^* \in \mathcal{Y}(t)$ as in (5.3). Then for any $\mathbf{u} \in \mathbf{U}(t)$, we have, using the DPE (4.13) and (5.4) with $r = T$,

$$\begin{aligned}\tilde{W}(p,t) &= -\int_t^T \left[\frac{\partial \tilde{W}}{\partial t}(p_s, s) + \langle \nabla_p \tilde{W}(p_s, s), F(p_s, \mathbf{u}[y^*](s), y^*(s))\rangle\right] ds + (p_T, \Phi)\\ &\le (p_T, \Phi),\end{aligned}$$

with equality for $\mathbf{u} = \mathbf{u}^*$ as in (5.2). Therefore $\tilde{W}(p,t) \le W(p,t)$.

Conversely, let $\mathbf{u} = \mathbf{u}^*$ and $\varepsilon > 0$. Then there exists $y \in \mathcal{Y}(t)$ such that

$$W(p,t) \le (p_T, \Phi) + \varepsilon.$$

Using (5.4) and $r = T$, this gives

$$\begin{aligned}W(p,t) &\le \tilde{W}(p,t) + \int_t^T \left[\frac{\partial \tilde{W}}{\partial t}(p_s, s) + \langle \nabla_p \tilde{W}(p_s, s), F(p_s, u^*(s), y(s))\rangle\right] ds + \varepsilon\\ &\le \tilde{W}(p,t) + \int_t^T \left[\frac{\partial \tilde{W}}{\partial t}(p_s, s) + \langle \nabla_p \tilde{W}(p_s, s), F(p_s, u^*(s), y(s))\rangle\right] ds + \varepsilon\\ &\le \tilde{W}(p,t) + \varepsilon.\end{aligned}$$

Hence $W(p,t) \le \tilde{W}(p,t)$.

We conclude that $W(p,t) = \tilde{W}(p,t)$, and in fact

$$\tilde{W}(\alpha, 0) = J(\mathbf{u}^*) = \inf_{\mathbf{u}\in\mathbf{U}(0)} J(\mathbf{u}),$$

proving the optimality of $\mathbf{u}^*$. □

6. Relation to certainty equivalence. In this section we explain how the certainty equivalence principle [5], [9], [6] fits into the general framework developed in this paper. This issue was treated in discrete-time in [24], [21] and in the case of continuous-time bilinear systems in [32], [33]; see also [6]. The certainty equivalence principle is as follows.

Consider a state feedback differential game with value function $V(x,t)$ satisfying the DPE

$$\begin{cases} \dfrac{\partial V}{\partial t} + \inf_{u\in U}\sup_{w\in\mathbf{R}^p}\{\nabla_x V\cdot(f(x,u)+g(x,w)) - \gamma^2\ell(w) + L(x,u)\} = 0 \text{ in } \mathbf{R}^n\times(0,T), \\ V(x,T) = \Phi(x) \quad \text{in } \mathbf{R}^n. \end{cases} \tag{6.1}$$

Equation (6.1) is a nonlinear first-order PDE and need not possess smooth solutions; thus (6.1) must also be interpreted in the viscosity sense in general. The value function $V(x,t)$ is the unique viscosity solution of (6.1) and is bounded Lipschitz continuous. Let $u^s(x,t)$ denote the optimal state feedback control (if it exists), i.e., the control value attaining the minimum in (6.1). The *minimum stress estimate* is defined by

$$\bar{x}(t) \in \operatorname*{argmax}_{x\in\mathbf{R}^n}(p_t(x) + V(x,t)). \tag{6.2}$$

In [9] it is proven (for a closely related problem) that the *certainty equivalence* controller

$$u^{ce}(t) = u^s(\bar{x}(t),t) \tag{6.3}$$

is optimal provided (i) $p_t(x)$ is a smooth solution of (3.3), (ii) $V(x,t)$ is a smooth solution of (6.1), and, most significantly, (iii) $\bar{x}(t)$ is *unique*.

The following theorem provides a new interpretation of the certainty equivalence controller (see also [6], [21], [24], [32], [33]).

THEOREM 6.1. *Fix a point $(p^1,t^1)\in\mathcal{D}\cap\mathcal{X}^1\times(0,T)$, and assume that*

(i) *$V(x,t) = V_t(x)$ is a smooth solution of* (6.1);

(ii) *the quantity*

$$\bar{x}_{t^1}(p^1) = \operatorname*{argmax}_{x\in\mathbf{R}^n}(p^1(x) + V_{t^1}(x)) \tag{6.4}$$

is unique (i.e., the maximum is attained at a unique point).
Then the function $\tilde{W}\in C(\mathcal{D}\cap\mathcal{X}\times[0,T])$ defined by

$$\tilde{W}(p,t) = (p, V_t) \tag{6.5}$$

(sup-pairing) is $\mathcal{X}$-Gateaux differentiable at (p^1,t^1) and satisfies the DPE (4.13) *at (p^1,t^1). Further, the optimal control at the point (p^1,t^1) is given by*

$$\mathbf{u}^*_{t^1}(p^1) = u^s(\bar{x}_{t^1}(p^1),t^1) = u^{ce}(t^1). \tag{6.6}$$

Proof. 1. We claim first that the Gateaux derivative $\partial_p\tilde{W}(p^1,t^1)$ is given by

$$\partial_p\tilde{W}(p^1,t^1) = E_{\bar{x}_{t^1}(p^1)}, \tag{6.7}$$

where $E_x\in\mathcal{X}^*$ is the evaluation map

$$\langle E_x, q\rangle = q(x), \quad (q\in\mathcal{X}). \tag{6.8}$$

For brevity, write $\bar{x} = \bar{x}_{t^1}(p^1)$, $\phi(p) = (p,V)$, $V(x) = V_{t^1}(x)$, and let $q\in\mathcal{X}$, $\varepsilon>0$. Since $p^1\in\mathcal{D}\cap\mathcal{X}$, $\phi(p^1+\varepsilon q)$ is finite for all ε sufficiently small, and $\phi(p^1+\varepsilon q)\to f(p^1)$,

$\bar{x}^\varepsilon \to \bar{x}$ as $\varepsilon \to 0$ by hypothesis (iii), where $\bar{x}^\varepsilon \in \operatorname{argmax}_{x\in\mathbf{R}^n}(p^1(x) + \varepsilon q(x) + V_{t^1}(x))$. We must show that

$$\lim_{\varepsilon\to 0} \frac{\phi(p^1 + \varepsilon q) - \phi(p^1)}{\varepsilon} = q(\bar{x}). \tag{6.9}$$

Now

$$\phi(p^1 + \varepsilon q) - \phi(p^1) \geq p^1(\bar{x}) + \varepsilon q(\bar{x}) + V(\bar{x}) - (p^1(\bar{x}) + V(\bar{x})) = \varepsilon q(\bar{x}),$$

and hence

$$\liminf_{\varepsilon\to 0} \frac{\phi(p^1 + \varepsilon q) - \phi(p^1)}{\varepsilon} \geq q(\bar{x}).$$

Similarly,

$$\phi(p^1 + \varepsilon q) - \phi(p^1) \leq p^1(\bar{x}^\varepsilon) + \varepsilon q(\bar{x}^\varepsilon) + V(\bar{x}^\varepsilon) - (p^1(\bar{x}^\varepsilon) + V(\bar{x}^\varepsilon)) = \varepsilon q(\bar{x}^\varepsilon),$$

and hence

$$\limsup_{\varepsilon\to 0} \frac{\phi(p^1 + \varepsilon q) - \phi(p^1)}{\varepsilon} \leq q(\bar{x}).$$

These two inequalities prove (6.9), establishing the claim.

2. It follows from Danskin's theorem (see [5, App.]) that

$$\begin{aligned} \frac{\partial \tilde{W}}{\partial t}(p^1, t^1) &= \frac{\partial V}{\partial t}(\bar{x}, t^1), \\ \nabla_x V(\bar{x}, t^1) &= -\nabla_x p^1(\bar{x}). \end{aligned} \tag{6.10}$$

The gradient $\nabla_x p^1$ is well defined since $p^1 \in \mathcal{X}^1$.

3. Next substitute the derivatives calculated above into the left-hand side of DPE (4.13) to yield

$$\begin{aligned} &\frac{\partial \tilde{W}}{\partial t}(p^1, t^1) + \inf_{u\in U} \sup_{y\in\mathbf{R}^p} \langle \partial_p \tilde{W}(p^1, t^1), F(p^1, u, y)\rangle \\ &= \frac{\partial V}{\partial t}(\bar{x}, t^1) + \inf_{u\in U} \sup_{y\in\mathbf{R}^p} \langle E_{\bar{x}}, -\nabla_x p^1 \cdot (f(\cdot, u) + g(\cdot, y-h)) + L(\cdot, u) - \gamma^2 \ell(h-y)\rangle \\ &= -\inf_{u\in U} \sup_{w\in\mathbf{R}^p} \left[\nabla_x V(\bar{x}, t^1) \cdot (f(\bar{x}, u) + g(\bar{x}, w)) - \gamma^2 \ell(w) + L(\bar{x}, u)\right] \\ &\quad + \inf_{u\in U} \sup_{y\in\mathbf{R}^p} \left[\nabla_x V(\bar{x}, t^1) \cdot (f(\bar{x}, u) + g(\bar{x}, y-h)) - \gamma^2 \ell(y-h) + L(\bar{x}, u)\right] \\ &= 0. \end{aligned} \tag{6.11}$$

This proves that $\tilde{W}$ satisfies the DPE at (p^1, t^1).

4. Finally, the above calculation yields explicitly the formula (6.6). □

Remark 6.2. If the minimum stress estimate (6.4) is not unique, i.e. contains more than one point, then in general the function $\tilde{W}(p, t)$ defined by (6.5) is not a solution of (4.13). To see this, we know from Lemma 4.2 that $\bar{x} = \bar{x}_{t^1}(p^1)$ is compact. The proof of Theorem 6.1 shows that

$$\partial_p \tilde{W}(p^1, t^1) \geq \max_{x\in\bar{x}_{t^1}(p^1)} E_x. \tag{6.12}$$

Hence in place of (6.11), Theorem 6.1, we have (for $x \in \bar{x}$)

$$
\begin{aligned}
&\frac{\partial \tilde{W}}{\partial t}(p^1, t^1) + \inf_{u \in U} \sup_{y \in \mathbf{R}^p} \langle \partial_p \tilde{W}(p^1, t^1), F(p^1, u, y)\rangle \\
&\geq \frac{\partial V}{\partial t}(x, t^1) + \inf_{u \in U} \sup_{y \in \mathbf{R}^p} \sup_{x \in \bar{x}} \langle E_x, -\nabla_x p^1 \cdot (f(\cdot, u) + g(\cdot, y - h)) + L(\cdot, u) - \gamma^2 \ell(h - y)\rangle \\
&= -\inf_{u \in U} \sup_{w \in \mathbf{R}^p} \left[\nabla_x V(x, t^1) \cdot (f(x, u) + g(x, w)) - \gamma^2 \ell(w) + L(x, u)\right] \\
&\quad + \inf_{u \in U} \sup_{y \in \mathbf{R}^p} \sup_{x \in \bar{x}} \left[\nabla_x V(x, t^1) \cdot (f(x, u) + g(x, y - h)) - \gamma^2 \ell(y - h) + L(x, u)\right] \\
&\geq -\inf_{u \in U} \sup_{w \in \mathbf{R}^p} \left[\nabla_x V(x, t^1) \cdot (f(x, u) + g(x, w)) - \gamma^2 \ell(w) + L(x, u)\right] \\
&\quad + \sup_{x \in \bar{x}} \inf_{u \in U} \sup_{y \in \mathbf{R}^p} \left[\nabla_x V(x, t^1) \cdot (f(x, u) + g(x, y - h)) - \gamma^2 \ell(y - h) + L(x, u)\right] \\
&\geq 0.
\end{aligned}
$$

(6.13)

This inequality can be strict in general. This calculation suggests that $\tilde{W}(p, t)$ is a *subsolution* of (4.13), but not in general a solution, and consequently

$$W(p, t) \geq \tilde{W}(p, t). \tag{6.14}$$

7. H_∞ control. As an application of the above results, we consider a relatively simple nonlinear H_∞ control problem, viz. finite-horizon disturbance attenuation. Some comments on the infinite-horizon problem will be made in §8.2. We follow closely the approach initiated in [4], [23]. We emphasize that we provide both necessary and sufficient conditions in terms of two PDEs—one defined on a finite-dimensional space $\mathbf{R}^n$, the other defined on an infinite-dimensional space $\mathcal{D} \cap \mathcal{X}$.

Associated with the system (2.1) is the performance output z (not measured) given by

$$z(t) = l(x(t), u(t)). \tag{7.1}$$

To maintain consistency with earlier notation, we set $L(x, u) = \frac{1}{2}|l(x, u)|^2$ and $\Phi = 0$. We assume that zero is an equilibrium; that is, $f(0, 0) = 0$, $g(0, 0) = 0$, $h(0) = 0$, $l(0, 0) = 0$.

Given $\gamma > 0$ and a fixed time interval $[0, T]$, the disturbance attenuation H_∞ problem is to find an output feedback control $\mathbf{u} \in \mathbf{U}(0)$ such that the resulting closed loop system $\Sigma^{\mathbf{u}}$ is *finite gain* $[0, T]$, i.e.,

$$\left\{ \frac{1}{2} \int_0^T |z(t)|^2 \, dt \leq \frac{\gamma^2}{2} \int_0^T \ell(w(t)) \, dt + \beta(x_0) \quad \text{for all } w \in \mathcal{W}(0), \right. \tag{7.2}$$

for some function $\beta(x) \geq 0$, $\beta(0) = 0$, $-\beta \in \mathcal{D} \cap \mathcal{X}$.

Clearly, $\Sigma^{\mathbf{u}}$ is finite gain on $[0, T]$ if and only if

$$J(\mathbf{u}) \leq 0 \ \text{ for } p_0 = \alpha = -\beta. \tag{7.3}$$

THEOREM 7.1. *If there exists a solution of the finite-time H_∞ problem, then there exist solutions of the PDEs* (3.3) *and* (4.13) *such that $p_0 = -\beta$ and $W(-\beta, 0) = 0$ for some $\beta(x) \geq 0$ with $\beta(0) = 0$. Conversely, if there exist smooth solutions of the PDEs* (3.3) *and* (4.13) *such that $p_0 = -\beta$ and $W(-\beta, 0) = 0$, for some $\beta(x) \geq 0$ with $\beta(0) = 0$, then the controller $\mathbf{u}^*$ defined by* (5.1), (5.2) *solves the finite-time H_∞ problem.*

Proof. 1. Assume that a control $\mathbf{u}^o \in \mathbf{U}(0)$ solves the finite-time H_∞ problem, and set $p_0 = -\beta^{\mathbf{u}^o}$. Then

$$J(\mathbf{u}^o) \leq 0,$$

and in fact

$$0 = (-\beta^{\mathbf{u}^o}, 0) \leq W(-\beta^{\mathbf{u}^o}, 0) \leq 0,$$

as in Lemma 4.1. Thus $W(-\beta^{\mathbf{u}^o}, 0) = 0$. Solutions to the PDEs (3.3) and (4.13) exist according to the results in earlier sections in the viscosity sense.

2. Conversely, if (3.3) and (4.13) have smooth solutions, then the verification Theorem 5.1 implies that the control $\mathbf{u}^*$ given by (5.1), (5.2) is optimal. Therefore, with $p_0 = -\beta$,

$$J(\mathbf{u}^*) = W(-\beta, 0) = 0,$$

which implies that $\Sigma^{\mathbf{u}^*}$ is finite gain on $[0, T]$. □

8. Remarks.

8.1. General partially observed differential games. We expect that the results developed in this paper will extend to much more general situations. However, additional technical difficulties arise. For instance, suppose (2.1) is replaced by

$$\text{(8.1)} \qquad \begin{cases} \dot{x}(t) &= f(x(t), u(t)) + g(x(t), w(t)), \\ y(t) &= h(x(t)) + v(t), \end{cases}$$

where $v(\cdot)$ is a second independent and unknown disturbance input. In this case, the function $F(p, u, y)$ governing the information state dynamics is nonlinear:

$$\text{(8.2)} \qquad F(p, u, y) = \sup_w \{-\nabla_x p \cdot (f(\cdot, u) + g(\cdot, w)) + L(\cdot, u) - \gamma^2 \ell(w, y - h)\}.$$

A consequence of this is that (3.3) does not in general have smooth solutions (even if α is smooth). This complicates substantially the proof that the value function $W(p, t)$ is a viscosity solution of the corresponding HJI equation (4.13).

8.2. Infinite-horizon H_∞ control. The theory of dissipative systems [18], [36] provides a powerful framework for treating infinite-horizon H_∞ problems, and many of the articles listed in the reference section make use of this theory. In the state feedback case, one is led to a partial differential inequality (PDI); see, e.g., [2], [18], [20], [34], [36]. In particular, it is shown in [2] and [20] that the PDI can be interpreted in the viscosity sense.

In the discrete-time case, the infinite-horizon output feedback H_∞ problem was discussed in [23], and an infinite-dimensional dissipation inequality was used. The continuous-time analogue of this inequality is an infinite-dimensional PDI, closely related to the steady-state version of (4.13). The PDI is

$$\text{(8.3)} \qquad \begin{cases} \inf_{u \in U} \sup_{y \in \mathbf{R}^p} \langle \nabla_p W, F(p, u, y) \rangle \leq 0 \text{ in } \mathcal{D} \cap \mathcal{X}^1, \\ W(p) \geq (p, 0) \text{ in } \mathcal{D} \cap \mathcal{X}, \\ W(-\beta) = 0. \end{cases}$$

A theory of infinite-horizon H_∞ control can be developed using this type of equation; see [16]. It is possible to prove the existence of a function $W(p)$ satisfying (8.3) in the viscosity sense (i.e., as a viscosity supersolution), using a stationary version of the definition given in §4. An explicit storage function for the closed-loop system is defined in [15].

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