Robust H_∞ Output Feedback Control for Nonlinear Systems

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***Abstract*— In this paper we present a new approach to the solution of the output feedback robust H_∞ control problem. We employ the recently developed concept of information state for output feedback dynamic games and obtain necessary and sufficient conditions for the solution to the robust control problem expressed in terms of the information state. The resulting controller is an information state feedback controller and is *intrinsically infinite dimensional*. Stability results are obtained using the theory of dissipative systems, and indeed, our results are expressed in terms of dissipation inequalities.**

I. Introduction

The modern theory of robust (or H_∞) control for linear systems originated in the work of Zames [38], which employed frequency domain methods (see also [8], [9], and [39]). After the publication of this work, there was an explosion of research activity which led to a rather complete and satisfying body of theory; see Doyle *et al.* [7]. In fact, the successful development of this theory is, to a large extent, due to the use of time-domain methods. In addition, significant advances in the theory depended on ideas from elsewhere; in particular, extensive use has been made of results concerning dynamic games, Riccati equations, the bounded real lemma (e.g., [4], [7], [24], [26], [27]), and risk-sensitive stochastic optimal control (e.g., [10], [35]). The solution to the output feedback robust H_∞ control problem has the structure of an observer and a controller and involves filter and control type Riccati equations.

The time-domain formulation of the robust H_∞ control problem has a natural generalization to nonlinear systems, since the H_∞ norm inequality $\|\Sigma\|_{H_\infty} < \gamma$ has an interpretation which in no way depends on linearity (of course, use of the term "norm" may not be appropriate for nonlinear systems). This inequality is related to the L_2 gain of the system and the bounded real lemma. The robust H_∞ control problem is to find a stabilizing controller which achieves this H_∞ norm bound and can be viewed as a dynamic game problem, with nature acting as a malicious opponent. A general and powerful framework for dealing with L_2 gains for nonlinear systems is Willems' theory of dissipative systems [37]. Using this framework, one can write down a nonlinear version of the bounded real lemma, which is expressed in terms of a dynamic programming inequality or a partial differential inequality, known as the dissipation inequality (see, e.g., [14]), which reduces to a Riccati inequality or equation in the linear context. Therefore, it is not surprising that in papers dealing with nonlinear robust H_∞ control, one sees dissipation inequalities and equations and dynamic game formulations (e.g., [2], [3], [16], [31]–[33]).

An examination of the references cited above reveals that the state feedback robust H_∞ control problem for nonlinear systems is reasonably well understood: one obtains the controller by solving the dissipation type inequality or equation which results from the dynamic game formulation (actually, controller synthesis remains a major difficulty for continuous-time systems, but the conceptual framework is in place). The output feedback case is not nearly so well developed, and no general framework for solving it is available in the literature. By analogy with the linear case, one expects the solution to involve a filter or observer in addition to a dissipation inequality/equation for determining the control. Several authors have proceeded by postulating a filter structure and solving an augmented game problem ([3], [16], [33]). These results yield sufficient conditions, which are in general not necessary conditions; that is, an output feedback problem may be solvable, but not necessarily by the means that have thus far been suggested.

In this paper we present a new approach to the solution of the output feedback robust H_∞ control problem for nonlinear systems. Our approach yields conditions which are both necessary and sufficient. The framework we present incorporates a separation principle, which in essence permits the replacement of the original output feedback problem by an equivalent one with full information, albeit infinite dimensional. It is this aspect which differentiates our results from other results in the literature. To express our ideas as simply as possible, we consider discrete-time systems, except in Section V where continuous-time systems are discussed. The system-theoretic ideas we introduce in this paper apply to both discrete- and continuous-time systems, with the main difference being the level of technical detail (see [20], [29], [18]).

Our approach to this problem was motivated by ideas from stochastic control and large deviations theory. In our earlier paper [19], we explored the connection between a partially observed risk-sensitive stochastic control problem

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and a partially observed dynamic game, and we introduced the use of an information state for solving such games. The information state for this game was obtained as an asymptotic limit of the information state for the risk-sensitive stochastic control problem. Historically, the information state we employ is related to the "past stress" used by Whittle [35] in solving the risk-sensitive problem for linear systems (see also [36]) and can be thought of as a modified conditional density or minimum energy estimator (c.f. [13], [25]). The framework developed in this paper to solve the output feedback robust H_∞ control problem involves a dynamic game formulation, and the use of the (infinite dimensional) information state dynamical system constitutes the above-mentioned separation principle. This idea of separation, using, say, the conditional density, is well known in stochastic control theory; see Kumar and Varaiya [23]. (We thank an anonymous referee for pointing out related results in [22].) Our results imply that if the robust H_∞ control problem is at all solvable by an output feedback controller, then it is solvable by an information state feedback controller. A different way of approaching output feedback games has been proposed by Basar and Bernhard [6], [4], using a certainty equivalence principle as in Whittle [35]. When this principle is applicable, the resulting controller is a special case of the information state controller, and the complexity of the solution is reduced considerably (which may be useful in practice); see [17].

The information state feedback controller we obtain has an observer/controller structure. The "observer" is the dynamical system for the information state $p_k(x)$

$$p_k = F(p_{k-1}, u_{k-1}, y_k)$$

(the notation is introduced in Section II). The "controller"

$$u_k = \bar{u}^*(p_k)$$

is determined by a dissipation inequality

$$W(p) \geq \inf_{u\in U} \sup_{y\in \boldsymbol{R}^p} \{W(F(p, u, y))\}$$

and the value function $W(p)$ solving it is a function of the information state. This dynamic programming inequality is an infinite dimensional relation defined for an infinite dimensional control problem, namely that of controlling the information state. Our solution is therefore an infinite dimensional dynamic compensator, Fig. 1. In a sense, the solution is "doubly infinite dimensional." Actually, there are three levels of equations and spaces involving i) the state of the plant x, ii) the information state p (a function of x), and iii) the value function W (a function of p). In some cases, the information state turns out to be finite dimensional; see [21] and [30].

While there is a separation principle, the task of "estimation" is not isolated from that of "control." The information state carries observable information that is relevant to the control objective and need not necessarily estimate accurately the state of the system being controlled. The control objective is taken into consideration and so the resulting state estimate is suboptimal, but nonetheless more suitable to achieving the control objective, relative to an observer designed with the exclusive aim of state estimation. Thus the information state

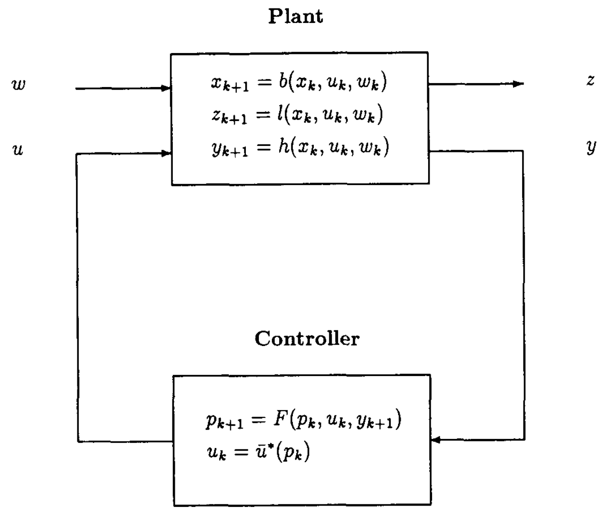

Fig. 1.

represents the optimal trade-off between estimation and control for the robust H_∞ control problem.

We begin in Section II by formulating the problem to be solved. Then in Section III we consider the state feedback problem; it is hoped that our treatment of this problem will clarify certain aspects of our solution to the output feedback problem, which is presented in Section IV. Note, however, that the solution to the state feedback problem is not used to solve the output feedback problem. Our results are obtained in a rather general context, and as a consequence the use of extended-real valued functions is necessary. We remark that while the key ideas for our solution were obtained from stochastic control theory, this paper makes no explicit use of that theory and is in fact self-contained and purely deterministic.

II. Problem Formulation

We consider discrete-time nonlinear systems (plants) Σ described by the state-space equations of the general form

$$\begin{cases} x_{k+1} = b(x_k, u_k, w_k), \\ z_{k+1} = l(x_k, u_k, w_k), \\ y_{k+1} = h(x_k, u_k, w_k). \end{cases} \tag{2.1}$$

Here, $x_k \in \boldsymbol{R}^n$ denotes the state of the system and is not in general directly measurable; instead an output quantity $y_k \in \boldsymbol{R}^p$ is observed. The additional output quantity $z_k \in \boldsymbol{R}^q$ is a performance measure, depending on the particular problem at hand. The control input is $u_k \in U \subset \boldsymbol{R}^m$, and $w_k \in \boldsymbol{R}^r$ is a disturbance input. For instance, w could be due to modeling errors, sensor noise, etc. The system behavior is determined by the functions $b: \boldsymbol{R}^n \times \boldsymbol{R}^m \times \boldsymbol{R}^r \to \boldsymbol{R}^n$, $l: \boldsymbol{R}^n \times \boldsymbol{R}^m \times \boldsymbol{R}^r \to \boldsymbol{R}^q$, $h: \boldsymbol{R}^n \times \boldsymbol{R}^m \times \boldsymbol{R}^r \to \boldsymbol{R}^p$. It is assumed that the origin is an equilibrium for system (2.1): $b(0, 0, 0) = 0$, $l(0, 0, 0) = 0$, and $h(0, 0, 0) = 0$.

The output feedback robust H_∞ control problem is: given $\gamma > 0$, find a controller $u = u(y(\cdot))$, responsive only to the observed output y, such that the resulting closed-loop system Σ^u achieves the following two goals:

i) Σ^u is asymptotically stable when no disturbances are present, and

ii) Σ^u is finite gain, i.e., for each initial condition $x_0 \in \boldsymbol{R}^n$ the input–output map $\Sigma^u_{x_0}$ relating w to z is finite gain, which means that there exists a finite quantity $\beta^u(x_0) \geq 0$ such that

$$\begin{cases} \sum_{i=0}^{k-1} |z_{i+1}|^2 \leq \gamma^2 \sum_{i=0}^{k-1} |w_i|^2 + \beta^u(x_0) \\ \text{for all } w \in l_2([0, k-1], \boldsymbol{R}^r) \text{ and all } k \geq 0. \end{cases} \quad (2.2)$$

Since $x_0 = 0$ is an equilibrium, we also require that $\beta^u(0) = 0$.

Of course, β will also depend on γ.

Note that we have specified the robust control problem in terms of the family of initialized input–output maps $\{\Sigma^u_{x_0}\}_{x_0 \in \boldsymbol{R}^n}$, whereas the conventional problem statement for linear systems refers only to the single map Σ^u_0. This is often expressed in terms of the H_∞ norm of Σ^u_0

$$\|\Sigma^u_0\|_{H_\infty} \triangleq \sup_{w \in l_2([0,\infty), \boldsymbol{R}^r), w \neq 0} \frac{\|z\|_{l_2([1,\infty), \boldsymbol{R}^q)}}{\|w\|_{l_2([0,\infty), \boldsymbol{R}^r)}}. \quad (2.3)$$

For linear systems, the linear structure means that the solvability of the robust control problem is equivalent to the solvability of a pair of Riccati difference equations (and a coupling condition), under certain assumptions, and so implicitly all the maps $\Sigma^u_{x_0}$ are considered. For nonlinear systems, our formulation seems natural and appropriate (see [14], [34]), since otherwise if we were to follow the linear systems formulation, one would need assumptions relating nonzero initial states x_0 to the equilibrium state zero (such as reachability). The formulation adopted here has also been used recently by van der Schaft [33]. A solution $u = u^*$ to this problem yields

$$\|\Sigma^u_0\|_{H_\infty} \leq \gamma \quad (2.4)$$

as is the case for linear systems. It is also apparent that in place of the l_2 norm used in the definition of the finite gain property, one could substitute any other l_q norm [34], or indeed, any other suitable function, and the corresponding theory would develop analogously.

III. The State Feedback Case

In this section we consider the special case where complete state information is available, i.e., where $h(x, u, w) \equiv x$. It is not assumed that the disturbance is measured. For an alternative presentation of the state feedback problem, see [2] and [4].

The system Σ is now described by

$$\begin{cases} x_{k+1} = b(x_k, u_k, w_k) \\ z_{k+1} = l(x_k, u_k, w_k) \end{cases} \quad (3.1)$$

where $u \in \mathcal{S}$ is a state feedback controller, i.e., those controllers for which $u_k = \overline{u}(x_k)$, where $\overline{u}: \boldsymbol{R}^n \to U$. The state feedback robust H_∞ control problem is: given $\gamma > 0$, find a feedback controller $u \in \mathcal{S}$ such that the resulting closed-loop system Σ^u is asymptotically stable and finite gain, as in Section II.

Before attacking the full problem, we consider the finite-time problem, where stability is not an issue.

A. Finite-Time Case

Let $\mathcal{S}_{k,l}$ denote the set of controllers u defined on the time interval $[k, l]$ such that for each $j \in [k, l]$ there exists a function $\overline{u}_j: \boldsymbol{R}^{(j-k+1)n} \to U$ such that $u_j = \overline{u}_j(x_{k,j})$. The finite-time state feedback robust H_∞ control problem is: given $\gamma > 0$ and a finite-time interval $[0, k]$, find a feedback controller $u \in \mathcal{S}_{0,k}$ such that the resulting closed-loop system Σ^u is finite gain, in the sense that (2.2) holds for the fixed value of k.

1) Dynamic Game: For $u \in \mathcal{S}_{0,k-1}$ and $x_0 \in \boldsymbol{R}^n$ we define the functional $J_{x_0,k}(u)$ for (3.1) by

$$J_{x_0,k}(u) = \sup_{w \in l_2([0,k-1], \boldsymbol{R}^r)} \cdot \left\{ \sum_{i=0}^{k-1} |z_{i+1}|^2 - \gamma^2 |w_i|^2 : x(0) = x_0 \right\}. \quad (3.2)$$

Clearly

$$J_{x_0,k}(u) \geq 0$$

and the finite gain property of $\Sigma^u_{x_0}$ can be expressed in terms of J as follows.

Lemma 3.1: $\Sigma^u_{x_0}$ is finite gain on $[0, k]$ if and only if there exists a finite quantity $\beta^u_k(x_0) \geq 0$ such that

$$J_{x_0,j}(u) \leq \beta^u_k(x_0), \quad j \in [0, k] \quad (3.3)$$

and $\beta^u_k(0) = 0$.

The state feedback dynamic game is to find a control $u^* \in \mathcal{S}_{0,k-1}$ which minimizes each functional $J_{x_0,k}$, $x_0 \in \boldsymbol{R}^n$. This will yield a solution to the finite-time state feedback robust H_∞ control problem.

2) Solution to the Finite-Time State Feedback Robust H_∞ Control Problem: The dynamic game can be solved using dynamic programming (see, e.g., [4]). The idea is to use the value function

$$V_j(x) = \inf_{u \in \mathcal{S}_{0,j-1}} \sup_{w \in l_2([0,j-1], \boldsymbol{R}^r)} \cdot \left\{ \sum_{i=0}^{j-1} |z_{i+1}|^2 - \gamma^2 |w_i|^2 : x(0) = x \right\} \quad (3.4)$$

and corresponding dynamic programming equation

$$\begin{cases} V_j(x) = \inf_{u \in U} \sup_{w \in \boldsymbol{R}^r} \{V_{j-1}(b(x, u, w)) \\ \quad + |l(x, u, w)|^2 - \gamma^2 |w|^2\} \\ V_0(x) = 0. \end{cases} \quad (3.5)$$

Remark 3.2: Note that we use a value function which evolves forward in time, contrary to standard practice. Our reason for this choice is that it simplifies transition to the infinite-time case.

Theorem 3.3 (Necessity): Assume that $u^s \in \mathcal{S}_{0,k-1}$ solves the finite-time state feedback robust H_∞ control problem. Then there exists a solution V to the dynamic programming equation (3.5) such that $V_j(0) = 0$ and $V_j(x) \geq 0$, $j \in [0, k]$.

(Sufficiency): Assume there exists a solution V to the dynamic programming equation (3.5) such that $V_j(0) = 0$ and $V_j(x) \geq 0$, $j \in [0, k]$. Let $u^* \in \mathcal{S}_{0,k-1}$ be a policy such that $u_j^* = \overline{u}_{k-j}^*(x_j)$, where $\overline{u}_j^*(x)$ achieves the minimum in (3.5); $j = 0, \cdots, k-1$. Then u^* solves the finite-time state feedback robust H_∞ control problem.

Proof: The proof uses standard arguments from dynamic programming, and we omit the details. □

B. Infinite-Time Case

We wish to solve the infinite-time problem by passing to the limit $V(x) = \lim_{k\to\infty} V_k(x)$, where $V_k(x)$ is defined by (3.4), to obtain a stationary version of the dynamic programming equation (3.5), viz.

$$V(x) = \inf_{u\in U} \sup_{w\in \boldsymbol{R}^r} \{V(b(x, u, w)) + |l(x, u, w)|^2 - \gamma^2|w|^2\}. \tag{3.6}$$

In many respects, this procedure is best understood in terms of the bounded real lemma [1], [27]. For instance, the finite gain property is captured in terms of a dissipation inequality [37] (or partial differential inequality in continuous time). Also, stability results are readily deduced [14], [37].

1) Bounded Real Lemma: We will say that Σ^u is finite gain dissipative if there exists a function (called a storage function [37]) $V(x)$ such that $V(x) \geq 0$, $V(0) = 0$, and satisfies the dissipation inequality

$$V(x) \geq \sup_{w\in \boldsymbol{R}^r} \{V(b(x, \overline{u}(x), w)) - \gamma^2|w|^2 + |l(x, \overline{u}(x), w)|^2\}. \tag{3.7}$$

The following result is one way of stating a version of the bounded real lemma. The proof is omitted.

Theorem 3.4 (Bounded Real Lemma): Let $u \in \mathcal{S}$. The system Σ^u is finite gain if and only if it is finite gain dissipative.

Under additional assumptions, stability results can be obtained for dissipative systems [37], [14]. We say that Σ^u is (zero state) detectable if $w \equiv 0$ and $\lim_{k\to\infty} z_k = 0$ implies $\lim_{k\to\infty} x_k = 0$. Σ^u is asymptotically stable if $w \equiv 0$ implies $\lim_{k\to\infty} x_k = 0$ for any initial condition.

Theorem 3.5: Let $u \in \mathcal{S}$. If Σ^u is finite gain dissipative and detectable, then Σ^u is asymptotically stable.

Proof: Since Σ^u is finite gain dissipative, (3.7) implies

$$V(x_k) + \sum_{i=0}^{k-1} |z_{i+1}|^2 \leq \gamma^2 \sum_{i=0}^{k-1} |w_i|^2 + V(x_0). \tag{3.8}$$

Setting $w \equiv 0$ in (3.8) and using the nonnegativity of V we get

$$\sum_{i=0}^{k-1} |z_{i+1}|^2 \leq V(x_0), \quad \text{for all } k > 0$$

for any initial condition x_0. This implies $\{z_k\} \in l_2$ $((0, \infty), \boldsymbol{R}^q)$, and so $\lim_{k\to\infty} z_k = 0$. By detectability, we obtain $\lim_{k\to\infty} x_k = 0$. □

Remark 3.6: In general, detectability (or observability) is a key property required for asymptotic stability as it is related to the positive definiteness of the storage functions [14], [34], but is difficult to check and will depend on the controller $u \in \mathcal{S}$. Detectability holds trivially in the uniformly coercive case: $-\nu_0 + \nu_1|x|^2 \leq |l(x, u, w)|^2$, where $\nu_0 \geq 0$, $\nu_1 > 0$.

2) Solution to the State Feedback Robust H_∞ Control Problem: It is clear from the previous section that the state feedback robust H_∞ control problem can be solved provided a stabilizing feedback controller can be found which renders the closed-loop system finite gain dissipative. The next theorem gives both necessary and sufficient conditions in terms of a controlled version of the dissipation inequality, under a suitable detectability condition. The proof is omitted.

Theorem 3.7 (Necessity): If a controller $u^s \in \mathcal{S}$ solves the state feedback robust H_∞ control problem then there exists a function $V(x)$ such that $V(x) \geq 0$, $V(0) = 0$, and

$$V(x) \geq \inf_{u\in U} \sup_{w\in \mathbf{R}^r} \{V(b(x, u, w)) - \gamma^2|w|^2 + |l(x, u, w)|^2\}. \tag{3.9}$$

(Sufficiency): Assume that V is a solution of (3.9) satisfying $V(x) \geq 0$ and $V(0) = 0$. Let $\overline{u}^*(x)$ be a control value which achieves the minimum in (3.9). Then the controller $u^* \in \mathcal{S}$ defined by $\overline{u}^*(x)$ solves the state feedback robust H_∞ control problem if the closed-loop system Σ^{u^*} is detectable.

Remark 3.8: The utility of this result is that the controlled dissipation inequality (3.9) provides (in principle) a recipe for solving the state feedback robust H_∞ control problem.

IV. The Output Feedback Case

We return now to the output feedback robust H_∞ control problem. As in the state feedback case, we start with a finite-time version. For the remainder of this paper we will assume the following

$$\text{for all } \xi, u, y, \text{ there exists } w \text{ such that } h(\xi, u, w) = y. \tag{4.1}$$

A. Finite-Time Case

Let $\mathcal{O}_{k,l}$ denote the set of output feedback controllers defined on the time interval $[k, l]$, so $u \in \mathcal{O}_{k,l}$ means that for each $j \in [k, l]$ there exists a function $\overline{u}_j : \boldsymbol{R}^{(j-k+1)p} \to U$ such that $u_j = \overline{u}_j(y_{k+1,j})$. For $u \in \mathcal{O}_{0,M-1}$, Σ^u denotes closed-loop system (2.1). The finite-time output feedback robust H_∞ control problem is: given $\gamma > 0$ and a finite-time interval $[0, k]$, find a controller $u \in \mathcal{O}_{0,k-1}$ such that the resulting closed loop system Σ^u is finite gain, in the sense that (2.2) holds for the fixed value of k.

1) Dynamic Game: Our aim in this section is to express the output feedback robust control problem in terms of a dynamic game.

We introduce the function space

$$\mathcal{E} = \{p : \boldsymbol{R}^n \to \boldsymbol{R}^*\}$$

and define for each $x \in \boldsymbol{R}^n$ a function $\delta_x \in \mathcal{E}$ by

$$\delta_x(\xi) \triangleq \begin{cases} 0 & \text{if } \xi = x, \\ -\infty & \text{if } \xi \neq x. \end{cases}$$

For $u \in \mathcal{O}_{0,k-1}$ and $p \in \mathcal{E}$, define the functional $J_{p,k}(u)$ for system (2.1) by

$$J_{p,k}(u) \triangleq \sup_{w \in l_2([0,k-1],\boldsymbol{R}^r)} \sup_{x_0 \in \boldsymbol{R}^n} \cdot \left\{ p(x_0) + \sum_{i=0}^{k-1} |z_{i+1}|^2 - \gamma^2 |w_i|^2 \right\}. \quad (4.2)$$

Remark 4.1: The quantity $p \in \mathcal{E}$ in (4.2) can be chosen in a way which reflects knowledge of any *a priori* information concerning the initial state x_0 of Σ^u. □

The finite gain property of Σ^u can be expressed in terms of J as follows.

Lemma 4.2: i) $\Sigma^u_{x_0}$ is finite gain on $[0, k]$ if and only if there exists a finite quantity $\beta^u_k(x_0) \geq 0$ such that

$$J_{\delta_{x_0},k}(u) \leq \beta^u_k(x_0) \quad (4.3)$$

and $\beta^u_k(0) = 0$. ii) Σ^u is finite gain on $[0, k]$ if and only if there exists a finite quantity $\beta^u_k(\cdot) \geq 0$ such that

$$J_{-\beta^u_k,k}(u) \leq 0 \quad (4.4)$$

and $\beta^u_k(0) = 0$.

It is of interest to know when $J_{p,k}(u)$ is finite. For a finite gain system Σ^u, we write

$$\operatorname{dom} J_{\cdot,k}(u) = \{p \in \mathcal{E} : J_{p,k}(u) \quad \text{finite}\}.$$

In what follows we make use of the "sup-pairing" [19]

$$(p, q) = \sup_{x \in \boldsymbol{R}^n} \{p(x) + q(x)\}. \quad (4.5)$$

Note that if $p \in \operatorname{dom} J_{\cdot,k}(u)$, then $-\infty < (p, 0)$.

Lemma 4.3: If each map $\Sigma^u_{x_0}$ is finite gain on $[0, k]$, then

$$(p, 0) \leq J_{p,k}(u) \leq (p, \beta^u_k) \quad (4.6)$$

and so $J_{p,k}(u)$ is finite whenever the lower and upper bounds in (4.6) are both finite.

Proof: Set $w \equiv 0$ in (4.2) to deduce $(p, 0) \leq J_{p,k}(u)$. Next, select $w \in l_2([0, k-1], \boldsymbol{R}^r)$ and $x_0 \in \boldsymbol{R}^n$. Then (2.2) implies

$$p(x_0) + \sum_{i=0}^{k-1} |z_{i+1}|^2 - \gamma^2 |w_i|^2 \leq p(x_0) + \beta^u_k(x_0) \leq (p, \beta^u_k).$$

This proves (4.6). □

The finite-time output feedback dynamic game is to find a control policy $u \in \mathcal{O}_{0,k-1}$ which minimizes the functional $J_{p,k}$. The idea then is that a solution to this game problem will solve the output feedback robust H_∞ control problem.

2) Information State Formulation: To solve the game problem, we borrow an idea from stochastic control theory (see, e.g., [5], [23]) and replace the original problem with a new one expressed in terms of a new state variable, viz., an information state [18]–[20].

For fixed $u_{0,j-1} \in l_2([0, j-1], U)$, $y_{1,j} \in l_2([1, j], \boldsymbol{R}^p)$ we define the information state $p_j \in \mathcal{E}$ by

$$\begin{aligned} p_j(x) \triangleq\ & \sup\nolimits_{w \in l_2([0,j-1],\boldsymbol{R}^r)} \sup\nolimits_{x_0 \in \boldsymbol{R}^n} \\ & \cdot \left\{ p_0(x_0) + \sum_{i=0}^{j-1} |z_{i+1}|^2 - \gamma^2 |w_i|^2 \right. \\ & \left. : x_j = x,\ h(x_i, u_i, w_i) = y_{i+1},\ 0 \leq i \leq j-1 \right\} \end{aligned} \quad (4.7)$$

where x_i, $i = 0, \cdots, j$ is the corresponding solution of (2.1). This quantity represents the worst cost up to time j that is consistent with the given inputs and the observed outputs subject to the end-point constraint $x_j = x$.

To write the dynamical equation for p_j, we define $F(p, u, y) \in \mathcal{E}$ by

$$F(p, u, y)(x) = \sup_{\xi \in \boldsymbol{R}^n} \{p(\xi) + B(\xi, x, u, y)\} \quad (4.8)$$

where the extended real valued function B is defined by

$$\begin{aligned} B(\xi, x, u, y) = \sup_{w \in \boldsymbol{R}^r} & \{|l(\xi, u, w)|^2 - \gamma^2 |w|^2 \\ & : b(\xi, u, w) = x,\ h(\xi, u, w) = y\}. \end{aligned} \quad (4.9)$$

Here, we use the convention that the supremum over an empty set equals $-\infty$.

Lemma 4.4: The information state is the solution of the following recursion

$$\begin{cases} p_j = F(p_{j-1}, u_{j-1}, y_j), & j \in [1, k], \\ p_0 \in \mathcal{E}. \end{cases} \quad (4.10)$$

Proof: The result is proven by induction. Assume the assertion is true for $0, \cdots, j-1$; we must show that p_j defined by (4.7) equals $F(p_{j-1}, u_{j-1}, y_j)$ defined by (4.8). Now

$$\begin{aligned} & F(p_{j-1}, u_{j-1}, y_j)(x) \\ & = \sup\nolimits_{\xi \in \boldsymbol{R}^n} \{p_{j-1}(\xi) + B(\xi, x, u_{j-1}, y_j)\} \\ & = \sup\nolimits_{\xi \in \boldsymbol{R}^n} \{p_{j-1}(\xi) + \sup\nolimits_{w_{j-1} \in \boldsymbol{R}^r} \\ & \quad \cdot (|l(\xi, u_{j-1}, y_j)|^2 - \gamma^2 |w_{j-1}|^2 : \\ & \quad b(\xi, u_{j-1}, w_{j-1}) = x,\ h(\xi, u_{j-1}, w_{j-1}) = y_j)\} \\ & = p_j(x) \end{aligned}$$

using the definition (4.7) for p_{j-1} and p_j. □

The next lemma concerns the finiteness of the information state. Allowing it to take infinite values is very important, as this encodes crucial information.

Lemma 4.5:

i) If Σ^u is finite gain, then

$$-\infty \leq p_j(x) \leq (p_0, \beta^u_k) < +\infty. \quad (4.11)$$

ii) If $-\infty < (p_0, 0)$, then $-\infty < (p_j, 0)$, $j = 1, \cdots, k$, and consequently, for each j there exists x for which $-\infty < p_j(x)$.

iii) If $-\infty < (p, 0)$, then

$$\sup_{y \in \boldsymbol{R}^p} (F(p, u, y), 0) \geq (p, 0) \quad \text{for all } u \in U. \quad (4.12)$$

Proof: The proof of (4.11) follows directly from definitions (4.7) and (2.2). To prove the second part of the lemma, it is enough to show that given any u, y, $-\infty < (p, 0)$ implies $-\infty < (F(p, u, y), 0)$. By definition, we have

$$(F(p, u, y), 0) = \sup_{x,\xi,w} \{p(\xi) + |l(\xi, u, y)|^2 - |w|^2 : b(\xi, u, w) = x, h(\xi, u, w) = y\}. \quad (4.13)$$

Since $-\infty < (p, 0)$, there exists ξ with $-\infty < p(\xi)$, and by assumption (4.1) there exists w such that $h(\xi, u, w) = y$. Set $x = b(\xi, u, w)$. Therefore the set over which the supremum in (4.13) is taken is nonempty, hence the supremum is greater than $-\infty$.

Finally, let $\epsilon > 0$ and choose ξ with $p(\xi) \geq (p, 0) - \epsilon > -\infty$, and set $w = 0$. Let $u \in U$ be arbitrary. Then let $y^u = h(\xi, u, 0)$ and $x^u = b(\xi, u, 0)$. Then from (4.13) we have

$$\begin{aligned}&\sup_{y\in\boldsymbol{R}^p} (F(p, u, y), 0)\\ &\geq (F(p, u, y^u), 0) \geq p(\xi) + |l(\xi, u, y)|^2\\ &\geq p(\xi) \geq (p, 0) - \epsilon.\end{aligned}$$

This proves (4.12). □

Remark 4.6: Note that we can write

$$p_j(x) \triangleq \sup_{\xi\in l_2([0,j],\boldsymbol{R}^n)} \cdot \left\{ p_0(\xi_0) + \sum_{i=0}^{j-1} B(\xi_i, \xi_{i+1}, u_i, y_{i+1}) : \xi_j = x \right\}. \quad (4.14)$$

□

We now state the following representation result.

Theorem 4.7: For $u \in \mathcal{O}_{0,j-1}$, $p \in \mathcal{E}$, such that $J_{p,j}(u)$ is finite, we have the representation

$$J_{p,j}(u) = \sup_{y_{1,j}\in l_2([1,j],\boldsymbol{R}^p)} \{(p_j, 0) : p_0 = p\}, \qquad j \in [0, k]. \quad (4.15)$$

Proof: We have

$$\begin{aligned}&\sup_{y_{1,j}\in l_2([1,j],\boldsymbol{R}^p)} \{(p_j, 0) : p_0 = p\}\\ &= \sup_{y_{1,j}\in l_2([1,j],\boldsymbol{R}^p)} \sup_{\xi\in l_2([0,j],\mathrm{R}^n)}\\ &\quad \cdot \{p(\xi_0) + \sum_{i=0}^{j-1} B(\xi_i, \xi_{i+1}, u_i, y_{i+1})\}\\ &= \sup_{w\in l_2([0,j-1],\boldsymbol{R}^r)} \sup_{x_0\in\boldsymbol{R}^n}\\ &\quad \cdot \{p(x_0) + \sum_{i=0}^{j-1} |z_{i+1}|^2 - \gamma^2|w_i|^2\}\\ &= J_{p,j}(u).\end{aligned}$$

□

Remark 4.8: This representation theorem is a separation principle and is similar to those employed in stochastic control theory; see [23], and in particular, [5], and [19]. □

Theorem 4.7 enables us to express the finite gain property of Σ^u in terms of the information state p, as the following corollary shows.

Corollary 4.9: For any output feedback controller $u \in \mathcal{O}_{0,k-1}$, the closed-loop system Σ^u is finite gain on $[0, k]$ if and only if the information state p_j satisfies

$$\sup_{y_{1,j}\in l_2([1,j],\boldsymbol{R}^p)} \{(p_j, 0) : p_0 = -\beta_k^u(\cdot)\} \leq 0, \qquad \text{for all } j \in [0, k] \quad (4.16)$$

for some finite $\beta_k^u(x_0) \geq 0$ with $\beta_k^u(0) = 0$.

Remark 4.10: In view of the above, the name "information state" for p is justified. Indeed, p_j contains all the information relevant to the key finite gain property of Σ^u that is available in the observations $y_{1,j}$. □

Remark 4.11: We now regard the information state dynamics (4.10) as a new (infinite dimensional) control system Ξ, with control u and disturbance y. The state p_j and disturbance y_j are available to the controller, so the original output feedback dynamic game is equivalent to a new one with full information. The cost is now the right-hand side of (4.15). The analogue in stochastic control theory is the dynamical equation for the conditional density (or variant), and y becomes white noise under a reference probability measure [19], [23]. □

Now that we have introduced the new state variable p, we need an appropriate class $\mathcal{I}_{i,l}$ of controllers which feedback this new state variable. A control u belongs to $\mathcal{I}_{i,l}$ if for each $j \in [i, l]$ there exists a map $\overline{u}_j$ from a subset of $\mathcal{E}^{j-i+1}$ (sequences $p_{i,j} = p_i, p_{i+1}, \cdots, p_j$) into U such that $u_j = \overline{u}(p_{i,j})$. Note that since p_j depends only on the observable information $y_{1,j}$, $\mathcal{I}_{0,j-1} \subset \mathcal{O}_{0,j-1}$.

3) Solution to the Finite-Time Output Feedback Robust H_∞ Control Problem: In this section we use dynamic programming to obtain necessary and sufficient conditions for the solution of the output feedback robust H_∞ control problem. We make use of the dynamic programming approach used in [19] to solve the output feedback dynamic game problem. The value function is given by

$$W_j(p) = \inf_{u\in\mathcal{O}_{0,j-1}} \sup_{y\in l_2([1,j],\boldsymbol{R}^p)} \{(p_j, 0) : p_0 = p\} \quad (4.17)$$

for $j \in [0, k]$, and the corresponding dynamic programming equation is

$$\begin{cases} W_j(p) = \inf_{u\in U} \sup_{y\in\boldsymbol{R}^p} \{W_{j-1}(F(p, u, y))\}, \\ \qquad p \in \operatorname{dom} W_j, \; j \in [1, k] \\ W_0(p) = (p, 0). \end{cases} \quad (4.18)$$

For a function $W : \mathcal{E} \to \boldsymbol{R}^*$, we write

$$\operatorname{dom} W = \{p \in \mathcal{E} : W(p) \text{ finite}\}.$$

Theorem 4.12 (Necessity): Assume that $u^o \in \mathcal{O}_{0,k-1}$ solves the finite-time output feedback robust H_∞ control problem. Then there exists a solution W to the dynamic programming equation (4.18) such that $\operatorname{dom} J_{\cdot,k}(u^o) \subset \operatorname{dom} W_j$, $W_j(-\beta^{u^o}) = 0$, $W_j(p) \geq (p, 0)$, $p \in \operatorname{dom} W_j$, $j \in [0, k]$.

Proof: For $p \in \operatorname{dom} J_{\cdot,k}(u^o)$, define $W_j(p)$ by (4.17), i.e.,

$$W_j(p) = \inf_{u \in \mathcal{O}_{0,j-1}} J_{p,j}(u).$$

Note the alternative expression for $W_j(p)$

$$W_j(p) = \inf_{u \in \mathcal{O}_{0,j-1}} \sup_{w \in l_2([0,j-1], \boldsymbol{R}^r)} \sup_{x_0 \in \boldsymbol{R}^n} \cdot \left\{ p(x_0) + \sum_{i=0}^{j-1} |z_{i+1}|^2 - \gamma^2 |w_i|^2 \right\}. \quad (4.19)$$

For $u = u^o$ we see that, using the finite gain property for Σ^{u^o}

$$\begin{aligned} W_j(p) &\le \sup_{w \in l_2([0,j-1], \boldsymbol{R}^r)} \sup_{x_0 \in \boldsymbol{R}^n} \cdot \left\{ p(x_0) + \sum_{i=0}^{j-1} |z_{i+1}|^2 - \gamma^2 |w_i|^2 \right\} \\ &\le (p, \beta_k^{u^o}). \end{aligned}$$

Thus $\operatorname{dom} J_{\cdot,k}(u^o) \subset \operatorname{dom} W_j$. Also, we have, for all $p \in \mathcal{E}$

$$W_j(p) \ge (p, 0).$$

Since $\beta_k^{u^o}(0) = 0$, $\beta_k^{u^o}(x) \ge 0$, then $(-\beta_k^{u^o}, 0) = 0$, and we have $W_j(-\beta_k^{u^o}) = 0$. Finally, the proof of Theorem 4.4 [19] shows that W_j is the unique solution of the dynamic programming equation (4.18). □

Theorem 4.13 (Sufficiency): Assume there exists a solution W to dynamic programming equation (4.18) on some nonempty domains $\operatorname{dom} W_j$ such that $-\beta \in \operatorname{dom} W_j$, $W_j(-\beta) = 0$ (for some $\beta \ge 0$ with $\beta(0) = 0$), $W_j(p) \ge (p, 0)$, for all $p \in \operatorname{dom} W_j$, $j \in [0, k]$. Assume $\overline{u}_j^*(p)$ achieves the minimum in (4.18) for each $p \in \operatorname{dom} W_j$, $j = 1, \cdots, k$. Let $u^* \in \mathcal{I}_{0,k-1}$ be a policy such that $u_j' = \overline{u}_{k-j}^*(p_j)$, where p_j is the corresponding trajectory with initial condition $p_0 = -\beta$, assuming $p_j \in \operatorname{dom} W_{k-j}$, $j = 0, \cdots, k$. Then u^* solves the finite-time output feedback robust H_∞ control problem.

Proof: Following the proof of Theorem 4.6 of [19], we see that

$$W_k(p) = J_{p,k}(u^*) \le J_{p,k}(u)$$

for all $u \in \mathcal{O}_{0,k-1}$, $p \in \operatorname{dom} W_k$. Therefore

$$\sup_{y \in l_2([1,k], \boldsymbol{R}^p)} \{(p_k, 0) : p_0 = -\beta, u = u^*\} \le W_k(-\beta) = 0$$

which implies by Corollary 4.9 that Σ^{u^*} is finite gain, and hence u^* solves the finite-time output feedback robust H_∞ control problem. □

Remark 4.14: Note that the controller obtained in Theorem 4.13 is an information state feedback controller. □

Corollary 4.15: If the finite-time output feedback robust H_∞ control problem is solvable by some output feedback controller $u^o \in \mathcal{O}_{0,k-1}$ (whose details are unspecified), then it is also solvable by an information state feedback controller $u^* \in \mathcal{I}_{0,k-1}$.

B. Infinite-Time Case

Again, we would like to solve the infinite-time problem by passing to the limit $W(p) = \lim_{k\to\infty} W_k(p)$, where $W_k(p)$ is defined by (4.17), to obtain a stationary version of the dynamic programming equation (4.18), viz.

$$W(p) = \inf_{u \in U} \sup_{y \in \boldsymbol{R}^p} \{W(F(p, u, y))\}. \quad (4.20)$$

For technical reasons, however, this is not quite what we do. Instead, we will minimize the functional

$$J_p(u) = \sup_{k \ge 0} J_{p,k}(u) \quad (4.21)$$

over $u \in \mathcal{O}$. Here, $\mathcal{O}$ denotes output feedback controllers u such that for each k, $u_k = \overline{u}_k(y_{1,k})$ for some map $\overline{u}_k$ from $\boldsymbol{R}^{pk}$ into U. This makes sense in view of the following lemma, whose proof is an easy consequence of the definitions (c.f. Corollary 4.9).

Lemma 4.16: For any output feedback controller $u \in \mathcal{O}$, the closed-loop system Σ^u is finite gain if and only if the information state p_k satisfies

$$\sup_{k \ge 0} \sup_{y_{1,k} \in l_2([1,k], \boldsymbol{R}^p)} \{(p_k, 0) : p_0 = -\beta^u\} \le 0 \quad (4.22)$$

for some finite $\beta^u(x_0) \ge 0$ with $\beta^u(0) = 0$.

Our results will be expressed in terms of an appropriate dissipation inequality, and so in the next section we formulate an appropriate version of the bounded real lemma for the information state system.

1) Bounded Real Lemma: Let $\mathcal{I}$ denote the class of information state feedback controllers u such that $u_k = \overline{u}(p_k)$, for some function $\overline{u}$ from a subset of $\mathcal{E}$ into U. We write

$$\operatorname{dom} J \cdot (u) = \{p \in \mathcal{E} : J_p(u) \quad \text{finite}\}.$$

From Lemma 4.16, we say that the information state system Ξ^u ((4.10) with information state feedback $u \in \mathcal{I}$) is finite gain if and only if $\overline{u}(p)$ is defined for all $p \in \operatorname{dom} J.(\overline{u})$ and the information state p_k satisfies (4.22) for some finite $\beta^u(x_0) \ge 0$ with $\beta^u(0) = 0$.

We say that the information state system Ξ^u is finite gain dissipative if there exists a function (called a storage function) $W(p)$ such that $\operatorname{dom} W$ contains $-\beta$ (where $\beta \ge 0$ and $\beta(0) = 0$), $\overline{u}(p)$ is defined for all $p \in \operatorname{dom} W$, $W(p) \ge (p, 0)$ for all $p \in \operatorname{dom} W$, $W(-\beta) = 0$, $p_k \in \operatorname{dom} W$, for all $k \ge 0$ whenever $p_0 = -\beta$ (for all sequences y_k), and W satisfies the dissipation inequality

$$W(p) \ge \sup_{y \in \boldsymbol{R}^p} \{W(F(p, \overline{u}(p), y))\} \; (p \in \operatorname{dom} W). \quad (4.23)$$

Theorem 4.17 (Bounded Real Lemma): Let $u \in \mathcal{I}$. Then the information state system Ξ^u is finite gain dissipative if and only if it is finite gain.

Proof: Assume that Ξ^u is finite gain dissipative. Then (4.23) implies

$$W(p_k) \le W(p_0) \tag{4.24}$$

for all $k > 0$ and all $y \in l_2([1, k], \mathbf{R}^p)$. Setting $p_0 = -\beta$ and using the inequality $W(p) \ge (p, 0)$ we get

$$(p_k, 0) \le W(-\beta) = 0$$

for all $k > 0$, $y \in l_2([1, k], \mathbf{R}^p)$. Therefore Ξ^u is finite gain.

Conversely, assume that Ξ^u is finite gain. Then

$$(p, 0) \le J_p(u) \le (p, \beta^u)$$

for all $p \in \mathcal{E}$. Write $W_a(p) = J_p(u)$, so that

$$(p, 0) \le W_a(p) \le (p, \beta^u),\ p \in \mathcal{E}.$$

This and (4.22) imply $W_a(-\beta^u) = 0$. We now show that W_a satisfies (4.23) and that if $p \in \operatorname{dom} W_a$, then $F(p, \overline{u}(p), y) \in \operatorname{dom} W_a$ for all y.

Fix $p \in \operatorname{dom} W_a$ and $y_1 \in \mathbf{R}^p$. Set $p_1 = F(p, \overline{u}(p), y_1)$. For any sequence $y_2, y_3, \cdots$ define a sequence $\tilde{y}_1, \tilde{y}_2, \cdots$ by

$$\tilde{y}_i = y_{i+1},\ i = 1, 2, \cdots.$$

Let p_k and $\tilde{p}_k$ denote the corresponding trajectories, and note that $\tilde{p}_k = p_{k+1}$. With y_1 fixed and maximizing over $k \ge 1$ and $y_{2,k}$ we have

$$\begin{aligned} W_a(p) &\ge \sup_{k\ge 1, y_{2,k}} \{(p_k, 0) : p_0 = p\} \\ &= \sup_{k \ge 0, \tilde{y}_{1,k}} \{(\tilde{p}_k, 0) : \tilde{p}_0 = p_1\} \qquad (= W_a(p_1)) \\ &\ge (p_1, 0). \end{aligned}$$

This implies, using Lemma 4.5, that $p_1 \in \operatorname{dom} W_a$ (since $p \in \operatorname{dom} W_a$ implies $-\infty < (p, 0)$ and $-\infty < (p_1, 0)$).

Since y_1 is arbitrary, we have

$$W_a(p) \ge \sup_{y \in \mathbf{R}^p} W_a(F(p, \overline{u}(p), y)).$$

This inequality implies that W_a solves (4.23). (Actually, W_a solves (4.23) with equality.) Thus Ξ^u is finite gain dissipative. □

Remark 4.18: The supremum in (4.21) can be replaced by a limit. To prove this, write $W_k(p) = J_{p,k}(u)$, so that

$$(p, 0) \le W_k(p) \le W_a(p) \le (p, \beta^u), \qquad k \ge 0,\ p \in \mathcal{E}.$$

Now W_k is monotone nondecreasing

$$W_{k-1}(p) \le W_k(p).$$

To see this, note that

$$W_k(p) = \sup_{w \in l_2([0, k-1], \mathbf{R}^r)} \sup_{x_0 \in \mathbf{R}^n} \cdot \left\{ p(x_0) + \sum_{i=0}^{k-1} |z_{i+1}|^2 - \gamma^2 |w_i|^2 \right\}.$$

Then given $\epsilon > 0$, choose $w' \in l_2([0, k-2], \mathbf{R}^r)$ and x_0' such that

$$W_{k-1}(p) \le p(x_0') + \sum_{i=0}^{k-2} |z_{i+1}'|^2 - \gamma^2 |w_i'|^2 + \epsilon$$

and define $w \in l_2([0, k-1], \mathbf{R}^r)$ by setting $w = w'$ on $[0, k-2]$ and $w_{k-1} = 0$, and let $x_0 = x_0'$. Then

$$\begin{aligned} W_k(p) &\ge p(x_0') + \sum_{i=0}^{k-1} |z_{i+1}|^2 - \gamma^2 |w_i|^2 \\ &\ge p(x_0') + \sum_{i=0}^{k-2} |z_{i+1}'|^2 - \gamma^2 |w_i'|^2 + |z_k|^2 \\ &\ge W_{k-1}(p) - \epsilon. \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, the monotonicity assertion is verified. Therefore the limit $\lim_{k\to\infty} W_k(p)$ exists and equals $W_a(p)$, for $p \in \operatorname{dom} W_a$.

Remark 4.19: The function W_a defined in the proof is called the available storage for the information state system. If Ξ^u is finite gain dissipative with storage function W, then $W_a \le W$, and W_a is also a storage function. W_a solves (4.23) with equality.

As in the case of complete state information, we can deduce stability results for the closed-loop system Σ^u. Stability here means internal stability, and so we must concern ourselves with the stability of the information state system as well. For the remainder of the paper, we will assume that h satisfies the linear growth condition

$$|h(x, u, w)| \le C(|x| + |w|). \tag{4.25}$$

We say that Σ^u is (zero state) z-detectable (respectively, $l_2 - z$-detectable) if $w \equiv 0$ and $\lim_{k\to\infty} z_k = 0$ implies $\lim_{k\to\infty} x_k = 0$ (respectively, $\{z_k\} \in l_2([0, \infty), \mathbf{R}^q)$ implies $\{x_k\} \in l_2([0, \infty), \mathbf{R}^n)$) and asymptotically stable if $w \equiv 0$ implies $\lim_{k\to\infty} x_k = 0$ for any initial condition. For $u \in \mathcal{O}$ and $y \in l_2([0, \infty), \mathbf{R}^p)$, Σ^u is uniformly (w, y)-reachable if for all $x \in \mathbf{R}^n$ there exists $0 \le \alpha(x) < +\infty$ such that for all $k \ge 0$ sufficiently large there exists $x_0 \in \mathbf{R}^n$ and $w \in l_2([0, k-1], \mathbf{R}^r)$ such that $x(0) = x_0$, $x(k) = x$, $h(x_i, u_i, w_i) = y_{i+1}$, $i = 0, \cdots, k-1$, and

$$|x_0|^2 + \sum_{i=0}^{k-1} |w_i|^2 \le \alpha(x). \tag{4.26}$$

Given inputs $u \in \mathcal{O}$ and $y \in l_2([0, \infty), \mathbf{R}^p)$, we say that the information state system Ξ^u is stable if for each $x \in \mathbf{R}^n$ there exists $K_x \ge 0$, $C_x \ge 0$ such that

$$|p_k(x)| \le C_x \quad \text{for all} \quad k \ge K_x \tag{4.27}$$

provided the initial value p_0 satisfies the growth conditions

$$-a_1'|x|^2 - a_2' \le p_0(x) \le -a_1|x|^2 + a_2 \tag{4.28}$$

where $a_1, a_1', a_2, a_2' \ge 0$.

Theorem 4.20: Let $u \in \mathcal{I}$. If Ξ^u is finite gain dissipative and Σ^u is z-detectable, then Σ^u is asymptotically stable. If Ξ^u is finite gain dissipative and Σ^u is $l_2 - z$-detectable and uniformly (w, y)-reachable, then Ξ^u is stable.

Proof: Inequality (4.24) implies

$$\sup_{w\in l_2([0,k-1],\boldsymbol{R}^r)} \sup_{x_0\in\boldsymbol{R}^n} \cdot\left\{p(x_0)+\sum_{i=0}^{k-1}|z_{i+1}|^2-\gamma^2|w_i|^2\right\}\le W(p) \quad (4.29)$$

for all $k>0$. Let $x_0\in\boldsymbol{R}^n$ and select $p=-\beta^u$. Then (4.29) gives, with $w\equiv 0$

$$\sum_{i=0}^{k-1}|z_{i+1}|^2\le W(-\beta^u)+\beta^u(x_0)<+\infty$$

for all $k>0$. This implies $\{z_k\}\in l_2((0,\infty),\boldsymbol{R}^q)$, and so $\lim_{k\to\infty} z_k=0$. By z-detectability, we obtain $\lim_{k\to\infty} x_k=0$. Therefore Σ^u is asymptotically stable. Also, l_2-z-detectability implies $\{x_k\}\in l_2([0,\infty),\boldsymbol{R}^n)$, and by assumption (4.25), $\{y_k\}\in l_2([1,\infty),\boldsymbol{R}^p)$ since $w\equiv 0$.

So now suppose that $\{y_k\}\in l_2([0,\infty),\boldsymbol{R}^p)$. We wish to show that Ξ^u is stable. The dissipation inequality implies

$$p_k(x)\le(p_k,0)\le W(p_0)<+\infty$$

for $p_0=-\beta$, $k\ge 0$. For the lower bound, the hypothesis imply, given x, for all k sufficiently large there exists x_0 and w such that $x(0)=x_0$, $x(k)=x$, and

$$|x_0|^2+\sum_{i=0}^{k-1}|w_i|^2\le\alpha(x)$$

for some finite nonnegative α. Thus

$$\begin{aligned}p_k(x)&\ge p_0(x_0)-\gamma^2\sum_{i=0}^{k-1}|w_i|^2\\&\ge-(a_1'+\gamma^2)\alpha(x)-a_2'\end{aligned}$$

for all k sufficiently large. Therefore Ξ^u is stable. □

Remark 4.21: The question of stability of the information state is a subtle and deep one and is the subject of current investigations [11]. The behavior we are attempting to capture here is that of eventual finiteness (and in fact boundedness) of the information state. The criteria used to imply stability are modeled on those used in the state feedback case and are of course difficult to check in practice. These conditions simplify greatly under appropriate nondegeneracy assumptions. Note that it is feasible that Σ^u is stable with Ξ^u unstable; this corresponds to an unstable stabilizing controller.

2) Solution to the Output Feedback Robust H_∞ Control Problem: We begin this section with a proposition which asserts that if the output feedback robust H_∞ control problem is solvable by an information state feedback controller, then there exists a solution to the dissipation inequality (4.30) below, using the bounded real lemma 4.17. This result is not adequate for a necessity theorem, however, since it is expressed *a priori* in terms of an information state feedback controller. The necessity theorem (Theorem 4.23 below) asserts the existence of a solution of the dissipation in equality assuming only that the output feedback robust H_∞ control problem is solved by some output feedback controller, which need not necessarily be an information state feedback controller.

Proposition 4.22: If a controller $u^i\in\mathcal{I}$ solves the output feedback robust H_∞ control problem, then there exists a function $W(p)$ such that $\operatorname{dom} W$ contains $-\beta^{u^i}$, $W(p)\ge(p,0)$ for all $p\in\operatorname{dom} W$, $W(-\beta^{u^i})=0$, and

$$W(p)\ge\inf_{u\in U}\sup_{y\in\boldsymbol{R}^p}\{W(F(p,u,y))\}\quad(p\in\operatorname{dom}W). \quad (4.30)$$

Proof: The Bounded Real Lemma 4.17 implies the existence of a storage function W_a satisfying the dissipation inequality (4.23)

$$\begin{aligned}W_a(p)&\ge\sup\nolimits_{y\in\boldsymbol{R}^p}\{W_a(F(p,\overline{u}^i(p),y))\}\\&\ge\inf\nolimits_{u\in U}\sup\nolimits_{y\in\boldsymbol{R}^p}\{W_a(F(p,u,y))\}.\end{aligned}$$

Therefore W_a satisfies (4.30). Also, we have $-\beta^{u^i}\in\operatorname{dom} W_a$, $W_a(p)\ge(p,0)$ for all $p\in\operatorname{dom}W_a$, and $W_a(-\beta^{u^i})=0$. □

Theorem 4.23 (Necessity): Assume that there exists a controller $u^o\in\mathcal{O}$ which solves the output feedback robust H_∞ control problem. Then there exists a function $W(p)$ which is finite on $\operatorname{dom} J.(u^o)$, satisfies $W(p)\ge(p,0)$ for all $p\in\operatorname{dom}W$, $W(-\beta^{u^o})=0$, and solves the dissipation inequality (4.30).

Proof: Define

$$W(p)=\inf_{u\in\mathcal{O}}J_p(u)$$

where $J_p(u)$ is defined by (4.21). Then we have

$$(p,0)\le W(p)\le J_p(u^o)\le(p,\beta^{u^o})$$

and so W is finite on $\operatorname{dom} J.(u^o)$. Clearly $W(p)\ge(p,0)$ for all $p\in\operatorname{dom}W$ and $W(-\beta^{u^o})=0$. It remains to show that W satisfies (4.30).

Fix $p\in\operatorname{dom}W$, and let $\epsilon>0$. Choose $u\in\mathcal{O}$ such that

$$W(p)\ge\sup_{k\ge 0,y\in l_2([1,k],\boldsymbol{R}^r)}\{(p_k,0):p_0=p\}-\epsilon. \quad (4.31)$$

Fix $y_1\in\boldsymbol{R}^p$, and set $p_1=F(p,u_0,y_1)$. For any sequence $y_2,y_3,\cdots$ define a sequence $\tilde{y}_1,\tilde{y}_2,\cdots$ by $\tilde{y}_i=y_{i+1}$, $i=1,2,\cdots$ Define a control $\tilde{u}\in\mathcal{O}$ by

$$\tilde{u}_i(\tilde{y}_1,\cdots,\tilde{y}_i)=u_{i+1}(y_1,\tilde{y}_1,\cdots,\tilde{y}_i).$$

Let p_j and $\tilde{p}_j$ denote the information state sequences corresponding to $p_0=p$, $u,y_1,y_2,\cdots$, and $\tilde{p}_0=p_1=F(p,u_0,y_1)$, $\tilde{u},\tilde{y}_1,\tilde{y}_2,\cdots$, respectively. Note that $p_k=\tilde{p}_{k-1}$. Then

$$\begin{aligned}W(p)&\ge\sup\nolimits_{k\ge 1,y_{2,k}}\{(p_k,0):p_0=p\}-\epsilon\\&=\sup\nolimits_{k\ge 0,\tilde{y}_{1,k}}\{(\tilde{p}_k,0):\tilde{p}_0=p_1\}-\epsilon\\&\ge(p_1,0)-\epsilon.\end{aligned}$$

That is

$$W(p)\ge W(F(p,u_0,y_1))-\epsilon.$$

Since y_1 was selected arbitrarily, we get

$$W(p)\ge\sup_{y\in\boldsymbol{R}^p}W(F(p,u_0,y))-\epsilon$$

and therefore

$$W(p)\ge\inf_{u\in U}\sup_{y\in\boldsymbol{R}^p}W(F(p,u,y))-\epsilon.$$

The infimum is finite in view of Lemma 4.5. From this, we see that W satisfies (4.30), since $\epsilon > 0$ is arbitrary. Also, it follows that $F(p, \overline{u}^*(p), y) \in \operatorname{dom} W$ for all y whenever $p \in \operatorname{dom} W$, where $\overline{u}^*(p) \in \operatorname{argmin}_{u \in U} \sup_{y \in \boldsymbol{R}^p} W(F(p, u, y))$. □

Theorem 4.24 (Sufficiency): Assume that W is a solution of (4.30) satisfying $-\beta \in \operatorname{dom} W$ (for some $\beta(\cdot) > 0$ with $\beta(0) = 0$, $W(p) \geq (p, 0)$ for all $p \in \operatorname{dom} W$ and $W(-\beta) = 0$. Let $\overline{u}^*(p)$ be a control value which achieves the minimum in (4.30) for all $p \in \operatorname{dom} W$. Assume $p_0 = -\beta$ is an initial condition for which $p_k \in \operatorname{dom} W$ for all $k = 1, 2, \cdots$ (for all sequences y_k). Then the controller $u' \in \mathcal{I}$ defined by $u_k^* = \overline{u}^*(p_k)$ solves the output feedback robust H_∞ control problem if the closed loop system Σ^{u^*} is $l_2 - z$-detectable and uniformly (w, y)-reachable.

Proof: The information state system Ξ^{u^*} is finite gain dissipative, since (4.30) implies (4.23) for the controller u^*. Hence by Theorem 4.17, Ξ^{u^*} is finite gain. Theorem 4.20 then shows that Σ^{u^*} is asymptotically stable and Ξ^{u^*} is stable. Hence u^* solves the information state feedback robust H_∞ control problem and hence the output feedback robust H_∞ control problem. □

Remark 4.25: As in the state feedback case (Section III-B2), the significance of this result is that the controlled dissipation inequality (4.30) provides (in principle) a recipe for solving the output feedback robust H_∞ control problem.

Corollary 4.26: If the output feedback robust H_∞ control problem is solvable by some output feedback controller $u^o \in \mathcal{O}$, then it is also solvable by an information state feedback controller $u^* \in \mathcal{I}$.

Remark 4.27: It follows from the above results that necessary and sufficient conditions for the solvability of the output feedback robust H_∞ problem can be expressed in terms of three conditions, viz., i) existence of a solution $p_k(x)$ to (4.10), ii) existence of a solution $W(p)$ to (4.30), and iii) coupling: $p_k \in \operatorname{dom} W$. Further developments of this theory can be found in [11] and [12].

V. Continuous Time

In this section, we discuss briefly the application of the above information state approach to continuous-time systems. This has been considered formally in [20] and [29] and in detail in [18].

The continuous-time version of system model (2.1) is

$$\begin{cases} \dot{x}(t) = b(x(t), u(t), w(t)), \\ z(t) = l(x(t), u(t), w(t)), \\ y(t) = h(x(t), u(t), w(t)) \end{cases} \tag{5.1}$$

and the output feedback robust H_∞ control problem is to find a controller $u = u(y(\cdot))$, responsive only to the observed output y, such that the resulting closed-loop system Σ^u is asymptotically stable when no disturbances are present, and Σ^u is finite gain, i.e., there exists a finite quantity $\beta^u(x_0)$ with $\beta(0) = 0$ such that

$$\begin{cases} \int_0^T |z(t)|^2\, dt \leq \gamma^2 \int_0^T |w(t)|^2\, dt + \beta^u(x_0) \\ \text{for all } w \in L_2([0, T], \boldsymbol{R}^r) \text{ and all } T \geq 0. \end{cases} \tag{5.2}$$

Applying the same methodology as in Section IV, one defines an information state $p_t(x)$ and a value function $W(p)$. The information state for this problem has dynamics

$$\dot{p}_t = F(p_t, u(t), y(t)) \tag{5.3}$$

where $F(p, u, y)$ is the nonlinear differential operator

$$\begin{aligned} F(p, u, y) = \sup_{w \in \boldsymbol{R}^r} \, [&-\nabla_x p \cdot b(\cdot, u, w) + |l(\cdot, u, w)|^2 \\ &- \gamma^2 |w|^2 + \delta_y(h(\cdot, u, w))]. \end{aligned} \tag{5.4}$$

Equation (5.3) is a first-order nonlinear partial differential equality (PDE) in $\boldsymbol{R}^n$. The dissipation partial differential inequality (PDI) is infinite dimensional (since it is defined on the infinite dimensional space $\mathcal{E}$)

$$\inf_{u \in U} \sup_{y \in \boldsymbol{R}^p} \nabla_p W(p) \cdot F(p, u, y) \leq 0 \text{ in } \mathcal{E}. \tag{5.5}$$

Here, $\nabla_p W(p)$ denotes, say, the Frechet derivative of W at p. If (5.3) and (5.5) possess sufficiently smooth solutions, then the control which attains the minimum in (5.5) defines a controller which solves the robust H_∞ control problem.

The technical difficulties concern the precise sense in which solutions to (5.3) and (5.5) are to be understood (since smooth solutions are not likely to exist in general) and existence of the minimizing controller. These difficulties are present even in very simple deterministic state feedback optimal control problems.

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