

Properties of the Multiscale Maxima and Zero-Crossings Representations

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Abstract—The analysis of a discrete multiscale edge representation is considered. A general signal description, called an inherently bounded adaptive quasi linear representation (AQLR), motivated by two important examples, namely, the wavelet maxima representation, and the wavelet zero-crossings representation, is introduced. This paper mainly addresses the questions of uniqueness and stability. It is shown, that the dyadic wavelet maxima (zero-crossings) representation is, in general, nonunique. Nevertheless, using the idea of the inherently bounded AQLR, two stability results are proven. For a general perturbation, a global BIBO stability is shown. For a special case, where perturbations are limited to the continuous part of the representation, a Lipschitz condition is satisfied.

I. INTRODUCTION

AN interesting and promising approach to signal representation is to make explicit important features in the data. The first example, taught in elementary calculus, is a “sketch” of a function based on extrema of a signal and possibly of its few derivatives. The second instance, widely used in computer vision, is an edge representation of an image. If the size of expected features is *a priori* unknown, the need for a multiscale analysis is apparent. Therefore, it is not surprising that multiscale sharp variation points are meaningful features for many signals, and they have been applied, for example, in edge detection [5], [18], signal compression [15], pattern matching [14], detection of transient signals [8], [11] and speech analysis [24].

Traditionally, multiscale edges are determined either as extrema of Gaussian-filtered signals [23] or as zero-crossings of signals convolved with the Laplacian of a Gaussian (see, e.g., [10] for a comprehensive review). Mallat in a series of papers [14], [15], [17] (the middle joint with Zhong) introduced zero-crossings and extrema of the wavelet transform as a multiscale edge representation. Two important advantages of this method are low algorithmic complexity and flexibility in choosing the basic filter. Moreover, [14], [15] propose reconstruction procedures and show accurate numerical reconstruction results from zero-crossings and maxima representations. In [14], [15], as in many other works in this area, the basic

algorithms were developed using continuous variables. The continuous approach gives an excellent background to motivate and justify the use of either local extrema or zero-crossings as important signal features. Unfortunately, in the continuous framework, analytic tools to investigate the information content of the representation are not yet available. The knowledge about properties of the representations is mainly based on empirical reconstruction results. From the theoretical point of view, there are still important open problems, e.g., stability, uniqueness, and structure of a reconstruction set (a family of signals having the same representation).

A. Previous Works

The multiscale edge representation has mainly been investigated in the zero-crossings case. The best-known result concerning the reconstruction of a signal from zero-crossings is the Logan Theorem [13]. This theorem basically states that zero-crossings uniquely define the signal within the family of band-pass signals having the property that the width of the band is smaller than the lower frequency of the band. Proving this theorem, Logan made an analytic extension of the signal and used standard properties of zeros of analytic functions. These tools are known as unstable and Logan has noticed that the reconstruction from zero-crossings appears to be very difficult and impractical. Under certain restrictions on the class of signals, usually polynomial data have been assumed, several additional proofs that zero-crossings form a complete (unique) signal representation have been published. All known proofs do not provide any stability results since they are based on unstable characterizations of analytic functions. The reader is referred to [10] for more details and further references.

In addition, in the case of general initial data, the restriction to polynomial data or even to band-limited signals may provide a poor approximation of the original signal. The situation is similar to the fact that a polynomial is determined by its zeros, but any nonzero value of a continuous function cannot be determined from zero-crossings of the function.

In spite of the last remark, there have been a number of attempts to reconstruct signals from multiscale zero-crossings, especially in image processing, e.g., [6], [21], [25]. They have been based on the belief that the restriction of a given reconstruction scheme into “natural” im-

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age data will be sufficiently stable and precise. Although good reconstruction results have been shown, stability results have not been proven.

Hummel and Moniot [10] have exhibited the stability problem by showing two significantly different signals having almost the same multiscale zero-crossings representations. In order to stabilize the reconstruction of a function from its zero-crossings, the authors have included the gradient along each zero-crossing. In fact, improved numerical results have been reported but stability has not been analyzed. The reconstruction algorithm in [10] is based on the solution of a Heat Equation; this approach is valid only for the Laplacian of a Gaussian filter and it is required to record the zero-crossings on a dense sequence of scales.

Aware of the above problems, Mallat [14] proposed to use the wavelet zero-crossings representation as a complete and stable signal description. In order to overcome the apparent instability of zero-crossings, he has included the values of the wavelet transform integral calculated between two consecutive zero-crossings. Using a reconstruction algorithm based on alternate projections, very accurate reconstruction results have been shown. In subsequent work, Mallat with Zhong [15] introduced the wavelet maxima representation as an alternative to the wavelet zero-crossings representation. As in the zero-crossings case, they have demonstrated very accurate reconstruction results. But, in both papers, neither uniqueness nor stability has been proven.

Recently two independent counterexamples for uniqueness have been published. One counterexample was given in Berman [3] and will be described later. The second example was given in a continuous context by Meyer [19]. His example is based on function $f_0(x)$ and a parametric family of functions $f_\alpha(x)$. Meyer [19] has shown that, for a particular wavelet transform, all functions of the form $f_0(x) + f_\alpha(x)$, for a suitable set of sequences $\alpha = \{\alpha_k\}_0^\infty$, have the same wavelet zero-crossings representation.

B. Outline of the Work

Open problems regarding uniqueness and stability have motivated this work. Our objective is to analyze these theoretical questions using a model of an actual implementation. The main assumption is that the data is discrete and finite. The discrete multiscale maxima and zero-crossings representations are defined in the general set-up of a linear filter bank, however, the main goal is to consider a particular case where the filter bank describes the wavelet transform. Since reconstruction sets of both maxima and zero-crossings representations have a similar structure, a general form is introduced and named adaptive quasi linear representation (AQLR). Moreover, many generalizations of the basic maxima and zero-crossings representations fit into the framework of the AQLR. This work uses the idea of the AQLR to investigate rigorously fundamental questions: uniqueness and stability.

Regarding the uniqueness question, first, conditions for uniqueness are presented. By applying these conditions to the wavelet transform-based representation, a conclusive result is obtained. It turns out, that neither the wavelet maxima representation nor the wavelet zero-crossings representation is, in general, unique. The proof is based on constructing a sinusoidal sequence, whose maxima (zero-crossings) representations cannot be unique for any dyadic wavelet transform. In view of additional counterexamples by Meyer [19], the importance of our result is mostly related to its generality. In addition, the question of how a particular representation can be tested for uniqueness is discussed. An example of a nonunique representation is described and analyzed.

The next subject is stability of the representation. This issue is of great importance because there are many known examples of unstable zero-crossings representations. Using the idea of the inherently bounded AQLR, we are able to prove stability results. For a general perturbation, global BIBO (bounded input, bounded output) stability is shown. For a special case, where perturbations are limited to the continuous part of the representation, a Lipschitz condition is satisfied. To the best of our knowledge, these are the first rigorous stability results established in the context of multiscale maxima (zero-crossings) representations.

In the framework of the inherently bounded AQLR one can obtain many further results. For example, in [2], [4] we propose a new reconstruction scheme and several modifications to the basic multiscale maxima representation.

II. BASIC DEFINITIONS

The multiscale maxima or zero-crossings representation is based on the following idea: smooth the signal at various scales, and detect, at every scale, sharp variation points. These points are chosen using local extrema or zero-crossings of the first or the second order derivative. This kind of reasoning requires continuous-time signal model. On the other hand, practical implementations demand discrete-time signal description. The relations between these different signal model types yield an active and interesting research area. See, for example, [14], [15] for a comprehensive discussion about different aspects of wavelet-based representations for continuous and discrete signals.

This work deals with signals and representations from the implementation point of view. Our study begins with the assumption that all signals are discrete and of finite duration. In other words, signals under consideration belong to \mathcal{L} , a linear space of real, finite sequences

$$\mathcal{L} \triangleq \{f: f = \{f(n)\}_{n=0}^{N-1}, f(n) \in \mathbb{R}\}.$$

Note that \mathcal{L} can be viewed as a vector space. Indeed, many concepts, like convex sets, null and range space of a linear operator, etc., will be used with standard notations of \mathbb{R}^N . However, to define local maxima, minima

and zero-crossings, the notion of a linear sequence space is required.

The definitions of the multiscale maxima and zero-crossings representations are based on those described in [14], [15]. However, there are two noticeable differences between the approach of [14], [15] and ours. First, we assume no knowledge about the continuous-time version of signals, namely, the original signal is assumed to be discrete (time) and all representation properties (uniqueness, stability, reconstruction) are considered with respect to this signal. Second, the wavelet transform, used in [14], [15], is generalized here to an arbitrary collection of linear operators.

Before the general structure is formulated, let us first introduce the discrete wavelet transform used in [14], [15]. The discrete dyadic wavelet transform is characterized by two discrete filters H and G , and the number of levels J . Let H_j, G_j denote the discrete filters obtained by putting $2^j - 1$ zeros between all two consecutive coefficients of the filters H and G , respectively. The following recursive formula defines the sequences $W_j f$ and $S_j f$:

$$W_{j+1} f = G_j * S_j f$$

$$S_{j+1} f = H_j * S_j f$$

where $j = 0, 1, 2, \dots, J-1$, $S_0 f = f$, $H_0 = H$, and $G_0 = G$. The discrete dyadic wavelet transform of a signal f consists of $J+1$ signals, $(W_j f)_{j=1}^J$ and Sf . We can interpret W_j as a linear operator, defined by consecutive convolutions with $H, H_1, \dots, H_{j-1}, G_j$. Similarly, S_j corresponds to a linear operator equivalent to successive convolutions with $H, H_1, \dots, H_{j-1}, H_j$. Henceforth, since J is fixed, the convention $W_{J+1} f = S_j f$ will be used.

In this work, the multiscale signal representation of a signal $f \in \mathcal{L}$ is assumed to be of the form

$$(W_j f)_{j=1}^{J+1}$$

such that

- W_j are linear operators
- $(W_j f)_{j=1}^{J+1}$ is a complete (unique) representation of a signal f .

Note, that the discrete wavelet transform and any complete filter bank are particular instances of this structure.

A. Multiscale Maxima

The multiscale maxima (extrema) representation, $R_m f$, is composed of local extrema (arguments and values) of sequences $W_j f$ ($j = 1, 2, \dots, J$). In addition one entire signal $W_{J+1} f$ is allowed to be a part of the representation. Let us explain why we use interchangeably the terms "the multiscale extrema representation" and "the multiscale maxima representation." The term "the wavelet maxima representation" has been introduced in [15], because the basic wavelet extrema representation has been further modified by removing local minima of absolute values of $W_j f$. The reason to preserve this notation is twofold: a "historical" rationale and the fact that the most important

extrema appear to be maxima of absolute value of signals at different scales.

Now, the goal is to derive a precise definition of the discrete multiscale maxima representations. Analysis of discrete extrema (zero-crossings) requires some carefulness and an additional notation will be introduced. The collection of local extrema of a signal $f \in \mathcal{L}$ and their arguments will be called one-level maxima representation and denoted $M(f)$. Let $X(f)$ and $Y(f)$ be sets of arguments of local maxima and local minima, respectively, of a sequence f . The formal definitions are

$$X(f) = \{k: f(k+1) \leq f(k) \text{ and } f(k-1) \leq f(k) \\ k = 0, 1, 2, \dots, N-1\} \quad (1)$$

$$Y(f) = \{k: f(k+1) \geq f(k) \text{ and } f(k-1) \geq f(k) \\ k = 0, 1, 2, \dots, N-1\}. \quad (2)$$

In this work, in order to avoid boundary problems, an N -periodic extension of finite sequences is assumed.

The set of arguments of local extrema is denoted $I_o(f)$ and defined as a pair of two sets

$$I_o f \triangleq (X(f), Y(f)).$$

Now, the sequence of local extrema, values of f at extreme points, $V_o(f)$ can be written as

$$V_o(f) \triangleq (f(k))_{k \in I_o(f)}.$$

Using this notation, one-level maxima representation, $M(f)$, is written as

$$M(f) \triangleq (I_o(f), V_o(f)).$$

Observe, that $I_o(f)$ consists of discrete indices, while $V_o(f)$ is a sequence of continuous values.

To illustrate one-level maxima representation let us consider the following example:

Example 1: Let f be the sequence ($N = 8$) depicted in Fig. 1. The values $f(k)$ are given as follows:

$$f(0) = 3.0 \quad f(1) = 2.3 \quad f(2) = 1.0 \quad f(3) = -2.2 \\ f(4) = -3.2 \quad f(5) = -2.1 \quad f(6) = 0.3 \quad f(7) = 1.6.$$

The one-level maxima representation, $M(f)$ consists of an index part, $I_o(f) = (\{0\}, \{4\})$, and a sequence of values, $V_o(f) = (f(0), f(4)) = (3.0, -3.2)$. Henceforth, curly brackets are used to indicate sets of integers, while parentheses describe sequences: of real numbers or sets.

Using the notation of the one-level maxima representation, $M(f)$, the multiscale maxima representation, $R_m f$, is defined immediately

$$R_m f \triangleq ((M(W_j f))_{j=1}^J, W_{J+1} f).$$

This representation can be cast to the following structure:

$$R_m f = (If, Vf)$$

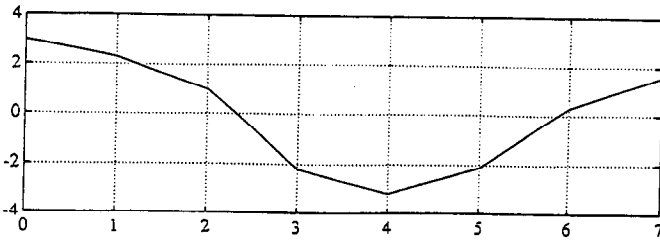


Fig. 1. The signal f in the maxima representation example.

where If is a set of all indices used in the representation, i.e.,

$$If = (I_o(W_j f))_{j=1}^J.$$

Vf is a collection of all continuous values included in $R_m f$

$$Vf = ((V_o(W_j f))_{j=1}^J, W_{J+1} f).$$

Consider $R_m(\alpha_1 f_1 + \alpha_2 f_2)$, the maxima representation of a linear combination of two signals. Neither the values at local extrema nor their arguments satisfy linear properties. Thus the multiscale maxima representation is a nonlinear signal transformation. However, for a fixed If , V can be considered as a linear operator. For all $h \in \mathcal{L}$, let us define Vh as a sequence of samples of $W_j h$ ($j = 1, 2, \dots, J, J+1$) at points specified by If . The detailed definition is

$$Vh \triangleq (((W_j h(k))_{k \in I_o(W_j f)}_{j=1}^J, W_{J+1} h).$$

The notation $R_m f = (If, Vf)$, due to its explicit linear part, is especially beneficial for the analysis of the reconstruction set. The reconstruction set is defined for a general signal representation.

For a given representation Rf , the reconstruction set $\Gamma(Rf)$ consists all sequences satisfying this representation, i.e.,

$$\Gamma(Rf) \triangleq \{h \in \mathcal{L} : Rh = Rf\}. \quad (3)$$

In other words, if the representation is described by an operator acting on sequences, then the reconstruction set is the inverse image of this operator at a given sequence.

Returning to our example, the reconstruction set of the given one-level maxima representation consists of all sequences $h(0), h(1), \dots, h(7)$ such that

$$h(0) = 3.0 \quad h(4) = -3.2$$

and

$$h(0) > h(1) > h(2) > h(3) > h(4)$$

$$h(4) < h(5) < h(6) < h(7) < h(0).$$

The fact, that the reconstruction set of the multiscale maxima representation can be described as a solution of a set of linear equalities and inequalities has a major role in this work. Moreover, there is a common structure which fits both the multiscale maxima and zero-crossings representations.

Definition 1: Let $Rf = \{If, Vf\}$ be a signal representation, such that for a fixed If , V is a linear operator de-

finied for all $h \in \mathcal{L}$. This representation is called an AQLR if for a fixed If , there exists a linear operator C and a sequence c such that

$$h \in \Gamma(Rf) \Leftrightarrow Vh = Vf \text{ and } Ch > c. \quad (4)$$

The reasoning behind the name ‘‘adaptive quasi linear representation’’ (AQLR) is the following: This representation is adaptive since V, C, c depend on the sequence f (via the set If). It is quasi linear because its reconstruction set is a solution of a set of linear equalities and inequalities.

In the sequel, instead of a ‘‘sequence’’ notation e.g., $h \in \mathcal{L}$, sometimes we will use a ‘‘vector notation,’’ then h will be a column vector, $h \in \mathbb{R}^N$ and linear operators, like C , will appear as corresponding matrices. The interpretation of (4) in matrix and vector context is obvious.

Proposition 1: Any multiscale maxima representation is an AQLR.

The proof requires additional notations and is given in Appendix A.

The origin of the next definition is a certain boundedness characteristic of the wavelet maxima representation. The property that the reconstruction set can be bounded by the linear part of the representation is implemented in the stability part of this work.

Note that f and Vf belong to linear spaces with different dimensions. Let $\|\cdot\|$ denote the Euclidean norm in an appropriate, finite dimensional linear space.

Definition 2: An AQLR is called inherently bounded if there exists a real $K > 0$ such that

$$h \in \Gamma(Rf) \Rightarrow \|h\| \leq K \|Vf\|. \quad (5)$$

The coefficient K can depend on the parameters of the representation e.g., $N, J, W_1, \dots, W_J, W_{J+1}$ but it must be independent of If and Vf .

Inequality (5) is very similar to a well known condition for stability in regular or irregular sampling theory. See, for example, a comprehensive paper by Benedetto [1]. Our situation is slightly dissimilar because for different signals, different linear operators are obtained. Nevertheless, as will be shown in Section IV, condition (5) implies stability.

Theorem 1: Any multiscale maxima representation is an inherently bounded AQLR.

Proof: Since $\{W_1, W_2, \dots, W_J, W_{J+1}\}$ provide a unique signal representation there exists $K_1 > 0$ such that for all $h \in \mathcal{L}$

$$\|h\|^2 \leq K_1 \left(\sum_{j=1}^{J+1} \|W_j h\|^2 \right). \quad (6)$$

Therefore it suffices to find $K_2 > 0$ such that for all $h \in \Gamma(Rf)$ and for $j = 1, 2, \dots, J, J+1$

$$\|W_j h\| \leq K_2 \|Vf\|. \quad (7)$$

Let $h \in \Gamma(Rf)$. $W_{J+1} h$ is included in Vh , hence

$$\|W_{J+1} h\| \leq \|Vf\|. \quad (8)$$

Consider,

$$|W_j h(n)| \leq \max_n |W_j h(n)| = \max_n |W_j f(n)| \leq \|Vf\|. \quad (9)$$

The middle equality holds because $W_j h$ has the same local extrema as $W_j f$, in particular it has the same global extrema as $W_j f$. The right inequality is valid since $\max_n |W_j h(n)|$ appears (with its original sign) as a component of Vf . Therefore, we conclude

$$\|W_j h\| \leq \sqrt{N} \|Vf\|. \quad \square \quad (10)$$

The next section describes a very similar treatment for the multiscale zero-crossings representation. The main observation is that also this representation satisfies the structure of the inherently bounded AQLR.

B. The Multiscale Zero-Crossings Representation

One-level zero-crossings representation, $Z_o(f)$, of a signal $f \in \mathcal{L}$, consists of indices at which the sequence f changes sign (zero-crossings) and sums of elements of f , calculated between two consecutive sign change points. The latter part has been proposed by Mallat in [14] to stabilize the zero-crossings representation. The multiscale zero-crossings representation is composed of one-level zero-crossings representations of the signals $(W_j f)_{j=1}^J$. As in the multiscale maxima representation case, one entire signal, $W_{J+1} f$, is allowed to a part of the representation.

To make the above idea rigorous, several notations are introduced. For any $f \in \mathcal{L}$, let $I_z(f)$ be the set of zero-crossings of f

$$I_z(f) \triangleq \{k: f(k-1) \cdot f(k) \leq 0, k = 0, 1, \dots, N-1\}. \quad (11)$$

For all $k \in I_z(f)$, the segment of k with respect to zero-crossings of f , $\mathcal{O}(f, k)$ is defined as follows:

$$\mathcal{O}(f, k) \triangleq \{k, k+1, \dots, k+r\} \quad (12)$$

such that $r \geq 0$, $k+r+1 \in I_z(f)$, and $k+1, k+2, \dots, k+r \notin I_z(f)$. Notice, that for fixed f and k , for all $k' \in \mathcal{O}(f, k)$, $f(k')$ has a constant sign. Since the segment $\mathcal{O}(f, k)$ consists of points between two consecutive zero-crossings of f , it is used to define $U_z(f)$, the sequence of sums of f calculated between two consecutive zero-crossings.

$$U_z(f) \triangleq \left(\sum_{k' \in \mathcal{O}(f, k)} f(k') \right)_{k \in I_z(f)}. \quad (13)$$

The one-level zero-crossings representation $Z_o(f)$ includes the set of zero-crossings $I_z(f)$ and the sequence of sums $U_z(f)$

$$Z_o f = (I_z(f), U_z(f)). \quad (14)$$

Example 2: Consider Fig. 2. It describes sequence $f(N)$

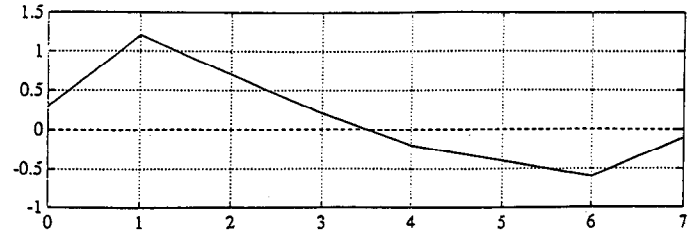


Fig. 2. The signal f in the zero-crossings representation example.

= 8) defined as follows:

$$\begin{aligned} f(0) &= 0.3 & f(1) &= 1.2 & f(2) &= 0.7 & f(3) &= 0.2 \\ f(4) &= -0.2 & f(5) &= -0.4 & f(6) &= -0.6 \\ f(7) &= -0.1 \end{aligned}$$

The one-level zero-crossings representation consists of the index set $I_o(f) = \{0, 4\}$ and of the values $U_z(f) = (\sum_{k=0}^3 f(k), \sum_{k=4}^7 f(k)) = (2.4, -1.3)$.

The multiscale zero-crossings representation, $R_z f$, can be written as

$$R_z f \triangleq ((Z_o(W_j f))_{j=1}^J, W_{J+1} f). \quad (15)$$

This representation can also be cast into the form

$$R_z f = (If, Vf) \quad (16)$$

where

$$\begin{aligned} If &= (I_z(W_j f))_{j=1}^J \\ Vf &= ((U_z(W_j f))_{j=1}^J, W_{J+1} f). \end{aligned}$$

Now, let us explain how V can be interpreted as a linear operator. The idea is to decompose $U_z(W_j f)$ to linear and nonlinear components. To simplify the notation, we assume $(W_j)_{j=1}^J$ fixed. For all $h \in \mathcal{L}$, and for any $k \in I_z(W_j f)$ we define $(U_j^f h)(k)$, the sum of values of $W_j h$ between two consecutive zero-crossings of $W_j f$.

$$(U_j^f h)(k) = \sum_{k' \in \mathcal{O}(W_j f, k)} W_j h(k').$$

Let $U_j^f h$ denote the sequence of sums of $W_j h$ between all consecutive zero-crossings of $W_j f$

$$U_j^f h = ((U_j^f h)(k))_{k \in I_z(W_j f)}. \quad (17)$$

U_j^f is a linear operator associated with $U_z(W_j)$. For all $h \in \mathcal{L}$, Vh is defined as follows:

$$Vh = ((U_j^f h)_{j=1}^J, W_{J+1} h)$$

and indeed V can be interpreted as a linear operator.

Regarding the representation set, first observe that an arbitrary signal h has the same one-level zero-crossings representation as the signal f described in the example, if and only if,

$$\begin{aligned} \sum_{k=0}^3 h(k) &= \sum_{k=0}^3 f(k) = 2.4 \\ \sum_{k=4}^7 h(k) &= \sum_{k=4}^7 f(k) = -1.3 \end{aligned}$$

and

$$\begin{aligned} h(0) > 0 \quad h(1) > 0 \quad h(2) > 0 \quad h(3) > 0 \\ h(4) < 0 \quad h(5) > 0 \quad h(6) < 0 \quad h(7) < 0. \end{aligned}$$

Likewise, the reconstruction set of the multiscale zero-crossings representation fits the structure of AQLR's. Moreover, the boundedness property is satisfied as well.

Theorem 2: Any multiscale zero-crossings representation is an inherently bounded AQLR.

Proof: See Appendix B.

III. UNIQUENESS

This section deals with a special case of the multiscale signal representation, the discrete wavelet transform. The main result established here states that, in general, the wavelet maxima (zero-crossings) representation is not unique. In addition, a uniqueness test for a representations based on a certain wavelet transform is discussed. This section concludes with an analysis of a particular nonunique signal representation.

A. Uniqueness Characterization

A representation $Rf = \{If, Vf\}$ is said to be unique, if the reconstruction set $\Gamma(Rf)$ consists of exactly one element. We have the following uniqueness characterization for AQLR's.

Lemma 1: Let $Rf = \{If, Vf\}$ be an AQLR. Then Rf is unique, if and only if, the kernel of the operator V is trivial, i.e., $\mathfrak{N}V = \{0\}$.

Proof: The lemma becomes obvious by topological arguments. Nevertheless, an elementary but constructive proof will be given. Initially, let us assume that the representation is not unique. Then there exists $h \neq f$ such that $Rh = Rf$. In particular, $Vh = Vf$, but then $0 \neq h - f \in \mathfrak{N}V$.

Next, consider the case where the kernel of V , $\mathfrak{N}V$ is not trivial. Let $h \neq 0$ be such that $Vh = 0$. Suppose $\alpha > 0$ and consider $f_\alpha \triangleq \alpha h + f$, as a candidate to belong to $\Gamma(Rf)$. Of course $Vf_\alpha = Vf$, therefore $f_\alpha \in \Gamma(Rf)$ is and only if $Cf_\alpha > c$ (see Definition 1). The latter is equivalent to

$$\alpha \cdot Ch > c - Cf. \quad (18)$$

Let $(c - Cf)_i$ be the i th component on the vector $c - Cf$. Since $f \in \Gamma(Rf)$, $(c - Cf)_i$ is negative for all i . Define

$$\alpha_0 \triangleq \min \left\{ \frac{(c - Cf)_i}{(Ch)_i} : (Ch)_i < 0 \right\}. \quad (19)$$

Note that $\alpha_0 > 0$. It is easy to show that for all α such that $0 < \alpha < \alpha_0$

$$Cf_\alpha > c. \quad (20)$$

Consequently, the representation Rf is not unique. \square

This claim has some significant consequences. Using the above lemma, an algorithm which tests for uniqueness can be developed. One option is to derive it from a rank

test of the operator V . Another, more ambitious, approach is to characterize, for a particular application, all sets If giving rise to a unique representation.

Perhaps the most important consequence of Lemma 1 is the fact that uniqueness of the representation Rf is equivalent to uniqueness of the underlying irregular sampling Vf . In other words, in the unique case, all the information about the signal is already contained in Vf . Neglecting for a moment stability issues, we can say that, in the unique case, since V is a nonsingular operator, additional constraints $Cf > c$ are redundant. On the other hand, from the signal compression, understanding and interpretation point of view, it seems to be desirable that little information would be specified explicitly by Vf and as much as possible information about a signal should be described implicitly by $Cf > c$. But the latter situation can arise only for nonunique cases. Therefore, in our opinion, the most important and interesting features of AQLR's appear in the nonunique case.

B. The Nonuniqueness Theorem

This subsection aims to show that, in general, the discrete dyadic wavelet maxima (zero-crossings) representation is not unique. The precise statement of the non-uniqueness theorem is as follows.

Theorem 3: Consider a discrete dyadic wavelet maxima (zero-crossings) representation based on a discrete filter H , whose transfer function is $H(\omega)$. If $H(\pi) = 0$, $J \geq 3$, and N is a multiple of 2^J then there exists a sequence f which has a nonunique maxima (zero-crossings) representation.

Let us point out that, although the hypothesis of the theorem may seem to be demanding, it is just a technical condition. Usually the number of levels, J , satisfies $J \geq 3$. In order to benefit from the fast wavelet transform, N has to be a multiple of 2^J . Since $H(\omega)$ is a low pass filter, it is natural to assume that $|H(\omega)|$ reaches its minimum of π . If this minimum is nonzero, then essentially $S_J f^1$ contains all information about f and the maxima (zero-crossings) information is redundant. Moreover $H(\pi) = 0$ is a well known condition for the regularity of the underlying scaling function $\Phi(\xi)$ (for more information, see [7] and [20]). Indeed, all filters used by Mallat, Zhong and many others fulfill the conditions of Theorem 3.

Most of this section describes the proof of the theorem, which will be divided to proofs of several propositions. The result is a consequence of Lemma 1, which relates uniqueness of the representation to the set $\mathfrak{N}V$, the kernel of the sampling information. The main idea is to construct a sequence f such that the set $\mathfrak{N}V$ corresponding to the representation Rf cannot be $\{0\}$. The construction of the counter example is based on the family of sequences \mathfrak{B} , defined as follows:

$$\mathfrak{B} = \{y_p\}_{p=1}^{2^J-1} \quad (21)$$

¹In the wavelet transform context, we prefer to use the notation S_J instead of W_{J+1} .

where

$$y_{2^p-1}(k) = \cos\left(\frac{2\pi pk}{2^J}\right) \quad p = 1, 2, \dots, 2^J-1 \quad (22)$$

$$y_{2^p}(k) = \sin\left(\frac{2\pi pk}{2^J}\right) \quad p = 1, 2, \dots, 2^J-1. \quad (23)$$

Proposition 2: The set \mathcal{B} is included in $\mathcal{N}S_J$, the kernel of the operator S_J .

Proof: See Appendix C.

Notice, that y_{2^j} does not appear in the set \mathcal{B} . The reason is that $y_{2^j} = 0$ and we claim that the set \mathcal{B} does not contain zero. It is easy to show that the set \mathcal{B} is orthogonal. Therefore the set \mathcal{B} is linearly independent.

As a generic example of nonuniqueness the following sequence is proposed.

$$f(k) = \cos\left(2\pi \frac{k}{2^J}\right) \quad k = 0, 1, \dots, N-1. \quad (24)$$

Observe that the same sequence is proposed for all dyadic wavelet transforms and for both the maxima representation and the zero-crossings representation.

The representation $Rf = \{If, Vf\}$ is unique if and only if $\mathcal{N}V = \{0\}$. Consequently, the nonuniqueness of Rf is easily deduced from the following proposition. Let $\text{span}(\mathcal{B})$ be a collection of all linear combinations of sequences from \mathcal{B} .

Proposition 3: The equation

$$Vh = 0 \quad h \in \text{span}(\mathcal{B}) \quad (25)$$

has a nontrivial solution.

Proof: Consider an arbitrary $h \in \text{span}(\mathcal{B})$.

$$h = \sum_{p=1}^{2^J-1} \alpha_p y_p. \quad (26)$$

The dimension of $\text{span}(\mathcal{B})$ is $2^J - 1$. The idea is to show that the set of equations $Vh = 0$ yields less than $2^J - 1$ independent equations with unknowns $\{\alpha_p\}$. Recall that Vh consists of $S_J h$ and of J components such that every component corresponds to $W_j h$ ($j = 1, 2, \dots, J$). For the maxima representation those component are values of $W_j h$ at local extrema of $W_j f$. For the zero-crossings representation those components are the sums of elements of $W_j h$ calculated between any two consecutive zero-crossings of $W_j f$.

From Proposition 2 we see that $S_J h = 0$ for all $\{\alpha_p\}$. Let j be fixed. Let us count independent equations related to $W_j h$. Recognize that f is a 2^J -periodic sinusoid, therefore $W_j f$ is a 2^J -periodic sinusoid as well. Moreover, y_p and $W_j y_p$ for $p = 1, 2, \dots, 2^J - 1$ are also 2^J -periodic. Therefore it suffices to consider only one 2^J period. But $W_j f$ has only two local extrema (zero-crossings) in a 2^J period. Consequently $W_j h$ implies only 2 independent

equations with $\{\alpha_p\}$ as unknowns! There are J levels ($j = 1, 2, \dots, J$) so the set of equations $Vh = 0$ contains, at most, $2J$ independent equations with $\{\alpha_p\}$ as unknowns. But,

$$2^J - 1 > 2J \quad \forall J \geq 3. \quad (27)$$

Accordingly, (25) has a nontrivial solution and the representation is not unique. \square

Some remarks need to be made at this point. From the proof, it turns out, that it is relatively easy to produce more examples of nonunique dyadic wavelet maxima (zero-crossings) representations using 2^p -periodic signals, where p is an integer. For example, consider $J = 5$ and let f be a 2^5 -periodic signal. When $W_j f$ ($j = 1, 2, \dots, 5$) are 2^5 -periodic as well. In this case, if $2^J - 1 = 31$ is greater than the total number of local extrema (zero-crossings) of $W_1 f, W_2 f, \dots, W_5 f$ per one 2^J period, then the representation is not unique. In other words, if $W_j f$'s have, in the mean, less than $31/5 = 6.2$ local extrema (zero-crossings) in one period, than Rf cannot be unique.

Hummel with Moniot [10], Mallat [14], and Mallat with Zhong [15] have reported that high frequency errors may occur in the discrete maxima (zero-crossings) representation. For these 2^J -periodic signals, components of the reconstruction error can appear as 2^p -periodic signals for $p = 1, 2, \dots, J$. Most of them cannot be related as high frequency errors.

From our simulations and from Mallat's results it turns out that for the vast majority of signals, the representation is unique. We even conjecture (without a proof) that the wavelet maxima (zero-crossings) representation is unique for a generic family of signals, i.e., if one chooses at random a sequence, he will get, with probability one, a unique wavelet maxima (zero-crossings) representation.

C. Test for Uniqueness

From the previous section we have learned that uniqueness is signal dependent. The next natural question to ask is what are the characteristics of a family of signals having a unique representation? This problem appears to be difficult and, unfortunately, we are not yet able to answer this question. Nevertheless, a given representation can be tested, quite efficiently, whether it is unique or it is not.

From Lemma 1, we already know that uniqueness depends only on the linear operator V which, in general, is signal dependent. The operator V can be divided into two parts, the first is the operator S_J , which is signal independent, and the second, remaining part, will be denoted by V^w . Since our approach is based on Lemma 1, the null space of V is investigated. The definition of V^w implies

$$\mathcal{N}V = \mathcal{N}S_J \cap \mathcal{N}V^w \quad (28)$$

where $\mathcal{N}V$, $\mathcal{N}S_J$, $\mathcal{N}V^w$ are the null spaces of operators V , S_J , V^w , respectively. Let $\{y_p\}_{p=1}^P$ be a basis of $\mathcal{N}S_J$. As a consequence of the fact that $\mathcal{N}V \subseteq \mathcal{N}S_J$, every $h \in \mathcal{N}V$

can be written as

$$h = \sum_{p=1}^P \alpha_p y_p. \tag{29}$$

Using this representation, we can say that

$$\begin{aligned} h \in \mathcal{N}V &\Leftrightarrow V^w \left(\sum_{p=1}^P \alpha_p y_p \right) = 0 \\ &\Leftrightarrow \sum_{p=1}^P \alpha_p V^w y_p = 0. \end{aligned} \tag{30}$$

In other words, $\mathcal{N}V = \{0\}$, if and only if, zero is the only vector $\mathbf{a} = (\alpha_1, \alpha_2, \dots, \alpha_P)'$ solving $\sum_{p=1}^P \alpha_p V^w y_p = 0$.

In essence, the remaining steps describe a basis of $\mathcal{N}S_j$, and to give a matrix interpretation to (30).

First, let us consider closely a particular wavelet transform on which the maxima representation is based. In order to be able to compare results, we use the same wavelets as [16]. Two cases were described in [16]. One corresponds to the cubic spline wavelet with

$$H(\omega) = \left(\cos \left(\frac{\omega}{2} \right) \right)^4$$

and the second corresponds to a Haar wavelet with

$$H(\omega) = \exp \left\{ -i \frac{\omega}{2} \right\} \cos \left(\frac{\omega}{2} \right).$$

Recall that the set \mathcal{B} , used in the proof of Theorem 3, is an independent set which is included in $\mathcal{N}S_j$. This set was constructed based on the fact that $H(\pi) = 0$. It turns out, that if $\omega = \pi$ is the only zero of $H(\omega)$ then the set \mathcal{B} is a basis for the space $\mathcal{N}S_j$. Since for both the cubic spline wavelet and the Haar wavelet, the only zeros of $H(\omega)$ appear at $\omega = \pi$, the set \mathcal{B} is a basis for $\mathcal{N}S_j$ in both cases.

A sequence h , which belongs to the null space of S_j can be written as

$$h(k) = \sum_{p=1}^{2^j-1} \alpha_p y_p(k).$$

By the linearity properties we can write

$$W_j h(k) = \sum_{p=1}^{2^j-1} \alpha_p W_j y_p(k).$$

Let us define the column vector of free coefficients

$$\mathbf{a} = (\alpha_1, \alpha_2, \dots, \alpha_{2^j-1})'.$$

For every $j = 1, 2, \dots, J$ and every $k = 0, 1, \dots, N - 1$, let ${}^w\mathcal{W}_j(k)$ denote the following row vector:

$${}^w\mathcal{W}_j(k) = (W_j y_1(k), W_j y_2(k), \dots, W_j y_{2^j-1}(k)).$$

The matrix ${}^w\mathcal{W}$ is defined as consisting of rows ${}^w\mathcal{W}_j(k)$ for all k which is a local extrema of $W_j f$. According to Lemma 1, f has a unique maxima representation, if and only if the only solution for

$${}^w\mathcal{W} \cdot \mathbf{a} = 0 \tag{31}$$

is the vector $\mathbf{a} = 0$. The latter condition is equivalent to

$$\text{rank} ({}^w\mathcal{W}) = 2^J - 1.$$

Theorem 4: The wavelet maxima representation Rf is unique, if and only if the rank of the matrix ${}^w\mathcal{W}$ is $2^J - 1$.

Conclusion 1: If the number of the extrema points is less than $2^J - 1$, the representation has to be nonunique. On the other hand, if the number of extrema points is equal or greater than $2^J - 1$ then uniqueness of the representation can be deduced from the rank(${}^w\mathcal{W}$). In the latter case, there may be situations in which analysis of the rank(${}^w\mathcal{W}$) can allow to eliminate some extrema from the representation.

D. An Example of a Nonunique Maxima Representation

In this section a sequence which has a nonunique maxima representation is described. As in the previous section, we assume $N = 256, J = 3$, and the cubic spline wavelet.

Consider,

$$f(k) = \cos \left(\frac{2\pi k}{8} + \frac{\pi}{6} \right).$$

This sequence is from the null space of S_3 . Here at every level we have 64 extrema points; they appear at regular distances. The basis for the null space of S_3 is

$$y_1(k) = \cos \left(\frac{2\pi k}{8} \right) \quad y_2(k) = \sin \left(\frac{2\pi k}{8} \right)$$

$$y_3(k) = \cos \left(\frac{4\pi k}{8} \right) \quad y_4(k) = \sin \left(\frac{4\pi k}{8} \right)$$

$$y_5(k) = \cos \left(\frac{6\pi k}{8} \right) \quad y_6(k) = \sin \left(\frac{6\pi k}{8} \right)$$

$$y_7(k) = \cos \left(\frac{8\pi k}{8} \right)$$

All the sequences y_p are 8-periodic. Linear operators preserve this periodicity. Therefore the rows ${}^w\mathcal{W}_j(k)$ are also 8-periodic in the sense

$${}^w\mathcal{W}_j(k) = {}^w\mathcal{W}_j(k + 8)$$

$$k = 0, 1, \dots, N - 1 \quad j = 1, 2, 3.$$

Thus every level contributes only two different rows to the matrix ${}^w\mathcal{W}$, and then there can be only six different rows in ${}^w\mathcal{W}$! The ultimate conclusion is that, in this case, the maxima representation is *not unique*. In essence, till now, we have just restored, for this particular case, the proof of Theorem 3. We continue to study this case in order to calculate exactly the reconstruction set of this representation.

It turns out that the rank of the matrix ${}^w\mathcal{W}$ is only 5. Consider

$${}^w\mathcal{W}_s = {}^w\mathcal{W}(1:5, [2, 3, 5, 6, 7]).$$

\mathbb{W}_s is the submatrix of \mathbb{W} consisting of the elements from the rows 1, 2, 3, 4, 5 and the columns 2, 3, 5, 6, 7. It is a regular matrix, with an inverse matrix not having large entries. Therefore we will use it for the representation of a general sequence satisfying $S_3 f = 0$ and giving zero samples at extreme points (i.e., from the space $\mathcal{H}V$).

Let, α_1^p and α_2^p be defined as

$$\alpha_1^p = -\mathbb{W}_s^{-1} \cdot \mathbb{W}^{c1}$$

$$\alpha_2^p = -\mathbb{W}_s^{-1} \cdot \mathbb{W}^{c4}$$

where \mathbb{W}^{c1} and \mathbb{W}^{c4} are the first and the fourth columns of the first five rows of \mathbb{W} . The space $\mathcal{H}V$ is spanned by the two following sequences

$$y_1^w = y_1 + \alpha_1^p(1) \cdot y_2 + \alpha_1^p(2) \cdot y_3 + \alpha_1^p(3)$$

$$\cdot y_5 + \alpha_1^p(4) \cdot y_6 + \alpha_1^p(5) \cdot y_7$$

$$y_2^w = y_4 + \alpha_2^p(1) \cdot y_2 + \alpha_2^p(2) \cdot y_3 + \alpha_2^p(3)$$

$$\cdot y_5 + \alpha_2^p(4) \cdot y_6 + \alpha_2^p(5) \cdot y_7.$$

Therefore, the general solution of $Vh = Vf$ is

$$h = f + a_1 \cdot y_1^w + a_2 \cdot y_2^w.$$

Conditions for $W_j h$ to be monotonic between extrema of $W_j f$ introduce 24 linear inequalities in a_1 and a_2 . Elementary analysis of this system leads to the following seven dominant inequalities:

$$\begin{aligned} 1.0000a_1 + 1.3059a_2 &> -0.4330 \\ 1.0000a_1 - 1.3059a_2 &> -0.4330 \\ 0.0000a_1 + 1.0000a_2 &> -0.1849 \\ 0.0000a_1 + 1.0000a_2 &< 0.1849 \\ 1.0000a_1 + 0.3574a_2 &< 0.1007 \\ 1.0000a_1 - 0.3574a_2 &< 0.1007 \\ 1.0000a_1 + 0.0000a_2 &< 0.0991. \end{aligned} \quad (32)$$

First, we pick up three pairs (0, 0), (0, -0.4), (-0.2, 0.16) which satisfy the system of inequalities (32). In order to visualize different sequences which the same wavelet maxima representation, let us define three sequences. The first is the original f and the next two are defined as

$$f_a = f - 0.4 \cdot y_1^w$$

$$f_b = f - 0.2 \cdot y_1^w + 0.16 \cdot y_2^w.$$

Fig. 3 shows these sequences and their first-level wavelet transforms. From the graphs one can indeed see that all have the same discrete wavelet maxima representation. Let us assume that f_a was reconstructed from the representation of f , in this case the noise to signal ratio is defined as

$$\frac{\|f - f_a\|^2}{\|f\|^2}$$

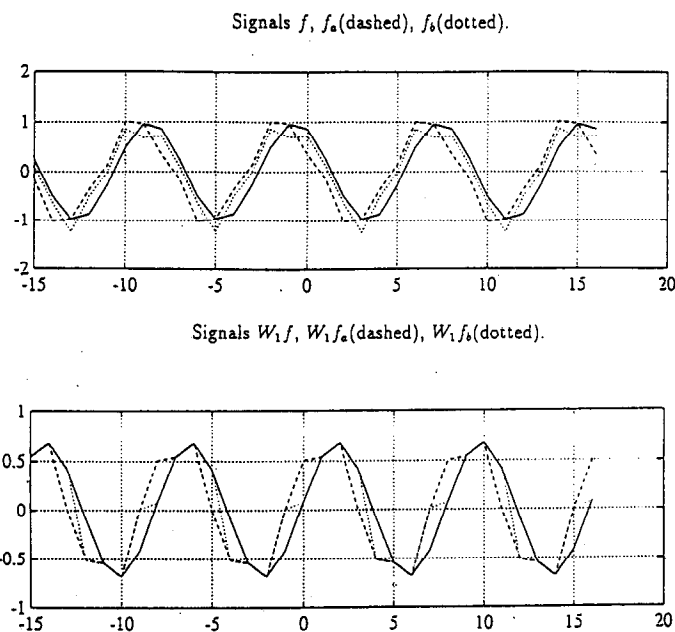


Fig. 3. The signals and their first level wavelet transforms.

and is equal to 0.345. In spite of the high N/S ratio, these signals have a very similar shape.

Set of inequalities (32) can be solved precisely. Figure 4 describes this solution by showing the boundary of set \mathcal{A} , the set of all pairs (a_1, a_2) satisfying set of inequalities (32).

Now, the reconstruction set of this representation is given as

$$\Gamma(Rf) = \{h: h = f + a_1 \cdot y_1^w + a_2 \cdot y_2^w, (a_1, a_2) \in \mathcal{A}\} \quad (33)$$

Fig. 5 was obtained by plotting all sequences from $\Gamma(Rf)$ on the same graph.

As was already mentioned, only a nonunique case may exhibit distinctive properties of the multiscale maxima representation. Therefore, it is worthwhile to examine carefully the reconstruction set $\Gamma(Rf)$. Since the cubic spline wavelet transform is based on wavelet $\Psi(\xi)$ which is a derivative of a smoothing function, all sequences in $\Gamma(Rf)$ have the same multiscale sharper variation points. Moreover, they all appear to have a very similar shape. This "shape" preserving property, perhaps expected, but not easily formulated and proven, is apparently the most interesting and promising feature of the multiscale maxima representation.

A nonunique representation can be viewed as an approximation, in this case we can, at least for the above example, make the following observation. The reconstruction set of the multiscale maxima representation, as a subset of \mathbb{R}^N , appears to be much less directionally homogeneous than reconstruction sets based on other standard approximations techniques like quantization or truncation.

The reconstruction set $\Gamma(Rf)$ was calculated for $J = 3$. It turns out that, in this particular case, $\Gamma(Rf) \subseteq \mathcal{H}S_3$.

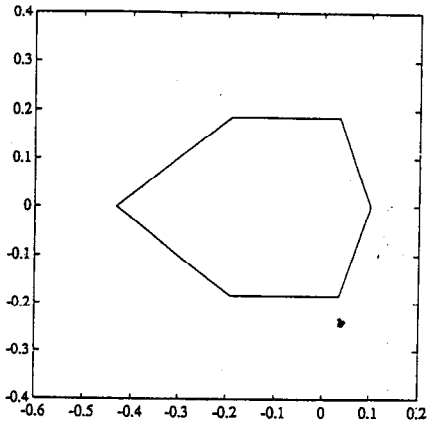


Fig. 4. The boundary of the set G on the plane (a_1, a_2) .

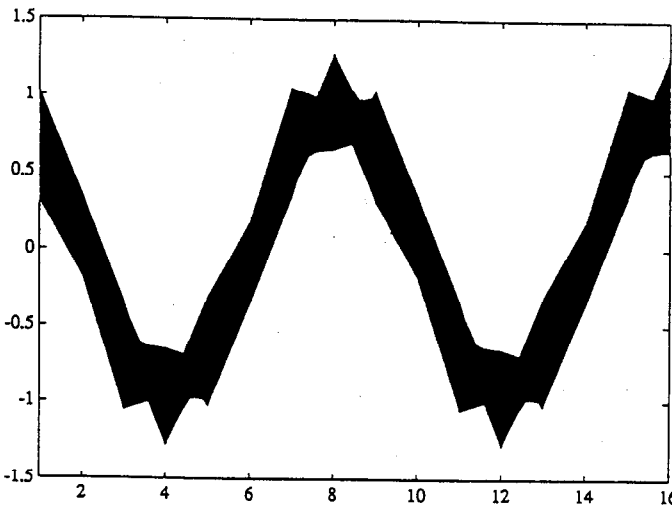


Fig. 5. All sequences belonging to $\Gamma(Rf)$.

But then, for all $h \in \Gamma(Rf)$ and for $j > 3$ the following is true:

$$W_j h = 0. \tag{34}$$

Therefore all sequences from this reconstruction set have the same wavelet maxima representation, even if different number of level is used, as long as J satisfies $J > 3$.

IV. STABILITY

In the signal processing community, the term stability is used for a variety of meanings. Often, poor reconstruction results are regarded as evidences of instability. The differences between stability, sensitivity, uniqueness, and accuracy are not always entirely evident. To explain the interpretation of stability used in this work, let us start with the citation from [14] "a representation is said to be unstable if a small perturbation of the representation may correspond to an arbitrary large perturbation of the original function." This definition refers to BIBO stability, shown by Proposition 4. Other citation from Hummel and Moniot [10] says "stability of the representation concerns continuity of the inverse map." Theorem 5 is exactly of this type.

In this section, we return to the general case of inherently bounded AQLR's and consider the problem of bounding possible variations in the reconstruction set due to perturbations in the representation.

A. BIBO Stability

To address the stability issue, the standard approach is to introduce the notion of perturbation of the representation, and of the reconstruction set. In addition, a distance measure for distinct representations and for different reconstruction sets should be defined. In general, it is not an easy task. Observe that If, Vf may have different sizes for different representations. Fortunately, for inherently bounded representations, the following characterization of BIBO (bounded input, bounded output) stability is easily verified.

Proposition 4: Let $Rf_i = \{If_i, V_if_i\}$, $i = 1, 2$ be inherently bounded AQLR's. Then for all $K_I > 0$ there exists K_O such that

$$\begin{aligned} \|V_if_i\| \leq K_I \quad (i = 1, 2) &\rightarrow \|h_1 - h_2\| \\ &\leq K_O \quad \forall h_i \in \Gamma(Rf_i). \end{aligned}$$

Proof: This claim is an immediate consequence of the definition of an inherently bounded AQLR. Indeed, using the definition of an inherently bounded AQLR and the hypothesis

$$h_i \in \Gamma(Rf_i) \rightarrow \|h_i\| \leq K \cdot \|V_if_i\| \leq K \cdot K_I$$

but then,

$$\|h_1 - h_2\| \leq \|h_1\| + \|h_2\| \leq 2K \cdot K_I. \quad \square$$

The above result is strong in the sense that it is valid regardless of the sets If_1, If_2 . It is weak in view of the fact that the bound on $\|h_1 - h_2\|$ is achieved by the bounds on absolute values of h_1, h_2 . In this case, a small perturbation in the representation does not necessary yield a small bound of $\|h_1 - h_2\|$. The next result is complimentary in the sense that a certain structure of the perturbation is assumed, but a bound, proportional to the size of the perturbation, is given.

B. A Lipschitz Condition

In many applications, the reasons for perturbations in a representation are arithmetic or quantization errors in a reconstruction algorithm. This kind of perturbations may change the continuous values of Vf but it preserves the discrete values of If . Therefore the perturbed representation, $(Rf)_p$, can be written as

$$(Rf)_p = \{If, Vf + \Delta(Vf)\}. \tag{35}$$

Let Γ_p be the corresponding reconstruction set. Our results are related to the following measure defined on (Γ, Γ_p) :

$$d(\Gamma, \Gamma_p) \triangleq \sup \{\|h - h_p\| : h \in \Gamma, h_p \in \Gamma_p\}.$$

Observe, that for inherently bounded AQLR's $d(\Gamma, \Gamma_p)$ is always finite. The measure of the perturbation in the reconstruction set is the difference between $d(\Gamma, \Gamma_p)$ and the size of Γ which is defined as follows:

$$s(\Gamma) \triangleq d(\Gamma, \Gamma) = \sup \{ \|h_1 - h_2\| : h_1, h_2 \in \Gamma \}. \quad (36)$$

$s(\Gamma)$ and $d(\Gamma, \Gamma_p)$ describe the largest possible Euclidean norm of a reconstruction error, from the original representation and from a perturbed one, respectively.

One remark is in order. In general, for an arbitrary $\Delta(Rf)$, the associated reconstruction set may be empty and then $d(\Gamma, \Gamma_p)$ would not be defined. In the sequel, it is assumed that this problem is treated by a reconstruction algorithm and hence $\Delta(Vf)$ yields a nonempty Γ_p . In this case, the following Lipschitz condition is satisfied.

Theorem 5: For all inherently bounded AQLR, there exists $K > 0$ such that

$$d(\Gamma, \Gamma_p) \leq K \cdot \|\Delta(Vf)\| + s(\Gamma). \quad (37)$$

Proof: See Appendix D.

Observe that the above result is global in the sense that as long as $\Delta(Vf)$ gives rise to a nonempty reconstruction set, the theorem holds regardless of the size of $\Delta(Vf)$.

Let us conclude this section with a remark concerning further research. The reconstruction accuracy from an AQLR is not easily evaluated from the stability results. However, quantitative analysis of the structure and the size of the reconstruction set, by applying the framework of AQLR's to a specific case, can provide important results related to reconstruction accuracy and sensitivity.

V. CONCLUSIONS

New theoretical results, regarding uniqueness and stability of the multiscale (wavelet) maxima and zero-crossings representations, have been presented. The concluding result, which states that the wavelet maxima (zero-crossings) representation is stable but nonunique, provides a new consideration of these signal descriptions. The standard multiscale zero-crossings (without any additional information) representation was assumed, at least for some family of signals, unique but unstable. Perhaps, this instability was the main obstacle to achieve, in spite of the first enthusiasm for zero-crossings in multiresolution representations (in the early 1980's), many engineering applications of this technique.

In our opinion, in addition to the actual results, there are three important consequences of this work.

The first is to show feasibility and capability of discrete analysis. In general, the discrete approach described here may be applied for a variety of representations and reconstruction algorithms, providing new insights into their properties. We believe that, even for complex algorithms, testing for uniqueness and computing a precise reconstruction set, even for a few examples, is worth the effort.

The second is the conclusion that, in order to benefit from novel characteristics, beyond properties of an adaptive irregular sampling, the multiscale maxima (zero-

crossings) representation should be considered in the non-unique context. Signal processing based on a unique representation has the advantage of possible separation between different processing units. This separation facilitates significantly analysis and design, but it does not improve system performance. Therefore, since the use of the multiscale maxima (zero-crossing) representation will require joint analysis and design of a whole system, one might expect involved analysis and difficult design with possible, as usual when local optimization is replaced by a global one, improvement in performance. However, in the signal processing community, the core of theoretical studies has been developed in the framework of unique representations. In our opinion, the need to develop more analytical tools and applications for non-unique representations is apparent.

The third is the framework of inherently bounded AQLR's. This structure makes possible to define, analyze, and reconstruct a wide family of signal representations. It should not be a surprise that this structure has already yielded additional results. One example is a new reconstruction algorithm based on an appropriate potential function. Modifications of the basic maxima representation are additional examples. For a reference see [2], [4].

Let us conclude with a citation from [10]. "The general methodology of studying a representation in terms of its mathematical properties, and developing reconstruction methods to evaluate the stability and variations in the fibers, in analogy with the study undertaken here, is highly recommended."

Using the word "here," Hummel and Moniot meant their work, but we hope that they would agree to use it in the context of our research as well.

APPENDIX A PROOF OF PROPOSITION 1

Let us assume that $R_m f$, a multiscale maxima representation of an arbitrary sequence $f \in \mathcal{L}$ is given. Proposition 1 will be proven in a constructive way, the exact conditions that an arbitrary sequence $h \in \mathcal{L}$ has to satisfy in order to belong to the reconstruction set of $R_m f$ will be specified. It is clear that $h \in \Gamma(R_m f)$, if and only if $\forall h = Vf$ and $W_j h$ ($j = 1, 2, \dots, J$) has local extrema at the same points as $W_j f$ does. Let us dwell upon the latter condition. In simple words, we have to assure that $W_j h$ is increasing in every segment starting at a local minimum of $W_j f$ and ending at a local maximum of $W_j f$ and it is decreasing otherwise.

The desired monotonic property can be achieved by enforcing an appropriate constraint on $W_j f(k+1) - W_j f(k)$ ($>0, \geq 0, <0, \leq 0$). If one of the points $k+1, k$ is not an extremum, such a constraint is a function of k and will be defined by the type of $k, t_j(f, k)$. If both k and $k+1$ are extreme points, the specific constraint cannot be defined solely either by k or by $k+1$. However, in the latter case, the sampling information assures the right relation-

ship between $W_j f(k + 1)$ and $W_j f(k)$. Consequently, the regular subset of $I_o(W_j f)$ is defined by

$$(I_o(W_j f))' \triangleq \{k \in I_o(W_j f) : k + 1 \notin I_o(W_j f)\}. \quad (38)$$

For all $k \in (I_o(W_j f))'$, the type of k with respect to extrema of f at level j , $t_j(f, k)$ is defined by

$$t_j(f, k) \triangleq \begin{cases} -1 & \text{if } k \in X(W_j f) \\ 1 & \text{otherwise.} \end{cases}$$

In words, for k a regular extremum, its type is -1 if k is a local maximum of $W_j f$ and it is 1 if k is a local minimum. To define a type of points which are not local extrema, the extremum segment is introduced. For any $k \in I_o(W_j f)$, the extremum segment of k with respect to f at level j , $P_j(f, k)$ is defined as

$$P_j(f, k) \triangleq \{k, k + 1, \dots, k + r\} \quad (39)$$

such that $r \geq 1$, $k + r \in I_o(W_j f)$, and $k + 1, \dots, k + r - 1 \notin I_o(W_j f)$. Note, that due to the N -periodic extension employed, $k + i$ is defined modulo N .

$P_j(f, k)$ is a segment which starts and ends with local extrema, but does not have any extremum as an interior point. Observe that for fixed j and f , for any $k' \notin I_o(W_j f)$ there exists exactly one $k \in I_o(W_j f)$ such that $k' \in P_j(f, k)$.

For all $k' \notin I_o(W_j f)$, the type of k' with respect to f at level j , $t_j(f, k')$ is introduced by

$$t_j(f, k') \triangleq \begin{cases} -1 & \text{if } k' \in P_j(f, k) \text{ and } k \in X(W_j f) \\ 1 & \text{otherwise} \end{cases}$$

In words, the type of k is 1 if either k is a local minimum or it belongs to a segment on which $W_j f$ is increasing.

Since $W_j h$ preserves monotonic properties of $W_j f$, the following theorem is easily verified.

Theorem 6: Let $R_m f$ be a given multiscale maxima representation. Then for an arbitrary $h \in \mathcal{L}$, $h \in \Gamma(Rf)$, if and only if

$$Vh = Vf \quad (40)$$

$$t_j(f, k) \cdot (W_j h(k + 1) - W_j h(k)) > 0. \quad (41)$$

The last inequality should be satisfied for $j = 1, 2, \dots, J$ and for all $k \in (I_o(W_j f))'$ and for all $k \notin I_o(W_j f)$.

We see that the theorem describes a particular case of the AQLR structure.

APPENDIX B

PROOF OF THEOREM 2

The proof is organized as follows: The first step is to show that the multiscale zero-crossings representation fits into the AQLR structure. Then, the required boundness property is shown.

Let $R_z f$ be a given multiscale zero-crossings representation. For an arbitrary $h \in \mathcal{L}$, the conditions on h to belong to $\Gamma(R_z f)$ are studied. Of course h needs to satisfy $Vh = Vf$, in addition $W_j h$ ($j = 1, 2, \dots, J$) has to sat-

isfy sign constraints yielding zero-crossings of $W_j h$ at $I_z(W_j f)$ points.

Let us study the latter condition. If, for some $k \in I_z(W_j f)$, $W_j f(k) = 0$ then $k + 1 \in I_z(W_j f)$ and $(U_j^f f)(k) = 0$. In this case, the zero-crossing point is preserved by satisfying the condition $Vh = Vf$. To take care for all other cases, the regular set of zero-crossings is defined

$$(I_z(W_j f))' \triangleq \{k \in I_z(W_j f) : k + 1 \notin I_z(W_j f)\}. \quad (42)$$

For all $k \in (I_z(W_j f))'$, the type of k with respect to f at level j is defined as

$$t_j(f, k) = \text{sgn}((U_j^f f)(k)).^2$$

Now careful, but straightforward, applications of the above definitions the following theorem can be proven.

Theorem 7: Let $R_z f$ be a given multiscale zero-crossings representation. For any $h \in \mathcal{L}$, $h \in \Gamma(R_z f)$, if and only if

$$Vh = Vf \quad (43)$$

$$t_j(f, k) \cdot W_j h(k') > 0. \quad (44)$$

such that $k' \in \mathcal{O}(W_j f, k)$. The last inequality should be satisfied for $j = 1, 2, \dots, J$ and for all $k' \in (I_z(W_j f))'$ and for all $k' \notin I_z(W_j f)$.

The above theorem has shown that $R_z f$ is an AQLR. Now we need to prove the boundness property. Consider an arbitrary $h \in \Gamma(R_z f)$, a fixed j and $k \in I_z(W_j f)$. Using $Vh = Vf$

$$\sum_{k' \in \mathcal{O}(W_j f, k)} W_j h(k') = \sum_{k' \in \mathcal{O}(W_j f, k)} W_j f(k') = (U_j^f f)(k). \quad (45)$$

Since $W_j h$ has the same zero-crossings as $W_j f$, for all $k' \in \mathcal{O}(W_j f, k)$ the values of $W_j h(k')$ have the same, fixed sign. Therefore,

$$\sum_{k' \in \mathcal{O}(W_j f, k)} |W_j h(k')| = |(U_j^f f)(k)|. \quad (46)$$

Applying,

$$\sum x_i^2 \leq \sum x_i^2 + 2 \cdot \sum x_i x_j = \left(\sum x_i \right)^2$$

for nonnegative x_i 's, we obtain

$$\sum_{k' \in \mathcal{O}(W_j f, k)} |W_j h(k')|^2 \leq |(U_j^f f)(k)|^2. \quad (47)$$

Now, using the fact that $\{W_1, W_2, \dots, W_J, W_{J+1}\}$ is a complete (unique) linear transformation and applying the

²For scrupulous definitions we need to assume that the information about these signs is included in the index set $I_z(W_j f)$.

definition of the Euclidian norm

$$\begin{aligned} \|h\|^2 &\leq K_1^2 \left(\|W_{J+1}h\|^2 + \sum_{j=1}^J \|W_jh\|^2 \right) \\ &= K_1^2 \left(\sum_{k'=0}^{N-1} |W_{J+1}h(k')|^2 \right. \\ &\quad \left. + \sum_{j=1}^J \sum_{k \in I(W_jf)} \sum_{k' \in \mathcal{O}(W_jf,k)} |W_jh(k')|^2 \right) \\ &\leq K_1^2 \left(\sum_{k'=0}^{N-1} |W_{J+1}h(k')|^2 \right. \\ &\quad \left. + \sum_{j=1}^J \sum_{k \in I(W_jf)} |(U_j^f f)(k)|^2 \right) = K_1^2 \|Vf\|^2. \end{aligned}$$

Thus, finally

$$\|h\| \leq K_1 \|Vf\|. \quad \square \quad (48)$$

APPENDIX C PROOF OF PROPOSITION 2

The discrete dyadic wavelet decomposition is based on two discrete filters, whose transfer functions are denoted $H(\omega)$ and $G(\omega)$ (for definitions and details see [15]).

As a straightforward consequence of the definition of the discrete dyadic wavelet transform, one can show that the discrete transfer function, corresponding to the operator S_J , is given by

$$(\widehat{S}_J)(k) = \prod_{p=0}^{J-1} \widehat{h}_p(k) \quad (49)$$

where

$$\widehat{h}_p(k) = H \left(2^{p+1} \frac{\pi k}{N} \right) \quad (50)$$

for $k = 0, 1, \dots, N-1$.

Now, let us consider

$$m(p) = \frac{pN}{2^J} \quad p = \pm 1, \pm 2, \dots, 2^{J-1}. \quad (51)$$

Since N is a multiple of 2^J , $m(p)$ is an integer. Notice that p can be written as $p = 2^l p_1$ where $0 \leq l \leq J-1$ and p_1 is an odd number. Observe that

$$\begin{aligned} \widehat{h}_{J-l-1}(m(p)) &= H \left(2^{J-l-1} 2\pi \frac{2^l p_1 N}{N 2^J} \right) \\ &= H(\pi p_1) = 0. \end{aligned} \quad (52)$$

Therefore, using (49), we obtain

$$\widehat{S}_J(m(p)) = 0. \quad (53)$$

The integers $m(p)$'s, as zeros of the transfer function S_J , will be used to define sequences belonging to the null

space of S_J . Let e_p be the following exponential sequence

$$\begin{aligned} e_p(k) &= \exp \left(2\pi i \frac{pk}{N} \right) \\ k &= 0, 1, \dots, N-1. \end{aligned} \quad (54)$$

Its Discrete Fourier Transform, \widehat{e}_p is given by

$$\widehat{e}_p(k) = N\delta_p(k) \quad (55)$$

where

$$\delta_p(k) = \begin{cases} 1 & \text{if } k = p \\ 0 & \text{otherwise.} \end{cases}$$

Combining together (53) and (55), one can conclude that $(S_J \widehat{e}_{m(p)}) = 0$, thus

$$S_J e_{m(p)} = 0. \quad (56)$$

The sequences y_p are expressed by $e_{m(p)}$'s in the subsequent way

$$\begin{aligned} y_{2p-1}(k) &= \cos \left(\frac{2\pi pk}{2^J} \right) = \frac{1}{2} (e_{m(p)} + e_{m(-p)}) \\ y_{2p}(k) &= \sin \left(\frac{2\pi pk}{2^J} \right) = \frac{1}{2i} (e_{m(p)} - e_{m(-p)}). \end{aligned}$$

Therefore $S_J y_p = 0$ for $p = 1, 2, \dots, 2^J - 1$. \square

APPENDIX D PROOF OF THEOREM 5

The proof is based on convex analysis and parametric linear programming. There are many relevant sources for the subject; we have mostly used [9], [22]. Due to the length of the proof it will be divided to several subsections.

A. The Structure of the Reconstruction Set

Let $Rf = \{If, Vf\}$ be an AQLR, its reconstruction set is given as

$$\Gamma = \{h: Vh = Vf, Ch > c\}. \quad (57)$$

The closure of the reconstruction set, $\bar{\Gamma}$ is the following convex polyhedron

$$\bar{\Gamma} = \{h: Vh = Vf, Ch \geq c\}. \quad (58)$$

Since every equality of the form $h_i = t_i$ can be replaced by two inequalities $h_i \geq t_i$, $-h_i \geq -t_i$, without loss of generality, we can assume that

$$\bar{\Gamma} = \{h: Bh \geq b\}$$

for a given $p \times N$ matrix B and a p -dimensional vector b .

For an inherently bounded AQLR's, the associated set $\bar{\Gamma}$ is bounded. Therefore as a special case of the theorem of Krein and Milman [12], the following holds.

Theorem 8: For an inherently bounded AQLR, the closure of the reconstruction set is the convex hull of its finitely many vertexes.

In the sequel, the following property of a polyhedron vertex will be used. Let $\{h: Bh \geq b\}$ be a polyhedron and v^i its vertex. Then, there exist N rows of B , which constitute a regular matrix $[B]^i$, such that

$$v^i = ([B]^i)^{-1} \cdot [b]^i \quad (59)$$

while $[b]^i$ is a subvector of b corresponding to these N rows. By inserting zero columns to the matrix $([B]^i)^{-1}$, the matrix D^i is obtained, such that

$$v^i = D^i b. \quad (60)$$

Returning to notations of Theorem 5, observe that there exist $v \in \bar{\Gamma}$ and $\bar{v} \in \bar{\Gamma}_p$ such that

$$\|v - \bar{v}\| = d(\bar{\Gamma}, \bar{\Gamma}_p) = d(\Gamma, \Gamma_p). \quad (61)$$

Moreover, it can be shown by a standard technique that v and \bar{v} are vertices of $\bar{\Gamma}$ and $\bar{\Gamma}_p$.

B. A Convex Representation

Let $\Delta(Vf)$ be fixed and arbitrary, such that Γ_p is non-empty. We define

$$(\mathcal{R}f)_p^\lambda \triangleq \{Vf, Vf + \lambda \cdot \Delta(Vf)\} \quad 0 \leq \lambda \leq 1 \quad (62)$$

with the underlying reconstruction set denoted by Γ_p^λ . From the definition of an adaptive quasi linear representation (AQLR)

$$\Gamma_p^\lambda = \{h: Vh = Vf + \lambda \cdot \Delta(Vf) \text{ and } Ch > c\}. \quad (63)$$

The above formula yields the following observation: if $h_0 \in \Gamma = \Gamma_p^0$ and $h_1 \in \Gamma_p = \Gamma_p^1$ then

$$h_0 + \lambda \cdot (h_1 - h_0) \in \Gamma_p^\lambda \quad 0 \leq \lambda \leq 1. \quad (64)$$

Therefore Γ_p^λ is nonempty for $0 \leq \lambda \leq 1$ and $d(\Gamma, \Gamma_p^\lambda)$ is well defined.

Next, notice that the closure of Γ_p^λ is given by

$$\begin{aligned} \overline{(\Gamma_p^\lambda)} &= \{h: Vh = Vf + \lambda \cdot \Delta(Vf), \\ Ch &\geq c\} = \{h: Bh \geq b + \lambda \Delta b\} \end{aligned} \quad (65)$$

where B is a $p \times N$ matrix and $b, \Delta b$ are p -dimensional vectors (see Section III-B). Since every quality of $Vh = Vf + \lambda \cdot \Delta(Vf)$ appears in two rows in $Bh \geq b + \lambda \Delta b$

$$\|\Delta b\| = 2 \|\Delta(Vf)\|. \quad (66)$$

We know that

$$d(\Gamma, \Gamma_p^\lambda) = \|v^\lambda - \bar{v}^\lambda\| \quad (67)$$

where v^λ is a vertex of $\bar{\Gamma}$ and \bar{v}^λ is a vertex of $\overline{\Gamma_p^\lambda}$. Using (60) we can write

$$v^\lambda = D^i b \text{ and } \bar{v}^\lambda = \bar{D}^i (b + \lambda \Delta b). \quad (68)$$

Both matrices D^i and \bar{D}^i are obtained from an inverse of a regular submatrix of B . Note that $\|D^i b - \bar{D}^i (b + \lambda \Delta b)\|$ is a continuous function of λ for any two matrices D, \bar{D} .

Therefore, if

$$\|v^\lambda - \bar{v}^\lambda\| < \|v_o - \bar{v}_o\| \quad (69)$$

for all pairs v_o, \bar{v}_o of vertexes of $\bar{\Gamma}, \overline{\Gamma_p^\lambda}$, respectively, which are different from $v^\lambda, \bar{v}^\lambda$, then there exists a segment $[\lambda_i, \lambda_{i+1}]$ such that

$$\begin{aligned} d(\Gamma, \Gamma_p^\lambda) &= \|D^i b - \bar{D}^i (b + \lambda \Delta b)\| \\ \forall \lambda &\in [\lambda_i, \lambda_{i+1}]. \end{aligned} \quad (70)$$

Furthermore, there exists another pair of vertexes, with associated matrices D^{i-1} and \bar{D}^{i-1} , such that

$$\begin{aligned} d(\Gamma, \Gamma_p^{\lambda_i}) &= \|D^i b - \bar{D}^i (b + \lambda_i \Delta b)\| \\ &= \|D^{i-1} b - \bar{D}^{i-1} (b + \lambda_i \Delta b)\|. \end{aligned} \quad (71)$$

Next observe that since the number of regular submatrices of B is finite, the number of possible pairs D, \bar{D} is finite as well. Consider $\|D^i b - \bar{D}^i (b + \lambda \Delta b)\|$ and $\|D_o b - \bar{D}_o (b + \lambda \Delta b)\|$ as two functions of λ . As square roots of quadratic forms, these expressions may coincide or be equal for at most two values of λ . Therefore, all possible pairs of these functions intersect at finitely many points. Consequently, there exist L points

$$0 = \lambda_0 < \lambda_1 \cdots < \lambda_{L-1} = 1 \quad (72)$$

and L pairs of matrices $(D^i, \bar{D}^i) i = 0, 1, \dots, L-1$ such that $D^i b$ is a vertex of Γ^c and $\bar{D}^i (b + \lambda \Delta b)$ is a vertex of $(\Gamma_p^\lambda)^c$ for all $\lambda \in [\lambda_i, \lambda_{i+1}]$. Moreover,

$$\begin{aligned} d(\Gamma, \Gamma_p^\lambda) &= \|D^i b - \bar{D}^i (b + \lambda \Delta b)\| \\ \forall \lambda &\in [\lambda_i, \lambda_{i+1}] \end{aligned} \quad (73)$$

$$\begin{aligned} d(\Gamma, \Gamma_p^{\lambda_i}) &= \|D^i b - \bar{D}^i (b + \lambda_i \Delta b)\| \\ &= \|D^{i-1} b - \bar{D}^{i-1} (b - \lambda_i \Delta b)\|. \end{aligned} \quad (74)$$

C. Auxiliary Proposition

Proposition 5:

$$d(\Gamma, \Gamma_p^{\lambda_i}) \leq d(\Gamma, \Gamma) + \sum_{k=0}^{i-1} (\lambda_{k+1} - \lambda_k) \cdot \|\bar{D}^k \Delta b\|. \quad (75)$$

Proof: By induction on i . Let $i = 1$

$$\begin{aligned} d(\Gamma, \Gamma_p^{\lambda_1}) &= \|D^1 b - \bar{D}^1 (b + \lambda_1 \Delta b)\| \\ &= \|D^0 b - \bar{D}^0 (b - \lambda_1 \Delta b)\| \\ &= \|D^0 b - \bar{D}^0 b + \bar{D}^0 b - \bar{D}^0 (b + \lambda_1 \Delta b)\| \\ &\leq \|D^0 b - \bar{D}^0 b\| + \|\bar{D}^0 b \\ &\quad - \bar{D}^0 (b + \lambda_1 \Delta b)\| \\ &= d(\Gamma, \Gamma) + \lambda_1 \cdot \|\bar{D}^0 \Delta b\|. \end{aligned}$$

Since $\lambda_0 = 0$, the above is exactly the claim for $i = 1$. By induction, let us assume that the proposition holds for $i - 1$. Consider,

$$\begin{aligned} d(\Gamma, \Gamma_p^{\lambda_i}) &= \|D^{i-1}b - \bar{D}^{i-1}(b + \lambda_i \Delta b)\| \\ &= \|D^{i-1}b - \bar{D}^{i-1}b \\ &\quad + \lambda_{i-1} \Delta b \\ &\quad + \bar{D}^{i-1}(b + \lambda_{i-1} \Delta b) \\ &\quad - \bar{D}^{i-1}(b + \lambda_i \Delta b)\| \\ &\leq \|D^{i-1}b - \bar{D}^{i-1}(b + \lambda_{i-1} \Delta b)\| \\ &\quad + \|\bar{D}^{i-1}(b + \lambda_{i-1} \Delta b) \\ &\quad - \bar{D}^{i-1}(b + \lambda_i \Delta b)\| \\ &= d(\Gamma, \Gamma_p^{\lambda_{i-1}}) + (\lambda_i - \lambda_{i-1}) \|\bar{D}^{i-1} \Delta b\| \leq \end{aligned}$$

(using the induction assumption)

$$\begin{aligned} &\leq d(\Gamma, \Gamma) + \sum_{k=0}^{i-2} (\lambda_{k+1} - \lambda_k) \|\bar{D}^k \Delta b\| \\ &\quad + (\lambda_i - \lambda_{i-1}) \|\bar{D}^{i-1} \Delta b\| \\ &= d(\Gamma, \Gamma) + \sum_{k=0}^{i-1} (\lambda_{k+1} - \lambda_k) \|\bar{D}^k \Delta b\|. \end{aligned}$$

This concludes the proof of the proposition. \square

D. Conclusion of the Proof

Using Proposition 6 we deduce that the distance between Γ and Γ_p satisfies

$$d(\Gamma, \Gamma_p) \leq d(\Gamma, \Gamma) + \sum_{k=0}^{L-1} (\lambda_{k+1} - \lambda_k) \|\bar{D}^k \Delta b\|. \quad (76)$$

Let $\|\bar{D}^i\|$ be the induced matrix norm of \bar{D}^i . Then,

$$\|\bar{D}^i \Delta b\| \leq \|\bar{D}^i\| \cdot \|\Delta b\|. \quad (77)$$

Since the number of possible matrix \bar{D}^i is finite, there exists $K_D > 0$ such that

$$\|\bar{D}^i\| \leq K_D \quad (78)$$

for all valid \bar{D}^i . Combining together (76)–(78) we show that

$$d(\Gamma, \Gamma_p) \leq d(\Gamma, \Gamma) + K_D \|\Delta b\|. \quad (79)$$

By taking $K = 2K_D$ and using (66), the desired relation is obtained

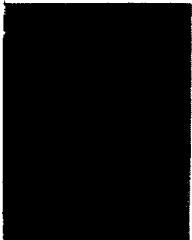
$$d(\Gamma, \Gamma_p) \leq d(\Gamma, \Gamma) + K \cdot \|\Delta(Vf)\|. \quad \square \quad (80)$$

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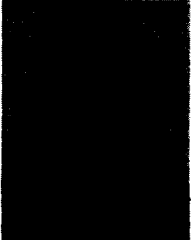
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