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DYNAMIC OBSERVERS AS ASYMPTOTIC LIMITS OF RECURSIVE FILTERS: SPECIAL CASES*

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DYNAMIC OBSERVERS AS ASYMPTOTIC LIMITS OF RECURSIVE FILTERS: SPECIAL CASES*

J. S. BARAS†, A. BENSOUSSAN‡, AND M. R. JAMES§

Abstract. A method for constructing observers for dynamical systems as asymptotic limits of filters is described. The program is carried out in detail for linear systems, and in addition an observer is obtained for a class of systems with nonlinear dynamics and linear observations. The method is motivated by some large deviation results of Hijab for certain conditional measures.

Key words. observers, filters, linear and nonlinear systems, large deviations

AMS(MOS) subject classifications. 93B07, 93E11, 60F10

1. Introduction. Our objective is to describe a method for constructing an observer for the dynamical system

$$\begin{aligned} &\dot{x}(t)=f(x(t))+\sum_{i=1}^{m} g_i(x(t))u_i(t), \qquad x(0)=x_0, \\ &y(t)=h(x(t)), \end{aligned} \tag{1}$$

as the asymptotic limit of nonlinear filters associated with the "noisy" version of (1):

$$\begin{aligned} &dx^\varepsilon(t)=f(x^\varepsilon(t))\,dt+\sum_{i=1}^{m} g_i(x^\varepsilon(t))u_i(t)+\sqrt{\varepsilon}N(x^\varepsilon(t))\,dw(t), \\ &x^\varepsilon(0)=x_0^\varepsilon, \\ &d\xi^\varepsilon(t)=h(x^\varepsilon(t))\,dt+\sqrt{\varepsilon}R\,dv(t), \qquad \xi^\varepsilon(0)=0, \end{aligned} \tag{2}$$

with $\varepsilon\to 0$. Here $x(t)\in\mathbb{R}^n$, $y(t)\in\mathbb{R}^p$ as usual. The method is motivated by some large deviation results of Hijab [4], [5] for the conditional measures $P^\varepsilon_{x|\xi}$ of (2).

In the present paper we present results of this general method as applied to the linear case and a certain class of nonlinear systems. The general nonlinear problem will be treated elsewhere.

2. Observers for linear systems. In this section we provide a complete description of the method as it applies to linear systems. The results are improvements and completions of earlier preliminary accounts provided in [1], [2].

The method constructs explicitly an observer for the linear system

$$\begin{aligned} &\dot{x}(t)=Ax(t)+Bu(t), \qquad x(0)=x_0, \\ &y(t)=Cx(t), \end{aligned} \tag{3}$$

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as the asymptotic limit of (Kalman) filters for a family of associated filtering problems

$$\begin{aligned} dx^\varepsilon(t) &= Ax^\varepsilon(t)\,dt + Bu(t)\,dt + \sqrt{\varepsilon}N\,dw(t), \qquad x^\varepsilon(0) = x_0^\varepsilon, \\ d\varepsilon^\varepsilon(t) &= Cx^\varepsilon(t)\,dt + \sqrt{\varepsilon}R\,dv(t), \qquad \xi^\varepsilon(0) = 0. \end{aligned} \tag{4}$$

Such a construction is suggested by the fact that for certain choices of $Q_0^\varepsilon = \operatorname{cov}(x_0^\varepsilon)$, the filters are independent of ε, as discussed in Baras and Krishnaprasad [1]. Also, the solutions of (4) converge in probability as $\varepsilon \to 0$ to the solution of (3).

The work of Hijab [4], [5] is indispensible here in deriving a large deviation principle for the conditional measures $P_{x|\xi}^\varepsilon$ (see § 2.3) and identifying the limit of the filters for (4) as an associated deterministic estimator.

2.1. Observers and filters. We assume as usual that $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^p$ and $t \mapsto u(t)$ is piecewise continuous.

Recall that the *observer* problem consists of constructing a dynamical system

$$\dot m(t) = Em(t) + Fu(t) + Gy(t), \qquad m(0) = m_0, \tag{5}$$

so that the error

$$e(t) = x(t) - Hm(t) \tag{6}$$

decays exponentially fast to zero, at a rate controlled by the designer, independent from the choice of m_0 and x_0. Here the matrices E, F, G, and H are possibly time-varying and the dimension of $m(t)$ is not necessarily n. This of course reflects the fact that the initial condition x_0 is unknown, and the best that can be done is to approximately estimate $x(t)$ by $Hm(t)$ in this way.

Solutions to this problem are well known and were first given by Luenberger [8]. In particular, if the pair (C, A) is *detectable*, then there exists a matrix Γ such that the matrix $A + \Gamma C$ has eigenvalues in the open left half plane. Then set

$$E = A + \Gamma C, \qquad F = B, \quad G = -\Gamma, \quad H = I.$$

In this case the error (6) satisfies

$$\dot e(t) = (A + \Gamma C)e(t), \qquad e(0) = x_0 - z_0,$$

and the eigenvalues of $A + \Gamma C$ can be arbitrarily assigned by the designer if and only if (C, A) is *observable.*

Consider the system (3). Define $\xi(t) = \int_0^t y(s)\,ds$, so that (3) becomes

$$\begin{aligned} \dot x(t) &= Ax(t) + Bu(t), \qquad x(0) = x_0, \\ \dot\xi(t) &= Cx(t), \qquad \xi(0) = 0. \end{aligned} \tag{7}$$

Then associate with (7) the family of filtering problems (4), where w, v are independent standard k-dimensional, respectively, p-dimensional, Brownian motions. The initial condition x_0^ε is Gaussian, independent from w, v with $E(x_0^\varepsilon) = \mu_0^\varepsilon$, $\operatorname{cov}(x_0^\varepsilon) = Q_0^\varepsilon$, where Q_0^ε is positive definite. Note that the (small) parameter ε controls the intensity of the noise. The matrix R is assumed positive definite.

As is well known, the minimum variance estimate $\hat x^\varepsilon(t) = E(x(t) \mid \xi^\varepsilon(s), 0 \leqq s \leqq t)$ for the linear Gaussian filtering problem (4) is given by the Kalman filter [3]

$$\begin{aligned} &d\hat x^\varepsilon(t) = A\hat x^\varepsilon(t)\,dt + Bu(t)\,dt + Q^\varepsilon(t)C'(RR')^{-1}(d\xi^\varepsilon(t) - C\hat x^\varepsilon(t)\,dt), \\ &\hat x^\varepsilon(0) = \mu_0^\varepsilon, \end{aligned} \tag{8}$$

where Q^ε satisfies the Riccati equation

$$\begin{aligned} &\dot Q^\varepsilon(t) = AQ^\varepsilon(t) + Q^\varepsilon(t)A' - Q^\varepsilon(t)C'(RR')^{-1}CQ^\varepsilon(t) + NN', \\ &Q^\varepsilon(0) = Q_0^\varepsilon/\varepsilon. \end{aligned} \tag{9}$$

Note that these filters depend on ε only via the matrix $Q_0^\varepsilon/\varepsilon$. In fact, if we choose $Q_0^\varepsilon = \varepsilon Q_0$, then all the filters are independent of ε and identical with the filter for $\varepsilon = 1$.

Following Hijab [4], it is convenient to consider the filter (8), (9) as a map

$$\mathscr{F}^\varepsilon : C([0, t], \mathbb{R}^p) \to \mathbb{R}^n, \qquad \xi(s), 0 \leqq s \leqq t \mapsto \hat{x}^\varepsilon(t).$$

2.2. Deterministic estimation. Following Mortensen [9] and Hijab [4], we associate with (7) the deterministic (noisy) system

$$\begin{aligned} \dot{z}(t) &= Az(t) + Bu(t) + Nw(t), \qquad z(0) = z_0, \\ \dot{\zeta}(t) &= Cz(t) + Rv(t), \qquad \zeta(0) = 0, \end{aligned} \tag{10}$$

and energy cost functional

$$J_t(z_0, w, v) = \frac{1}{2}(z_0 - \mu)Q_0^{-1}(z_0 - \mu) + \frac{1}{2}\int_0^t (w(s)'w(s) + v(s)'v(s))\, ds, \tag{11}$$

where $t \mapsto w(t) \in \mathbb{R}^k$, $t \mapsto v(t) \in \mathbb{R}^p$ are piecewise continuous, the rank of N is n and Q_0 is positive definite.

A minimum energy input triple (z_0^*, w^*, v^*) given $\zeta(s)$, $0 \leqq s \leqq t$, is a triple that minimises J_t subject to (10) and produces the given output record $\zeta(s)$, $0 \leqq s \leqq t$. The *deterministic* or minimum energy *estimate* of $z(t)$ given $\zeta(s)$, $0 \leqq s \leqq t$, is the endpoint $\hat{z}(t)$ of the trajectory $z^*(s)$, $0 \leqq s \leqq t$, of (10) corresponding to a minimum energy input triple: $\hat{z}(t) = z^*(t)$.

According to Krener [7], $\hat{z}$ is the solution of the Kalman filter equations

$$\begin{aligned} \dot{\hat{z}}(t) &= A\hat{z}(t) + Bu(t) + Q(t)C'(RR')^{-1}(\dot{\zeta}(t) - C\hat{z}(t)), \\ \hat{z}(0) &= \mu, \end{aligned} \tag{12}$$

$$\begin{aligned} \dot{Q}(t) &= AQ(t) + Q(t)A' - Q(t)C'(RR')^{-1}CQ(t) + NN', \\ Q(0) &= Q_0. \end{aligned} \tag{13}$$

As in the stochastic case (§ 2.1), it is convenient to consider the deterministic filter (12), (13) as a map

$$\mathscr{F} : C^1([0, t], \mathbb{R}^p) \to \mathbb{R}^n, \qquad \zeta(s), 0 \leqq s \leqq t \mapsto \hat{z}(t).$$

Note that the deterministic filter coincides with the stochastic filter for $\varepsilon = 1$, that is, $\mathscr{F}^1$. Also, $\hat{z}(t)$ is obtained from an optimal control problem, and is determined by a Hamilton-Jacobi-Bellman equation [4], [6], [9].

We now prove that as $\varepsilon \to 0$ the stochastic filter $\mathscr{F}^\varepsilon$ (8), (9) converges to the deterministic filter $\mathscr{F}$ (12), (13).

THEOREM 1. *Suppose that* (4) *has initial conditions* x_0^ε *Gaussian with mean* μ^ε *and covariance* Q_0^ε *satisfying*

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} Q_0^\varepsilon = Q_0, \qquad \lim_{\varepsilon \to 0} \mu^\varepsilon = \mu,$$

where Q_0 *is positive definite. Then*

$$\lim_{\varepsilon \to 0} E\,|\hat{x}^\varepsilon(t) - \hat{z}(t)|^2 = 0$$

uniformly in $t \in [0, T]$.

Proof. Now $Q^\varepsilon(t) \to Q(t)$ uniformly in $t \in [0, T]$. Applying Itô's rule to $|\hat{x}^\varepsilon(t) - \hat{z}(t)|^2$ and taking expectations, we find that, given $\delta > 0$,

$$E|\hat{x}^\varepsilon(t) - \hat{z}(t)|^2 \leqq (|\mu^\varepsilon - \mu|^2 + \delta K(1 + |\mu^\varepsilon|) + \varepsilon K)e^{Kt},$$

for all ε sufficiently small, where $K>0$. The desired result follows from this inequality. □

2.3. Large deviations. Consider the stochastic differential equation (4), with initial condition $x_0^\varepsilon = x_0$ for all $\varepsilon > 0$. In this section we take $u \equiv 0$. Let P_x^ε be the probability measure induced on $\Omega^n = C([0, T], \mathbb{R}^n)$ by the diffusion x^ε. It is well known from the theory of Ventcel-Friedlin (see Varadhan [10]) that the family of measures P_x^ε satisfies a large deviation principle. Moreover, as $\varepsilon \to 0$, P_x^ε converges weakly to the degenerate measure concentrated on the unique solution x of (3).

We now consider the observation equation in (4). Let $Q^\varepsilon_{x|(\xi, x_0)}$ be an unnormalised conditional measure on Ω^n of x^ε given $\xi \in \Omega^p = C([0, T], \mathbb{R}^p)$ where the diffusions are initialised as above. For a "control" $t \to w(t)$, let z_w denote the unique solution to (10). The function v is defined by $R^{-1}(\dot{\xi}(t) - Cz_w(t))$ when ξ is C^1. Hijab [5] proved the following large deviation result for $Q^\varepsilon_{x|(\xi, x_0)}$.

THEOREM 2. *For any open subset* $\mathcal{O}$ *and any closed subset* $\mathcal{C}$ *of* Ω^n,

$$\liminf_{\varepsilon \to 0} \varepsilon \log Q^\varepsilon_{x|(\xi, x_0)}(\mathcal{O}) \geqq -I(x_0, \xi, \mathcal{O}),$$

$$\limsup_{\varepsilon \to 0} \varepsilon \log Q^\varepsilon_{x|(\xi, x_0)}(\mathcal{C}) \leqq -I(x_0, \xi, \mathcal{C})$$

where for $\mathcal{A} \subset \Omega^n$,

$$I(x_0, \xi, \mathcal{A}) = \inf_w \left\{ \frac{1}{2} \int_0^T (w(s)'w(s) + z_w(s)'C'Cz_w(s))\, ds - \int_0^T z_w(s)'C'\, d\xi(s) \,\Big|\, z_w(0) = x_0,\ z_w \in \mathcal{A} \right\}. \tag{14}$$

Proof. Define, for each $\xi \in \Omega^p$, $\omega \in \Omega^n$,

$$\phi(\omega, \xi) = -\xi(T)'C\omega(T) + \int_0^T \left(\xi(t)'CA\omega(t) + \frac{1}{2}\omega(t)'C'C\omega(t) - \frac{1}{2}\xi(t)'CNN'C'\xi(t) \right) dt.$$

There exist constants A, B depending only on ξ, such that

$$-\phi(\omega, \xi) \leqq A + B\|\omega\|.$$

Then arguing as in Varadhan [11],

$$\lim_{R \to \infty} \limsup_{\varepsilon \to 0} \varepsilon \log \int_{\{\omega : -\phi(\omega, \xi) \geqq R\}} \exp\left(-\frac{1}{\varepsilon} \phi(\omega, \xi) \right) dP_x^\varepsilon = -\infty.$$

But this estimate is enough to prove the theorem. See Hijab [5] and Varadhan [11] for details. □

The minimisation in (14) is an optimal control problem similar to the one in § 2.2, but with fixed initial condition x_0. James and Baras [6] have made a simple generalisation of Theorem 2 in which the variational problem arising is exactly the optimal control problem for deterministic estimation in § 2.2.

Assume that the initial conditions x_0^ε of (4) have (unnormalised) density

$$q^\varepsilon(x_0) = \exp\left(-\frac{1}{2\varepsilon}(x_0 - \mu)'Q_0^{-1}(x_0 - \mu) \right).$$

Let $Q^\varepsilon_{(x, x_0)|\xi}$ be an unnormalised joint conditional measure of $(x^\varepsilon, x_0^\varepsilon)$ on $\Omega^n \times \mathbb{R}^n$ given $\xi \in \Omega^p$. The following result is quoted from [6].

THEOREM 3. *For any open subset $\mathcal{O}$ and any closed subset $\mathcal{C}$ of Ω^n, and for any open subset $\mathcal{O}_0$ and any closed subset $\mathcal{C}_0$ of $\mathbb{R}^n$, we have*

$$\liminf_{\varepsilon \to 0} \varepsilon \log Q^{\varepsilon}_{(x,x_0)|\xi}(\mathcal{O} \times \mathcal{O}_0) \geqq -J(\mathcal{O} \times \mathcal{O}_0, \xi),$$

$$\limsup_{\varepsilon \to 0} \varepsilon \log Q^{\varepsilon}_{(x,x_0)|\xi}(\mathcal{C} \times \mathcal{C}_0) \leqq -J(\mathcal{C} \times \mathcal{C}_0, \xi)$$

where for $\mathcal{A} \times \mathcal{A}_0 \subset \Omega^n \times \mathbb{R}^n$,

$$J(\mathcal{A} \times \mathcal{A}_0, \xi) = \inf_{x_0 \in \mathcal{A}_0} \left\{ \frac{1}{2}(x_0 - \mu)' Q_0^{-1}(x_0 - \mu) + I(x_0, \xi, \mathcal{A}) \right\}. \tag{15}$$

This theorem implies that if $\xi = \zeta$ is an actual output record of the system (10), then as $\varepsilon \to 0$, $Q^{\varepsilon}_{(x,x_0)|\xi}$ converges weakly to a degenerate measure concentrated on the optimal initial condition z_0^* and optimal trajectory $z^*(s)$, $0 \leqq s \leqq T$ of (10) corresponding to a minimum energy input triple. As pointed out in § 2.2, the deterministic estimate of the state at time T is a functional of this optimal path, namely its value at T: $\hat{z}(T) = z^*(T)$.

Thus in a weak sense, $\hat{x}^{\varepsilon}(T) \to \hat{z}(T)$, and the large deviation principle for $Q^{\varepsilon}_{(x,x_0)|\xi}$ characterises the limiting filter as the deterministic filter.

2.4. Observer design. From §§ 2.2 and 2.3 it is plain that the deterministic estimator (12), (13) is a natural candidate for an observer for the linear system (3). We make the natural assumption that the pair (C, A) is *detectable*. Recall that N has rank n and R is positive definite. The design parameters are Q_0, N, R and μ.

Then from (12) we define

$$\begin{aligned} \dot{m}(t) &= Am(t) + Bu(t) + Q(t)C'(RR')^{-1}(y(t) - Cm(t)), \\ m(0) &= m_0 = \mu, \end{aligned} \tag{16}$$

where $Q(t)$ is the solution of the Riccati equation (13). The inverse $P(t)$ of $Q(t)$ is the solution of

$$\begin{aligned} \dot{P}(t) &= -P(t)A - A'P(t) - P(t)NN'P(t) + C'(RR')^{-1}C, \\ P(0) &= P_0 = Q_0^{-1}. \end{aligned} \tag{17}$$

Since we are interested in the asymptotic behaviour of $e(t) = x(t) - m(t)$, it is important to obtain bounds for $\|Q(t)\|$, $\|P(t)\|$. To this end we interpret $Q(t)$, $P(t)$ in terms of control problems. Write $H = R^{-1}C$.

Consider the control problem

$$-\dot{\eta} = A'\eta + H'v, \qquad \eta(T) = h, \tag{18}$$

where h is given and v is the control. We minimise

$$J_1(v) = \eta(0)'Q_0\eta(0) + \int_0^T (v(t)'v(t) + \eta(t)'NN'\eta(t))\, dt. \tag{19}$$

then the optimal control for (18), (19) is given by the following algorithm.

Consider the system of equations:

$$\begin{aligned} \dot{\hat{\lambda}} &= A\hat{\lambda} + NN'\hat{\eta}, \qquad \hat{\lambda}(0) = Q_0\hat{\eta}(0), \\ -\dot{\hat{\eta}} &= A'\hat{\eta} - H'H\hat{\lambda}, \qquad \hat{\eta}(T) = h. \end{aligned} \tag{20}$$

Then an optimal control is $\hat{v}(t) = -H\hat{\lambda}(t)$. Moreover,

$$\min J_1(v) = h'Q(T)h = h'\hat{\lambda}(T). \tag{21}$$

In addition, the following relation holds:

$$\hat{\lambda}(t)=Q(t)\hat{\eta}(t) \quad \text{for all } t, \tag{22}$$

where $Q(t)$ is the solution of the Riccati equation (13).

Similarly, consider the control problem

$$\dot{\lambda}=A\lambda+Nv, \qquad \lambda(T)=h. \tag{23}$$

Again h is given and v is the control. We minimise

$$J_2(v)=\lambda(0)'P_0\lambda(0)+\int_0^T (v(t)'v(t)+\lambda(t)'H'H\lambda(t))\,dt. \tag{24}$$

The system of necessary conditions is given by

$$\begin{aligned} &\dot{\hat{\lambda}}=A\hat{\lambda}+NN'\hat{\eta}, \qquad \hat{\lambda}(T)=h,\\ &-\dot{\hat{\eta}}=A'\hat{\eta}-H'H\hat{\lambda}, \qquad \hat{\eta}(0)=P_0\hat{\lambda}(0), \end{aligned} \tag{25}$$

and an optimal control is $\hat{v}(t)=N'\hat{\eta}(t)$, with

$$\min J_2(v)=h'P(T)h=h'\hat{\eta}(T), \tag{26}$$

and

$$\hat{\eta}(t)=P(t)\hat{\lambda}(t) \quad \text{for all } t, \tag{27}$$

where $P(t)$ is the solution of (17).

Since R is positive definite, in particular nonsingular, the pair (H, A) is detectable. We assume that there exists a matrix Λ such that

$$\eta'(A+\Lambda H)\eta \leqq -\alpha_0|\eta|^2, \qquad \alpha_0>0. \tag{28}$$

Also, since N has rank n, the pair (A, N) is controllable, and we assume that there exists a matrix Γ such that

$$\lambda'(A+N\Gamma)\lambda \geqq \beta_0|\lambda|^2, \qquad \beta_0>0. \tag{29}$$

THEOREM 4. *Under the above assumptions, we have*

$$\|Q(T)\| \leqq \left(\|Q_0\|+\frac{\|N\|^2+\|\Lambda\|^2}{2\alpha_0}\right)\equiv q, \tag{30}$$

$$\|P(T)\| \leqq \left(\|P_0\|+\frac{\|H\|^2+\|\Gamma\|^2}{2\beta_0}\right)\equiv p. \tag{31}$$

Proof. Consider in (18) a feedback control

$$v(t)=\Lambda'\eta(t).$$

The corresponding state is the solution of

$$-\dot{\eta}=(A'+H'\Lambda')\eta, \qquad \eta(T)=h. \tag{32}$$

Therefore

$$h'Q(T)h \leqq \eta(0)'Q_0\eta(0)+\int_0^T \eta(t)'(NN'+\Lambda\Lambda')\eta(t)\,dt. \tag{33}$$

From (32) it follows that

$$|\eta(0)|^2-2\int_0^T \eta(t)'(A'+H'\Lambda')\eta(t)\,dt=|h|^2,$$

and from (28) we deduce that $|\eta(0)|^2 \leqq |h|^2$ and

$$\int_0^T |\eta(t)|^2\, dt \leqq \frac{|h|^2}{2\alpha_0}.$$

Therefore from (33) it follows that

$$h'Q(T)h \leqq h'\left(\|Q_0\| + \frac{\|N\|^2 + \|\Lambda\|^2}{2\alpha_0}\right)h$$

which proves (30).

Next, consider in (23) the feedback control

$$v(t) = \Gamma\lambda(t).$$

Then

(34) $$\dot{\lambda} = (A + N\Gamma)\lambda, \qquad \lambda(T) = h,$$

and we have

(35) $$h'P(T)h \leqq \lambda(0)'P_0\lambda(0) + \int_0^T \lambda(t)'(\Gamma'\Gamma + H'H)\lambda(t)\, dt.$$

Using (29) and (34) it follows that

$$|h|^2 \geqq |\lambda(0)|^2 + 2\beta_0 \int_0^T |\lambda(t)|^2\, dt,$$

and hence

$$\int_0^T |\lambda(t)|^2\, dt \leqq \frac{|h|^2}{2\beta_0}.$$

This together with (35) yields (31). □

Remark. This theorem is true if rank $N < n$, provided that (A, N) is *controllable.*

2.5. Convergence of the linear observer. We now use the bounds (30), (31) to prove the following.

THEOREM 5. *The dynamical system* (16), (13) *is an observer for the linear control system* (3) *provided that* (C, A) *is detectable and the above assumptions hold. That is, there exists constants* $K > 0$, $\gamma > 0$ *such that*

$$|x(t) - m(t)| \leqq K|x_0 - m_0|\, e^{-\gamma t}$$

for all $t > 0$.

Proof. From (30), (31) it follows that

(36) $$|P(t)\lambda| \geqq \frac{|\lambda|}{q}$$

and

(37) $$\frac{|\lambda|^2}{q} \leqq \lambda'P(t)\lambda \leqq p|\lambda|^2.$$

Now $e(t) = x(t) - m(t)$ satisfies

$$\dot{e}(t) = (A - Q(t)H'H)e(t).$$

Using (17), (36) we deduce

$$\frac{d}{dt} e(t)'P(t)e(t) = -e(t)'(P(t)NN'P(t) + H'H)e(t)$$

$$\leqq -e(t)'P(t)NN'P(t)e(t)$$

$$\leqq -\frac{r_0}{q^2}|e(t)|^2,$$

where $NN' \geqq r_0 I$, $r_0 > 0$. This together with (37) implies that

$$\frac{d}{dt} e(t)'P(t)e(t) \leqq -\frac{r_0}{pq^2} e(t)'P(t)e(t).$$

Set $\gamma = r_0/2pq^2$. Therefore

$$e(t)'P(t)e(t) \leqq e(0)'P_0 e(0)e^{-2\gamma t}$$

and

$$|e(t)|^2 \leqq q e(0)'P_0 e(0) e^{-2\gamma t}$$

from which the desired result follows. □

Finally, we state the following result which is a consequence of standard facts concerning the Riccati equation (13).

THEOREM 6. *Given the linear system* (3), *where* (C, A) *is detectable, an* $n \times m$ *matrix* N *such that* (A, N) *is stabilisable, and a positive definite matrix* R, *then there exists a unique nonnegative definite solution* $\bar{Q}$ *to the algebraic Riccati equation*

$$A\bar{Q} + \bar{Q}A' - \bar{Q}C'(RR')^{-1}C\bar{Q} + NN' = 0,$$

the matrix $A - \bar{Q}C'(RR')^{-1}C$ *is exponentially stable, and the system*

$$\dot{m}(t) = Am(t) + Bu(t) + \bar{Q}C'(RR')^{-1}(y(t) - Cm(t)),$$
$$m(0) = m_0,$$

is a time-invariant observer for the given system.

3. Observers for nonlinear systems. We consider a nonlinear dynamical system with linear observations:

$$\dot{x}(t) = f(x(t)), \qquad x(0) = x_0,$$
$$y(t) = Cx(t). \tag{38}$$

We assume that $f: \mathbb{R}^n \to \mathbb{R}^n$ is smooth with bounded derivatives, and we write

$$A(x) = Df(x)$$

for the $n \times n$ matrix of first derivatives. Set

$$\|A\| = \sup_{x \in \mathbb{R}^n} \|A(x)\|.$$

3.1. Observer design. Motivated by the linear design, we construct an observer for (38) as an approximation to the corresponding deterministic estimator. Associate the following system with (38):

$$\dot{z}(t) = f(z(t)) + Nw(t), \qquad z(0) = z_0,$$
$$\dot{\zeta}(t) = Cz(t) + Rv(t), \qquad \zeta(0) = 0, \tag{39}$$

where rank $N = n$, R is positive definite, and the energy functional

$$J_t(z_0, w, v) = \frac{1}{2}(z_0 - \mu)'P_0(z_0 - \mu) + \frac{1}{2}\int_0^t (w(s)'w(s) + v(s)'v(s))\,ds, \tag{40}$$

where P_0 is positive definite.

According to Hijab [4], the deterministic estimate $\hat{z}$ is the solution of

$$\begin{aligned} \dot{\hat{z}}(t) &= f(\hat{z}(t)) + Q(t)C'(RR')^{-1}(\dot{\zeta}(t) - C\hat{z}(t)), \\ \hat{z}(0) &= \mu, \end{aligned} \tag{41}$$

where

$$Q(t)^{-1} = D^2S(\hat{z}(t), t),$$

and $S(z, t)$ is the solution of the Hamilton–Jacobi–Bellman equation

$$\begin{aligned} \frac{\partial}{\partial t} S(z, t) + H(z, t, DS(z, t)) &= 0, \\ S(z, 0) &= \tfrac{1}{2}(z - \mu)'P_0(z - \mu), \end{aligned} \tag{42}$$

where

$$H(z, t, \alpha) = \alpha f(z) + \tfrac{1}{2}\alpha NN'\alpha' - \tfrac{1}{2}z'C'(RR')^{-1}Cz + z'C'(RR')^{-1}\dot{\zeta}(t).$$

In the linear case the solution of (42) is a quadratic form and $Q(t) = P(t)^{-1}$ satisfies a Riccati equation. However, in the general nonlinear case, solutions are not smooth and must be interpreted in the viscosity sense. Thus (41) is not well defined in the large. We seek therefore to "approximate" $S(z, t)$ by a quadratic form, and replace the Hamilton–Jacobi equation (42) by a simpler Riccati equation. In this way we will obtain a well-defined observer. Write $H = R^{-1}C$.

Suppose that S is smooth in a neighbourhood of $(\mu, 0)$. Denoting components by superscripts and partial derivatives by subscripts, and using the summation convention, for small t we have at $(\hat{z}(t), t)$

$$\frac{d}{dt} S_{ij}(\hat{z}(t), t) = S_{ijk}\dot{\hat{z}}^k - S_{kij}f^k - 2S_{ki}f^k_j - S_{ki}(NN')^{kl}S_{lj} + (H'H)^{ij},$$

using the fact that $S_k(\hat{z}(t), t) = 0$ from the definition of $\hat{z}(t)$. If S were quadratic, the third-order terms vanish. This suggests that we replace (41) by

$$\begin{aligned} \dot{m}(t) &= f(m(t)) + Q(t)C'(RR')^{-1}(y(t) - Cm(t)), \\ m(0) &= m_0 = \mu, \end{aligned} \tag{43}$$

where now $Q(t) = P(t)^{-1}$, and $P(t)$ satisfies the Riccati equation

$$\begin{aligned} \dot{P}(t) &= -P(t)A(m(t)) - A(m(t))'P(t) - P(t)NN'P(t) + H'H, \\ P(0) &= P_0. \end{aligned} \tag{44}$$

Also $Q(t)$ is the solution of

$$\begin{aligned} \dot{Q}(t) &= A(m(t))Q(t) + Q(t)A(m(t))' - Q(t)H'HQ(t) + NN', \\ Q(0) &= Q_0 = P_0^{-1}. \end{aligned} \tag{45}$$

Once again it is important to obtain bounds for $\|Q(t)\|$, $\|P(t)\|$. To recover estimates similar to (30), (31), we assume that the pair $(H, A(x))$ is *uniformly detectable*, that is, there exists a bounded Borel matrix-valued function $\Lambda(x)$ such that

$$\eta'(A(x) + \Lambda(x)H)\eta \leqq -\alpha_0|\eta|^2, \qquad \alpha_0 > 0, \tag{46}$$

for all $x \in \mathbb{R}^n$. In addition, we assume that the pair $(A(x), N)$ is *uniformly controllable*, that is, there exists a bounded Borel $\Gamma(x)$ such that

$$\lambda'(A(x)+N\Gamma(x))\lambda \geqq \beta_0|\lambda|^2, \qquad \beta_0>0, \tag{47}$$

for all $x \in \mathbb{R}^n$. Set

$$\|\Lambda\| = \sup_{x\in\mathbb{R}^n} \|\Lambda(x)\|, \qquad \|\Gamma\| = \sup_{x\in\mathbb{R}^n} \|\Gamma(x)\|.$$

Then using the methods of § 2.4, the following generalisation of Theorem 4 can be proven.

THEOREM 7. *Under the above assumptions, we have*

$$\|Q(T)\| \leqq \left(\|Q_0\| + \frac{\|N\|^2+\|\Lambda\|^2}{2\alpha_0} \right) \equiv q, \tag{48}$$

$$\|P(T)\| \leqq \left(\|P_0\| + \frac{\|H\|^2+\|\Gamma\|^2}{2\beta_0} \right) \equiv p. \tag{49}$$

3.2. Asymptotic convergence. We wish to prove that the system (43), (45) is an observer for the nonlinear system (38). This is possible provided that we can bound the region where the initial condition lies and provided the second derivative of f is not too large.

Consider $DA(x) = D^2f(x)$. For any x, $D^2f(x) \in L(\mathbb{R}^n, L(\mathbb{R}^n, \mathbb{R}^n))$ and we denote $\|D^2f\|$ the supremum over x of the norm of the linear operator $D^2f(x)$.

Note that the numbers p and q defined by (48), (49) are functions of the design parameters P_0, N, and R, and the given data f and C. The designer is free to choose the parameters within the stated constraints. Also $NN' \geqq r_0 I$ for some $r_0 > 0$. Define

$$\varphi(P_0, N, R) = \frac{r_0}{q^2 p^{1/2} \|P_0^{1/2}\|}. \tag{50}$$

THEOREM 8. *Assume that*

$$|x_0 - m_0|\, \|D^2 f\| < \max_{P_0, N, R} \varphi(P_0, N, R). \tag{51}$$

Then the dynamical system (43), (45) *is an observer for the nonlinear system* (38) *provided that* $(C, A(x))$ *is uniformly detectable and the above assumptions hold. That is, there exist constants* $K>0$, $\gamma>0$, *such that*

$$|x(t)-m(t)| \leqq K|x_0 - m_0| e^{-\gamma t}$$

for all $t>0$.

Proof. Now $e(t) = x(t) - m(t)$ satisfies

$$\dot{e}(t) = f(x(t)) - f(m(t)) - Q(t)H'He(t).$$

From (44) we deduce

$$\begin{aligned} \frac{d}{dt} e(t)'P(t)e(t) &= -e(t)'(2P(t)A(m(t)) + P(t)NN'P(t) - H'H)e(t) \\ &\quad + 2e(t)'P(t)(f(x(t)) - f(m(t)) - Q(t)H'He(t)) \\ &= -e(t)'(P(t)NN'P(t) + H'H)e(t) \\ &\quad + 2e(t)'P(t) \int_0^1 \int_0^1 rD^2 f(m(t)+rse(t))e(t)^2 \, dr\, ds \end{aligned} \tag{52}$$

$$\leqq e(t)' \left(-\frac{r_0}{q^2} + |P^{1/2}(t)e(t)| p^{1/2} \|D^2 f\| \right) e(t). \tag{53}$$

By (51) we can find P_0, N, R such that

$$|e(0)|\,\|D^2 f\| < \varphi(P_0, N, R);$$

hence

$$|P_0^{1/2} e(0)|\,\|D^2 f\| < \frac{r_0}{q^2 p^{1/2}},$$

or

$$-\frac{r_0}{q^2} + |P_0^{1/2} e(0)|\, p^{1/2} \|D^2 f\| < 0.$$

Since $P^{1/2}(t)e(t)$ is continuous, there exists an interval $[0, t_0)$ such that

$$-\frac{r_0}{q^2} + |P^{1/2}(t)e(t)|\, p^{1/2} \|D^2 f\| < 0,$$

on $[0, t_0)$. But from (53),

$$\frac{d}{dt} |P^{1/2}(t)e(t)| < 0$$

on $[0, t_0)$, and thus

$$|P^{1/2}(t)e(t)| \leqq |P_0^{1/2} e(0)|$$

on $[0, t_0)$. By continuity we have

$$|P^{1/2}(t_0)e(t_0)| \leqq |P_0^{1/2} e(0)|,$$

and we can proceed from t_0 on. Therefore, in fact,

$$|P^{1/2}(t)e(t)| \leqq \frac{1}{p^{1/2}\|D^2 f\|}\left(\frac{r_0}{q^2} - \delta\right), \qquad \delta > 0,$$

for all $t > 0$, and (53) implies that

$$\frac{d}{dt}\, e(t)'P(t)e(t) \leqq -\delta |e(t)|^2.$$

But from (49),

$$e(t)'P(t)e(t) \leqq \|P(t)\|\,|e(t)|^2 \leqq p|e(t)|^2;$$

hence

$$\frac{d}{dt}\, e(t)'P(t)e(t) \leqq -\frac{\delta}{p}\, e(t)'P(t)e(t)$$

which implies that

$$e(t)'P(t)e(t) \leqq e(0)'P_0 e(0) e^{-(\delta/p)t}.$$

Therefore, using (48),

$$\begin{aligned} |e(t)|^2 &\leqq \|Q(t)\| e(t)'P(t)e(t) \\ &\leqq q e(t)'P(t)e(t) \\ &\leqq q e(0)'P_0 e(0)\, e^{-(\delta/p)t}, \end{aligned}$$

from which we deduce the desired result. □

Remark. By the mean value theorem,

$$f(x(t))-f(m(t))=\int_0^1 Df(sx(t)+(1-s)m(t))\,ds(x(t)-m(t)).$$

This yields the estimate

$$2e(t)'P(t)(f(x(t))-f(m(t)))\leqq 2p\|A\|\,|e(t)|^2.$$

Using this in (52) we obtain

$$\frac{d}{dt}e(t)'P(t)e(t)\leqq e(t)'\left(-\frac{r_0}{q^2}+4p\|A\|\right)e(t).$$

Hence if the design parameters P_0, N, R were chosen so that

$$0<\delta\equiv\frac{r_0}{q^2}-4p\|A\|,$$

then (51) is unnecessary. Then (43), (45) is an observer for (38) *independent* of the initial conditions. Unfortunately this inequality is at best difficult to achieve.

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