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ACCURATE EVALUATION OF STOCHASTIC WIENER INTEGRALS WITH APPLICATIONS TO SCATTERING IN RANDOM MEDIA AND TO NONLINEAR FILTERING*

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Abstract. In 1973 A. J. Chorin [Math. Comp., 27 (1973), pp. 1–15] reported quadrature formulas for the approximation of a class of Wiener path integrals by n-fold ordinary integrals with an error $O(n^{-2})$. Similar formulas are presented here for certain stochastic Wiener function space integrals in which n-fold integral approximations result in errors $O(n^{-2})$ or $O(n^{-3/2})$. Stochastic Wiener integrals arise in the physics of wave propagation in random media and in signal estimation problems in communication theory. The present formulas provide simple, accurate computational tools for these applications, among others. This point is illustrated by several examples in the paper.

1. Introduction. Function space integrals—Wiener integrals—arise in a number of problems in mathematical physics including the analysis of wave scattering in random media [2] and representations of the solutions of the "Schrödinger equation" [3], [4]

$$\frac{\partial}{\partial t}u(t,x)=\frac{1}{2}\alpha\frac{\partial^2}{\partial x^2}u(t,x)+V(x)u(t,x), \tag{1.1}$$

$$u(0,x)=f(x),\qquad 0\leqq t\leqq T,\quad \alpha\in C.$$

That is,

$$u(t,x)=\mathscr{E}_x^\alpha\left[\exp\left(\int_0^t -V(x(s))\,ds\right)f(x(t))\right] \tag{1.2}$$

where $\{x(t), t\geqq 0\}$ is a path of Brownian motion and $\mathscr{E}_x^\alpha$ is the Wiener function space integral over all paths starting at $x(0)=x$. The integral in (1.2) may be represented explicitly in the following way. Let $C([0,t])$ be the space of real-valued continuous functions $x(s)$ on $[0,t]$ with $x(0)=0$, and let W be a Wiener measure on C. The integral in (1.2) is a functional, say $F(x)$, on $C([0,t])$, and the Wiener integral is abstractly

$$I=\int_C F(x)\,dW(x). \tag{1.3}$$

This is defined as the "sequential" limit [5]

$$I=\lim_{\substack{\max|t_j-t_{j-1}|\to 0\\ l\leqq j\leqq n}}\int_R\overset{(n)}{\cdots}\int_R da_1\cdots da_n\,F(z_{sx})\prod_{j=1}^n\frac{\exp\left[-(a_j-a_{j-1})^2/2\alpha(t_j-t_{j-1})\right]}{[2\pi\alpha(t_j-t_{j-1})]^{n/2}}, \tag{1.4}$$

where $0<t_1<t_2<\cdots<t_n=t$ and $z_{s,x}$ is a polygonal function on $[0,t]$ passing through x at $s=0$ and a_j at t_j, $j=1,2,\cdots,n$.

Except for a few simple cases, it is impossible to evaluate Wiener integrals explicitly. Approximation formulas amenable to numerical computations are, therefore, of considerable interest in applications. In 1973 A. J. Chorin [1] presented some remarkably simple formulas for the accurate approximation of Wiener integrals. His

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results were based on the use of certain parabolas to interpolate the Wiener paths and on the expansion of the nonlinear functional F in a Taylor series with the quadrature formula adjusted to optimize the approximation of the first two terms. Chorin constructed approximation formulas of the form

(1.5)
$$\int_C F(x)\,dW(x)=\pi^{-n/2}\int_{R^n} F_n(u_1,u_2,\cdots,u_n)\cdot\exp(-u_1^2-\cdots-u_n^2)\,du_1\cdots du_n+O(n^{-2}).$$

In the specific case $F(x)=G(\int_0^t V[x(s)]\,ds)$ with G and V smooth Chorin obtained [1, eq. 9]

(1.6)
$$I_0=\int_C G\left(\int_0^t V[x(s)]\,ds\right)dw(x)$$
$$=\pi^{-n/2}\int_{R^n}\left\{G\left[\sum_{i=1}^n \frac{t}{n}V\left(x_{i-1}+\frac{vt}{(2n)^{1/2}}\right)\right]\right\}\cdot\exp(-u_1^2-\cdots-u_{n-1}^2-v^2)\,du_1\cdots du_{n-1}\,dv+O(n^{-2})$$

where $x_{i-1}=t(u_1+\cdots+u_{i-1})/\sqrt{n}$.

Our objective here is to extend the formulas (1.5), (1.6) to stochastic Wiener integrals. Two types of integrals are considered. Let $\{\xi(s),\ 0\leqq s\leqq T\}$ be a real-valued random process on some probability space $(\Omega,\mathcal{F},P)$ which has continuous paths $\xi(\cdot,\omega)$ almost surely, and which satisfies $E\xi^2(s)<\infty$, $0\leqq s\leqq T$. With $x\in C([0,t])$ we define

(1.7)
$$F^{(1)}(x,\omega)=G\left(\int_0^t V[x(s)]\xi(s,\omega)\,ds\right)$$

and the *stochastic Wiener integral of type* 1

(1.8)
$$I^{(1)}(\omega)=\int_C F^{(1)}(x,\omega)\,dW(x).$$

The notation emphasizes the fact that $I^{(1)}$ is an R-valued random variable on $(\Omega,\mathcal{F},P)$.

Integrals of type 1 arise in scattering problems and in filtering problems; see §§ 3 and 4 below.

Now let $\{y(s),\ 0\leqq s\leqq T\}$ be an Ito process generated by

(1.9)
$$dy(s)=f(s)\,ds+g(s)\,dw(s),\qquad y(0)=0,\qquad 0\leqq s\leqq T,$$

where $\{w(s)\}$ is an R-valued Wiener process on $(\Omega,\mathcal{F},P)$ and $\{f(s)\}\{g(s)\}$ are R-valued processes on $(\Omega,\mathcal{F},P)$ which are nonanticipative with respect to $\{w(s)\}$ and which have continuous paths almost surely. With $x\in C[0,t]$, $0\leqq t\leqq T$ fixed, we define

(1.10)
$$F^{(2)}(x,\omega)=G\left(\int_0^t V[x(s)]\,dy(s,\omega)\right)$$

and the *stochastic Wiener integral of type* 2

(1.11)
$$I^{(2)}(\omega)=\int_C F^{(2)}(x,\omega)\,dW(x).$$

Integrals of type 2 arise in filtering problems; see § 4.

We shall construct approximation formulas[1] of the form

$$I^{(j)}(\omega)=(2\pi)^{-n/2}\int_{R^n} F_n^{(j)}(u_1,\cdots,u_n;\omega)\cdot\exp[(u_1^2-\cdots-u_n^2)/2]\,du_1\cdots du_n+e_n^{(j)},\qquad j=1,2, \tag{1.12}$$

where the integrand $F_n^{(j)}$ is a simple random variable similar in form to (1.6), and the error $e_n^{(j)}$ is small. In the two cases we show that

$$(E|e_n^{(1)}|^2)^{1/2}=O(n^{-2}),\qquad (E|e_n^{(2)}|^2)^{1/2}=O(n^{-3/2})\text{ or }O(n^{-2}). \tag{1.13}$$

The paper is organized as follows. In § 2 we present the approximation formulas in a series of four theorems. In §§ 3 and 4 we apply these formulas to specific problems in scattering theory and nonlinear filtering theory, respectively. In § 5 we prove Theorems 1 and 2, which provide the approximation formulas for type 1 integrals. In § 6 we establish the results for type 2 integrals. Our proofs, which are similar in form to Chorin's argument, differ substantially in detail. For example, in the treatment of type 2 integrals simple Schwartz type estimates used by Chorin fail, and it is necessary to estimate several high order moments of the random truncation error very precisely. This portion of the argument is our main technical contribution.

The examples of scattering and nonlinear filtering problems presented exploit the longstanding relationship between Wiener–Feynman integrals and differential equations as established by Feynman [3] and Kac [4]; see [6], [7] for some recent results and the survey [8, § 4] for a complete history. Stochastic function space integrals of the types considered here have been used by Chow and others [2], [7], [9], [10] in studies of wave propagation in random media (especially type 1 integrals) and by Kushner, Pardoux, Davis and us [11]–[14] in studies of nonlinear filtering problems (type 2 integrals). The approximation results presented here for specific problems in these areas are intended as illustrations of the use of the formulas rather than definitive statements about wave propagation or filtering problems.

We have adapted Chorin's treatment to the stochastic case because of the particularly simple form of the finite dimensional approximations. Other possibilities merit consideration. For instance, Cameron obtained accurate ($O(n^{-2})$) approximations to Wiener integrals using a version of Simpson's rule [15]. In treating nonlinear filtering problems Kushner obtained recursive approximations by exploiting weak convergence arguments based on appropriate Markov chain structures [16]. Levieux [17], among others, has directly discretized the nonlinear stochastic PDE which occurs in the filtering of diffusion processes. Our work should be regarded as complementing these methods in specific applications.

2. The Approximation formulas. Our first two results are approximation formulas for stochastic Wiener integrals of type 1.

THEOREM 1. *Let $\{\xi(s), 0\leqq s\leqq T\}$ be a real-valued random process on $(\Omega, \mathcal{F}, P)$, with right continuous paths and*

$$\sup_{0\leqq s\leqq T} E|\xi(s)|^2<\infty \tag{A1}$$

[1] For type 2 integrals we are only able to treat the (important) case $G(\cdot)=\exp(\cdot)$.

Suppose that $V: R \to R$ *has derivatives up to order* 4 *with*

$$\int_t^{t+1/n} |V''''(f(s))|^2 \, ds = O(n^{-1}) \tag{A2}$$

for all $t \in [0, T)$ *and any continuous* $f: [0, T] \to R$. *Then, for any fixed* $t \in (0, T)$,

$$\begin{aligned} I^{(11)} &= \int_C \left[\int_0^t V(x(s))\xi(s) \, ds \right] dW(x) \\ &= (2\pi)^{-n/2} \sum_{i=1}^n \Xi_{i-1} \left(\int_{R^n} V\left(x_{i-1} + \frac{vt}{\sqrt{2n}} \right) \right. \\ &\qquad \left. \cdot \exp\left[\frac{-u_1^2 - \cdots - u_{n-1}^2 - v^2}{2} \right] du_1 \cdots du_{n-1} \, dv \right) + e_n^{(11)}, \end{aligned} \tag{2.1}$$

where

$$\begin{aligned} x_i &= \frac{t(u_1 + \cdots + u_i)}{n}, \qquad i = 1, 2, \cdots, n, \\ \Xi_{i-1} &= \int_{t_{i-1}}^{t_i} \xi(s) \, ds, \qquad t_i = \frac{it}{n}, \end{aligned} \tag{2.2}$$

and, with expectation on the paths of ξ,

$$(E|e_n^{(11)}|^2)^{1/2} = O(n^{-2}). \tag{2.3}$$

THEOREM 2. *Let* $\{\xi(s), 0 \leqq s \leqq T\}$ *and* $V: R \to R$ *satisfy* (A1) *and* (A2) *of Theorem* 1. *Let* $G: R \to R$ *have derivatives up to order* 4 *with*

$$E\left| G''''\left(\int_s^t f(r)\xi(r) \, dr \right) \right|^4 < \infty \tag{A3}$$

for all $0 \leqq s \leqq t \leqq T$ *and any continuous* $f: [0, T] \to R$. *Then, with* x_i, Ξ_i *as in Theorem* 2,

$$\begin{aligned} I^{(12)} &= \int_C G\left[\int_0^t V(x(s))\xi(s) \, ds \right] dW(x) \\ &= (2\pi)^{-n/2} \int_{R^n} \left\{ G\left[\sum_{i=1}^n V\left(x_{i-1} + \frac{vt}{(2n)^{1/2}} \right) \Xi_{i-1} \right] \right\} \\ &\qquad \cdot \exp\left[\frac{-u_1^2 - \cdots - u_{n-1}^2 - v^2}{2} \right] du_1 \cdots du_{n-1} \, dv + e_n^{(12)}, \end{aligned} \tag{2.4}$$

where

$$(E|e_n^{(12)}|^2)^{1/2} = O(n^{-2}). \tag{2.5}$$

THEOREM 3. *Let* $\{w(s), 0 \leqq s \leqq t\}$ *be an* R*-valued standard Wiener process on* $(\Omega, \mathcal{F}, P)$ *and let* $\{f(s), g(s), 0 \leqq s \leqq t\}$ *be* R*-valued random processes nonanticipative with respect to* w *which have continuous paths almost surely and second moments uniformly bounded in* $s \in [0, t]$. *Let*

$$\begin{aligned} dy(s) &= f(s) \, ds + g(s) \, dw(s), \\ y(0) &= 0, \qquad 0 \leqq s \leqq t. \end{aligned} \tag{A4}$$

Suppose that $V: R \to R$ *satisfies* (A2) *of Theorem* 1. *Then*

$$I^{(21)} = \int_C \left[\int_0^t V(x(s))\,dy(s)\right] dW(x)$$

$$= (2\pi)^{-n/2} \sum_{i=1}^{n} \Delta y_{i-1}\left[\int_{R^n} V\left(x_{i-1} + \frac{vt}{\sqrt{2n}}\right) \cdot \exp\left[\frac{-u_1^2 - \cdots - u_{n-1}^2 - v^2}{2}\right] du_1 \cdots du_{n-1}\,dv\right] + e_n^{(21)}, \tag{2.6}$$

where $t_i = it/n$, $i = 0, 1, 2, \cdots, n$, *and*

$$\Delta y_{i-1} = y(t_i) - y(t_{i-1}). \tag{2.7}$$

The approximation error is

$$(E|e_n^{(21)}|^2)^{1/2} = O(n^{-3/2}). \tag{2.8}$$

As stated in the Introduction, we are not yet able to produce the counterpart of Theorem 2 for the approximation of

$$I^{(22)} = \int_C G\left[\int_0^t V(x(s))\,dy(s)\right] dW(x)$$

when y is an Ito process and G is an arbitrary smooth function. However, by specializing to the case $G(v) = \exp(v)$, we can obtain a useful approximation formula. Since this case arises in the nonlinear filtering problem (see § 4), our primary application [14], we report it here.

THEOREM 4. *Let* w, f, g, *and* V *satisfy the hypotheses of Theorem* 3. *Then*

$$I_e^{(22)} = \int_C \exp\left[\int_0^t V(x(s))\,dy(s)\right] dW(x)$$

$$= (2\pi)^{-n/2} \int_{R^n} \left\{\exp\left[\sum_{i=1}^{n} V\left(x_{i-1} + \frac{vt}{(2n)^{1/2}}\right)\Delta y_{i-1}\right]\right\} \cdot \exp\left[\frac{-u_1^2 - \cdots - u_{n-1}^2 - v^2}{2}\right] du_1 \cdots du_{n-1}\,dv + e_n^{(22)}, \tag{2.9}$$

where

$$(E|e_n^{(22)}|^2)^{1/2} = O(n^{-2}). \tag{2.10}$$

Remark. Since the measure dW is Gaussian, the restriction $G = \exp$ allows us to exploit the characteristic function of the Gaussian and its moment generating properties to obtain the estimate (2.10). Since the exponential matches the Gaussian so well, we obtain a sharper estimate than one would expect from Theorem 3. When G is an arbitrary smooth function we expect that the approximation will be $O(n^{-3/2})$; after all $G(x) = x$ in Theorem 3 leads to this. To date we have not been able to establish this conjecture.

3. Wave propagation in random media. The problems of wave propagation in random or turbulent media are frequently modeled by hyperbolic systems with random coefficients. Typical cases include acoustic or electromagnetic waves propagating in turbulent atmosphere, ocean, or plasma [18], [19] or light in imperfect optical fibers [20]. Most work on these problems begins with random scattering phenomena governed by the scalar wave equation

$$\frac{\partial^2 v}{\partial t^2} - c^2(x, \omega)\Delta v = f(t, x, \omega), \tag{3.1}$$

where c is the local speed of propagation characterizing the random medium, f is a randomly distributed source, and ω is the random event. Assuming time harmonic processes ($v(t, x) = e^{i\nu t}u(x)$) or by using Laplace transforms, etc., the wave equation is usually reduced to the random Helmholtz equation

$$\Delta u + k^2 n^2(x, \omega)u = q(x, \omega) \tag{3.2}$$

which must satisfy the physical radiation condition

$$\lim_{|x|\to\infty} |x|\left(\frac{\partial u}{\partial |x|}\right) - iku = 0. \tag{3.3}$$

Here the wave number k is complex, Im $k > 0$, and the index of refraction, $n(x, \omega)$, is inversely proportional to $c(x, \omega)$.

An especially simple version of this problem is the scattering of a time-harmonic wave by a random half-space [2]. It is supposed that the wave propagates in a medium homogeneous for $x < 0$ and randomly inhomogeneous for $x \geqq 0$. Here $\underline{x} = (x, \underline{r})$ is the space variable with $\underline{r}$ the transverse coordinate. The wave function u satisfies

$$\begin{aligned} &\Delta u + k^2 n^2(\underline{x})u = 0, \qquad x > 0,\\ &\Delta u + k^2 u = 0, \qquad x < 0, \end{aligned} \tag{3.4}$$

and the requirements that u and u_x be continuous at $x = 0$ and that u be outgoing at infinity. The refractive index is assumed to be $n^2(\underline{x}) = 1 + \eta(x, \underline{r})$. For a wave $f(\underline{r})\, e^{ikx}$ incident from the left, one seeks a solution $u(\underline{x}) = v(\underline{x})\, e^{ikx}$ in $x > 0$. Substituting this in (3.4), and neglecting the term v_{xx} in the resulting equation, leads to

$$\frac{\partial v}{\partial x} = \frac{i}{2k}(\Delta_T + k^2\eta)v, \qquad x > 0, \tag{3.5a}$$

where Δ_T is the transverse Laplacian. The "initial" condition

$$v(0, \underline{r}) = f(\underline{r}) \tag{3.5b}$$

determines the solution in the context of the approximation. The reduction of the full wave problem (3.4) to the initial value problem (3.5) in which x behaves as a time-like variable is called the parabolic equation approximation. It is a singular perturbation approximation, valid for large k [21].

Evidently, (3.5) is a special case of the generalized heat equation

$$\begin{aligned} &\frac{\partial u}{\partial t} = [\tfrac{1}{2}\alpha\Delta + \xi(t, \underline{x}, \omega)]u, \qquad t > 0,\\ &u(0, \underline{x}) = f(\underline{x}), \qquad \underline{x} \in R^d, \end{aligned} \tag{3.6}$$

where α is a complex number with Re $(\alpha) \geqq 0$, Δ is the Laplacian in $\underline{x} \in R^d$ and ξ is a random field. Other scattering phenomena (e.g., radiation from a point source in a random medium [2, § 2]) may also be described by (3.6) (with an appropriate source term appended).

In a different application,[2] when α is real and nonnegative, (3.6) models random heat generation in an active conducting medium, where the randomly generated source is proportional to the local temperature.

Formally, the solution to (3.6) is given by

$$u(t, \underline{x}) = \mathscr{E}_z^\alpha\left\{\exp\left[\int_0^t \xi(\tau, z(\tau)+\underline{x})\, d\tau\right] f[z(t)+\underline{x}]\right\} \tag{3.7}$$

where $\mathscr{E}_z^\alpha$ is the *sequential Wiener integral*

$$\mathscr{E}_z^\alpha\{F[z]\} = \lim_{\substack{\max|\tau_j-\tau_{j-1}|\to 0\\ 0\leqq j\leqq m}} \int_{R^d} \overset{(m)}{\cdots} \int_{R^d} dy_1 \cdots dy_m \cdot F[z_{\tau,y}] \prod_{j=1}^m \frac{\exp\{-|y_j-y_{i-1}|^2/2\alpha(\tau_j-\tau_{j-1})\}}{[2\pi\alpha(\tau_j-\tau_{j-1})]^{n/2}}. \tag{3.8}$$

Here $F: C([0, t]; R^d) \to R$, $0<\tau_1<\cdots<\tau_m = t$, and $z_{\tau,y}$ is a polygonal function of τ on $[0, t]$ passing through (y_j, τ_j), $j = 1, 2, \cdots, m$, and z at $\tau = 0$.

Conditions for the existence of the integral in (3.8) for F deterministic and α complex are given in Cameron's paper [5]. If α is real and positive, then $\mathscr{E}_z^\alpha$ reduces to an ordinary Wiener integral [5, Thm. 1]. If α is purely imaginary, then $\mathscr{E}_z^\alpha$ is a "Feynman integral" [5, p. 127], and the sequential formulation is essential. When the variance α is complex with Re $(\alpha)>0$ and Im $\alpha \neq 0$, then the Wiener "measure" in (3.8) has infinite variation, even though the "measure" of the whole space is finite. Therefore, no integration theory in the usual measure theoretic sense is possible for nonreal variances.

This fact affects the scattering problem (3.6) since the generic case (3.5) has an imaginary variance parameter and requires a (stochastic) Feynman integral for its solution. Existence and uniqueness theorems for (3.6) when α is real and nonnegative, are in [9]. However, as Chow points out [12], [9], the objects of interest for (3.5), (3.6) are the moments

$$\begin{aligned} \Gamma_1(t, \underline{x}) &= E[u(t, \underline{x})], \\ \Gamma_2(t_1, t_2; \underline{x}_1, \underline{x}_2) &= E[u(t_1, \underline{x}_1)u(t_2, \underline{x}_2)], \end{aligned} \tag{3.9}$$

where E is expected value with respect to the distribution of the random field ξ. To date no one has established a Fubini-type theorem for exchanging the order of the Lebesgue integral E and the sequential Wiener integral $\mathscr{E}_z^\alpha$. Since this step is important in computing Γ_i, $i = 1, 2, \cdots$, it is usually taken even without justification in the course of the analysis. For example, Γ_1 becomes

$$\Gamma_1(t, \underline{x}) = \mathscr{E}_z^\alpha\left\{E\left[\exp\left(\int_0^t \xi(\tau, z(\tau)+\underline{x})\, d\tau\right)\right] f(\underline{x}+z(t))\right\} \tag{3.10}$$

and the characteristic function for $\xi(\tau, \underline{x})$ may be used to evaluate the inner integral, especially if $\xi(\tau, \underline{x})$ is Gaussian.

By bringing to bear the approximation formula in Theorem 2 both before and after the change in the order of the integrals, we can produce two types of approximations to the moments. Note that since $\xi(t, \underline{x})$ in (3.6) is not generally separable $(\xi(t, \underline{x}) \neq V(\underline{x})\xi(t))$ as assumed in Theorems 1 and 2, changes in the approximation formulas (2.1), (2.4) are necessary. These modifications entail no particular problems.

[2] We thank the anonymous referee for suggesting this application.

Assume that the random process $\xi(t, \underline{x})$ satisfies

(A5) $$\sup_{0 \leqq t \leqq T} E|\xi(t, \underline{x})|^2 < \infty \quad \forall \underline{x} \in R^d,$$

and that each sample path of ξ as a function of $\underline{x}$ has continuous derivatives up to order 4 and that

(A6) $$\left(\int_t^{t+1/n} E|\xi_{ijkl}(s, \underline{z}(s))|^2 \, ds \right)^{1/2} = O(n^{-1})$$

for each continuous $\underline{z}: [0, T] \to R^d$. (This replaces (A2) in Theorems 1 and 2.) Then the hypotheses of Chow's existence and uniqueness criterion for (3.6) are satisfied [9, eq. (2.10), (2.11)], and (3.7) is the solution. Using our Theorem 2 with $G(x) = \exp(x)$, and $d = 1$ (for simplicity), we have

THEOREM 5. *Under the hypotheses* (A5), (A6), $d = 1$, *the solution* $u(t, x)$ *of* (3.6), *with* α *real and nonnegative, satisfies*

(3.11) $$\begin{aligned} u(t, x) = (2\pi)^{-n/2} \int_{R^n} \Big\{ &\exp\Big[\sum_{i=1}^n \Xi_{i-1}\Big(x + x_{i-1} + \frac{vt}{(2n)^{1/2}} \Big) \Big] \\ &\cdot f\Big(x + x_{n-1} + \frac{vt}{(2n)^{1/2}} \Big) \Big\} \\ &\cdot \exp[(-u_1^2 - \cdots - u_{n-1}^2 - v^2)/2] \, du_1 \cdots du_{n-1} \, dv + O(n^{-2}), \end{aligned}$$

provided that f is twice continuously differentiable and

(A7) $$\left| \int_{-\infty}^{\infty} f''(y) \exp(-y^2) \, dy \right| < \infty.$$

Here

(3.12) $$\Xi_{i-1}(x) = \int_{t_{i-1}}^{t_i} V(s, x) \, ds, \qquad i = 1, 2, \cdots, n, \qquad t_i = \frac{it}{n}.$$

To compute the moments (3.9) of u, we must evaluate expectations of the form

$$E\Big(\exp\Big[\sum_{i-1}^{n} \Xi_{i-1}(\underline{\tilde{x}}_{i-1}) \Big] \Big).$$

If we assume that $\xi(t, \underline{x})$ is a centered Gaussian field (as in [2], [9]) which is stationary and homogeneous, then its covariance R satisfies

(A8) $$R(t, \underline{x}) = R(\pm t, \pm \underline{x}), \qquad |t| < \infty, \quad \underline{x} \in R^d.$$

The moment Γ_1 in (3.10) is

(3.13) $$\Gamma_1(t, \underline{x}) = \mathscr{E}_z^\alpha \Big\{ \exp\Big[\frac{1}{2} \int_0^t \int_0^t R(\tau - \mu, z(\tau) - z(\mu)) \, d\tau \, d\mu \Big] f(z(t) + \underline{x}) \Big\}.$$

Chorin's method may be applied directly to (3.13); however, the double integral in (3.13) apparently leads to an approximation error larger than $O(n^{-2})$. By computing the expectation directly from (3.11), we obtain a more accurate expansion (since $O(n^{-2})$ in (3.11) already includes the expectation). Let

$$X_n = [x_0, x_1, \cdots, x_{n-1}]^T \in R^n,$$

and $R_n(y, X)$ be the $n \times n$ matrix with elements r_n^{ij}, $i, j = 1, 2, \cdots, n$ given by (again, $d = 1$ is assumed)

$$r_n^{ij}(y, X) = E[\Xi_{i-1}(y + x_{i-1})\Xi_{j-1}(y + x_{j-1})],$$

(3.14)
$$y = x + \frac{vt}{(2n)^{1/2}}.$$

Finally, let $\varepsilon = [1, 1, \cdots, 1]^T \in R^n$. With this notation we have

(3.15)
$$\Gamma_1(t, x) = (2\pi)^{-n/2} \int_{R^n} \left\{ \exp\left[\frac{1}{2}\varepsilon^T R_n\left(x + \frac{vt}{(2n)^{1/2}}, [x_0, \cdots, x_{n-1}]\right)\varepsilon\right] \cdot f\left(x + x_{n-1} + \frac{vt}{(2n)^{1/2}}\right)\right\} \cdot \exp\left[\frac{-u_1^2 - \cdots - u_{n-1}^2 - v^2}{2}\right] du_1 \cdots du_{n-1}\, dv + O(n^{-2}).$$

A similar but more complex formula could be given for Γ_2 (or any higher order moment). Indeed, one could obtain an approximation to the characteristic functional[3]

(3.16)
$$\Psi(\rho) = E\{\exp[i\langle \rho, u\rangle]\}$$

of the random variable $u(t, x, \cdot)$ in terms of a functional Taylor (Volterra) series (which is absolutely convergent in the Gaussian case, among others) by expanding the exponential function in (3.16) in a power series

(3.17)
$$\Psi(\rho) = E\left\{\sum_{m=0}^{\infty} \frac{(i)^m}{m!}\langle \rho, u\rangle^m\right\}$$

and evaluating $E\langle \rho, u\rangle^m$. For example, if the solution space of (3.6) consists of bounded, continuous functions on $[0, T] \times R^d = D_T$ which vanish as $|x| \to \infty$ (assume $f(x) \to 0$ as $|x| \to \infty$), then

(3.18)
$$\langle \rho, u\rangle = \int_0^T \int_{R^d} u(t, x)\, d\rho\,(t, x),$$

where $\rho(t, x)$ is a function of bounded variation on $\bar{D}_T$ (see [9, p. 395]). It follows that

(3.19)
$$E\langle \rho, u\rangle^m = \left(\int_0^T \int_{R^d}\right) \overset{(m)}{\cdots} \left(\int_0^T \int_{R^d}\right) \Lambda_m(t_1, \cdots, t_m; \theta_1, \cdots, \theta_m)\, d\rho\,(t_1, \theta_1) \cdots d\rho\,(t_m, \theta_m),$$

where Λ_m is the mth order moment of $u(t, x)$. Using (3.19) in (3.17) results in the functional series for Ψ. It is evident that this series together with the approximation (3.11) could be used to produce simple estimates accurate to $O(n^{-2})$ for $\Psi(\rho)$. Since the pair (u, Ψ) determines the process u completely, one acquires in this way a complete approximation theory for the random solutions of (3.6) in terms of n-fold ordinary integrals. This possibility was anticipated by Chow [9, p. 381, Remark 3].

We wish to emphasize that the approximation is rigorously established here only for α real and nonnegative in (3.6). For α complex the formal representation of the

[3] Here $\langle \rho, u\rangle$ is the value of the linear functional ρ on the solution space of (3.6) at u.

solution of (3.6) by the path integral (3.7) is commonly accepted, although to establish it rigorously certain steps still remain open (see [2], [9]). Thus the formal representation of the scattered wave by (3.7) is at the same level as formal calculations widely used in random wave propagation studies [2]. The authors share with others the understanding that the representation (3.7) (3.8) can be rigorously established for a rather wide class of random PDE's like (3.6). With the approximation formula of Theorem 5 a second difficulty arises, however: namely to rigorously establish a finite dimensional approximation to the sequential integral (3.8). This will require quite different techniques from those presented here.

4. Nonlinear filtering of diffusion processes. We consider here two examples which we believe to be generic to a class of nonlinear filtering problems. First, consider the pair of scalar Ito equations

$$\begin{gathered} dx(t)=\tfrac{1}{2}x(t)\,dt+x(t)\,dw(t),\\ dy(t)=x(t)\,dt+dv(t),\\ x(0)=x_0,\qquad y(0)=0,\qquad 0\leqq t\leqq T, \end{gathered}\tag{4.1}$$

where w and v are independent, standard Brownian motions and x_0 is a positive-valued random variable with a smooth density p_0 independent of w and v. We call x the signal and y the observation. The filtering problem is to determine an estimate of $x(t)$ given observations of $y(s)$, $s\leqq t$; specifically, the σ-algebra Y_t generated by $\{y(s), 0\leqq s\leqq t\}$. For the estimate one usually takes the conditional mean $\hat{x}(t)=E[x(t)\,|\,Y_t]$, since this produces the minimum mean square error.

To compute $\hat{x}(t)$ it suffices to know the conditional density $p(t,x\,|\,Y_t)$ of $x(t)$ given Y_t, if it exists. This satisfies a complex nonlinear stochastic partial differential equation which is difficult to analyze or to treat numerically [11], [17]. Alternatively, one can write

$$p(t,x\,|\,Y_t)=\frac{\phi(t,x)}{\int_{-\infty}^{\infty}\phi(t,x')\,dx'}\tag{4.2}$$

where the unnormalized conditional density ϕ satisfies

$$\begin{gathered} d\phi(t,x)=\left[\frac{1}{2}\frac{\partial^2}{\partial x^2}(x^2\phi)-\frac{1}{2}\frac{\partial}{\partial x}(x\phi)\right]dt+x\phi\,dy(t),\\ \phi(0,x)=p_0(x),\qquad 0\leqq t\leqq T, \end{gathered}\tag{4.3}$$

a linear stochastic PDE discovered by Zakai [22] (for general Ito diffusion processes).[4]

Let L be the second order operator in (4.3). Using the density of the Markov diffusion generated by L (that is, $x(t)$ in (4.1)), one can write down a stochastic function space integral (like those of type 2 in § 2) for the solution of (4.3). However, since L is not the Laplacian, i.e., $x(t)$ is not a Wiener process, this function space integral will not be a "pure" Wiener integral (of type 2) since the underlying measure is not a Wiener measure on $C[0, T]$. For this reason it is difficult to verify that the formal expression for the function space integral makes sense[5] or to evaluate it numerically.

---

[4] Actually, R. Mortensen and, independently, T. Duncan also derived evolution equations for the unnormalized conditional density, see [22] for remarks on this point.

[5] Especially since the coefficients in (4.3) are unbounded. Otherwise, the general theory of Pardoux [12] would guarantee existence and uniqueness.

A simple coordinate change in (4.1) remedies both problems. Let

$$z(t) = \ln x(t). \tag{4.4}$$

Then, using the Ito calculus, we have

$$\begin{aligned} dz(t) &= dw(t), \\ dy(t) &= e^{z(t)}\,dt + dv(t), \\ z(0) = z_0 = \ln x_0, \qquad y(0) &= 0, \qquad 0 \leqq t \leqq T. \end{aligned} \tag{4.5}$$

In the new coordinates the signal $z(t)$ is a Wiener process which is observed through a nonlinear processor. The Zakai equation for the unnormalized density $u(t, z)$ of $z(t)$ given Y_t satisfies

$$\begin{aligned} du(t, z) &= \frac{1}{2}\frac{\partial^2}{\partial z^2}u(t, z)\,dz + e^z u(t, z)\,dy(t), \\ u(0, z) &= q_0(z) = \text{density of } z_0, \qquad 0 \leqq t \leqq T. \end{aligned} \tag{4.6}$$

In effect, the coordinate change transfers the complexity from the second order operator L in (4.3) to the "potential" term $e^z\dot{y}(t)$ in (4.6).

Since the Laplacian appears in (4.6), we may write its solution in terms of a "pure" Wiener stochastic function space integral of type 2; namely,

$$u(t, z) = \mathscr{E}\left\{\left[\exp\left(\int_0^t e^{z(s)+z}\,dy(s) - \frac{1}{2}\int_0^t e^{2z(s)+2z}\,ds\right)\right]q_0(z(t)+z)\right\}. \tag{4.7}$$

It is not obvious that (4.7) is well defined; however, using some results of Meyer [23] on stochastic integrals with respect to semi-martingales or the results in [24, pp. 211–226], one can show that (4.7) is finite almost surely (samples of y). Using Ito's formula on $u(t, z)$, one can show that it satisfies (4.6). Also, $u(t, z)$ is the minimum positive solution of (4.6); a stronger uniqueness property is not yet available (see [14] for details).

Applying Theorem 4 to (4.7), we have

$$\begin{aligned} u(t, z) = (2\pi)^{-n/2}\int_{R^n}\Bigg\{&\exp\Bigg[\sum_{i=1}^n \exp\left(z + x_{i-1} + \frac{vt}{(2n)^{1/2}}\right)\Delta y_{i-1} \\ &\qquad - \frac{1}{2}\sum_{i=1}^n \exp\left(2\left(z + x_{i-1} + \frac{vt}{(2n)^{1/2}}\right)\right)\frac{t}{n}\Bigg] \\ &\qquad \cdot q_0\left(z + x_{n-1} + \frac{vt}{(2n)^{1/2}}\right)\Bigg\} \\ &\cdot \exp\left[\frac{-u_1^2 - \cdots - u_{n-1}^2 - v^2}{2}\right]du_1 \cdots du_{n-1}\,dv + O(n^{-2}), \end{aligned} \tag{4.8}$$

where $\Delta y_{i-1} = y(it/n) - y((i-1)t/n)$.[6]

The approximation (4.8) is not recursive, and is thus less attractive than the evolution equation for the conditional distribution or mean as a state estimator; however, its high accuracy suggests the possibility of rapid implementation based on a few terms.

[6] Numerical integration formulas for Ito equations are reported in [24], [25].

Another class of filtering problems which appears to be generic and to admit treatment by our approximation formulas is

$$
\begin{aligned}
&dx(t)=f[x(t)]\,dt+dw(t),\\
(4.9)\qquad &dy(t)=x(t)\,dt+dv(t),\\
&x(0)=x_0,\qquad y(0)=0,\qquad 0\leqq t\leqq T.
\end{aligned}
$$

Here w, v, x_0 are as above and $f:P\to R$ is smooth. The nonlinear drift makes $x(t)$ non-Gaussian and complicates the problem of estimating $x(t)$ given $\sigma\{y(s), s\leqq t\}\equiv Y_t$. If, however, f satisfies $f_x+f^2=ax^2+bx+c$ for some constants, then Beneš [27] has shown that the optimal least squares estimator for a transformed version of the system is the linear, finite dimensional Kalman–Bucy filter. Several others, including Mitter and Ocone [28] and Brockett [29], have studied this system which has some interesting algebraic structure.

The Zakai equation for the unnormalized conditional density of $x(t)$, given Y_t, is

$$
\begin{aligned}
&du(t,x)=\left[\frac{1}{2}\frac{\partial^2}{\partial x^2}-f(x)\frac{\partial}{\partial x}-f'(x)\right]u(t,x)\,dt+xu(t,x)\,dy(t),\\
(4.10)\qquad &u(0,x)=p_0(x)=\text{density of } x_0.
\end{aligned}
$$

To this we apply the "gauge transformation" as suggested by Mitter [28],

$$
(4.11)\qquad v(t,x)=\exp\left(\int_0^x -f(r)\,dr\right)u(t,x).
$$

This yields

$$
\begin{aligned}
&dv(t,x)=\frac{1}{2}\frac{\partial^2}{\partial x^2}v(t,x)\,dt-V(x)v(t,x)\,dt+xv(t,x)\,dy(t),\\
(4.12)\qquad &V(x)=\tfrac{1}{2}[f'(x)+f^2(x)].
\end{aligned}
$$

By adapting certain comparison theorems one can show that this equation admits a well-defined nonnegative solution for some f's [28].

To apply our technique, we regard the last two terms on the right in (4.12) as a potential. The isolation of the Laplacian via (4.11) permits us to write the solution to (4.12) as a Wiener stochastic function space integral of type 2, namely,

$$
\begin{aligned}
&v(t,x)=\mathscr{E}\left\{\exp\left[\int_0^t (x(s)+x)\,dy_s-\frac{1}{2}\int_0^t (x(s)+x)^2\,ds\right]\right.\\
(4.13)\qquad &\left.\cdot\exp\left[\int_0^t -V(x(s)+x)\,ds\right]p_0(x(t)+x)\right\}.
\end{aligned}
$$

From this formula it is evident that, if

$$
(4.14)\qquad V(x)=\tfrac{1}{2}[f'(x)+f^2(x)]=ax^2+bx+c
$$

for some constants a, b, c with $(a+1)\geqq 0$, then we can evaluate the integral in closed form. This is the essence of Beneš's result [27], though his argument is very different.

Even if (4.14) does not hold, the gauge transformation (4.11) has the beneficial effect of making the underlying measure in (4.13) Wiener measure. Hence, Theorem 4 applies and yields

$$
\begin{aligned}
v(t,x)=(2\pi)^{-n/2}\int_{R^n}\Big\{\exp\Big[\sum_{i=1}^{n}\Big(x+x_{i-1}+\frac{\nu t}{(2n)^{1/2}}\Big)\Delta y_{i-1}\\
-\frac{1}{2}\sum_{i=1}^{n}\Big(x+x_{i-1}+\frac{\nu t}{(2n)^{1/2}}\Big)^2\frac{t}{n}\\
+\sum_{i=1}^{n}-V\Big(x+x_{i-1}+\frac{\nu t}{(2n)^{1/2}}\Big)\frac{t}{n}\Big]\\
\cdot p_0\Big(x+x_{n-1}+\frac{\nu t}{(2n)^{1/2}}\Big)\Big\}\\
\cdot\exp\Big[\frac{-u_1^2-\cdots-u_{n-1}^2-\nu^2}{2}\Big]du_1\cdots du_{n-1}\,d\nu+O(n^{-2}).
\end{aligned}
\tag{4.15}
$$

Note that, if $V(x)$ is a quadratic form and p_0 is normal, then the n-fold integral can be evaluated explicitly and recursively in t, if desired.

The two filtering problems (4.1) and (4.9) have as a common feature the existence of a transformation ((4.4) or (4.11), respectively) which "isolates" the Laplacian in the associated evolution equation and causes the corresponding function space integral to be a Wiener integral. This is crucial for the application of our approximation formulas in their simplest form. In other, more complex cases,

$$
\begin{aligned}
dx(t)&=f(x(t))\,dt+g(x(t))\,dw(t),\\
dy(t)&=h(x(t))\,dt+dv(t),
\end{aligned}
\tag{4.16}
$$

one may be able to employ a Girsanov–Cameron–Martin transformation [24, Chapt. 7] on the probability distribution of the x-process so that with respect to the new measure x is a Wiener process. Relative to the new measure, y is an observation in noise of a nonlinear functional of a Wiener process. In this setting the Girsanov transformation behaves like a gauge transformation, and creates the possibility of applying approximation formulas like those in § 2 to the transformed system. Any such formulas will be useful only when the transformation itself can be computed.

5. Proofs of Theorems 1 and 2.

5.1. Proof of Theorem 1. We use three lemmas. In each case V and $\{\xi(s), 0\leqq s\leqq t\}$ satisfy the hypotheses (A1), (A2) in Theorem 1.

LEMMA 5.1.

$$
\begin{aligned}
I^{(11)}&=\int_C\Big(\int_0^t V[x(s)]\xi(s)\,ds\Big)\,dW\\
&=\int_C\Big(\sum_{i=1}^{n}\Big[V(x_{i-1})\int_{t_{i-1}}^{t_i}\xi(s)\,ds\\
&\qquad+\frac{1}{2}V''(x_{i-1})\int_{t_{i-1}}^{t_i}\xi(s)(\Delta x_i(s))^2\,ds\Big]\Big)\,dW+O(n^{-2}),
\end{aligned}
\tag{5.1}
$$

where $t_i=it/n$, $\xi_{i-1}=\xi(t_{i-1})$ *and*

$$
\begin{aligned}
\Delta x_i(s)&=\sqrt{n}(u_i(s-t_{i-1})+b_i[(s-t_{i-1})(t_i-s)]^{1/2}),\qquad t_{i-1}\leqq s\leqq t_i,\\
x_{i-1}&=\frac{t(u_1+\cdots+u_{i-1})}{\sqrt{n}},\qquad i=1,2,\cdots,n.
\end{aligned}
\tag{5.2}
$$

The b_i are independent, zero-mean Gaussian random variables with unity variance.

Remark. The interpolation (5.2) for Brownian paths is due to P. Lévy [30].

Proof. For $s \in [t_{i-1}, t_i]$, $x(s) = x_{i-1} + \Delta x_i(s)$ and

$$\begin{aligned}\int_{t_{i-1}}^{t_i} & V[x_{i-1} + \Delta x_i(s)]\xi(s)\,ds \\ &= \int_{t_{i-1}}^{t_i} \left[V(x_{i-1}) + V'(x_{i-1})\Delta x_i(s) + \frac{1}{2} V''(x_{i-1})(\Delta x_i(s))^2 \right] \xi(s)\,ds \\ &\quad + \frac{1}{6} \int_{t_{i-1}}^{t_i} V'''(x_{i-1})(\Delta x_i(s))^3 \xi(s)\,ds \\ &\quad + \frac{1}{24} \int_{t_{i-1}}^{t_i} V''''(x_{i-1} + \theta \Delta x_i(s))(\Delta x_i(s))^4 \xi(s)\,ds,\end{aligned} \tag{5.3}$$

for some θ, $0 \leqq \theta \leqq 1$. Consider the last term in (5.3),

$$e_i = \int_{t_{i-1}}^{t_i} V''''(x_{i-1} + \theta \Delta x_i(s))(\Delta x_i(s))^4 \xi(s)\,ds. \tag{5.4}$$

Evidently (omitting the arguments),

$$\begin{aligned} Ee_i^2 &= E\left(\int_{t_{i-1}}^{t_i} V''''(-)(\Delta x_i)^4 \xi\,ds \right)^2 \\ &\leqq \frac{t}{n} \int_{t_{i-1}}^{t_i} [V''''(-)]^2 (\Delta x_i)^8 E\xi^2(s)\,ds. \end{aligned} \tag{5.5}$$

Now $E\xi^2(s) \leqq c$ for all $s \in [0, T]$, and

$$\sup_{t_{i-1} \leqq s \leqq t_i} |\Delta x_i(s)| \leqq \frac{(|u_i| + |b_i|)t}{\sqrt{n}} \tag{5.6}$$

Hence,

$$E(e_i^2) \leqq c(|u_i| + |b_i|)^8 t^9 \cdot (n^{-5}) \cdot \int_{t_{i-1}}^{t_i} |V''''(-)|^2\,ds \tag{5.7}$$

and so

$$[E(e_i^2)]^{1/2} = O(n^{-3}). \tag{5.8}$$

Consider the next to last term in (5.3),

$$d_i = \int_{t_{i-1}}^{t_i} V'''(x_{i-1})(\Delta x_i(s))^3 \xi(s)\,ds. \tag{5.9}$$

Using a Fubini theorem, we have

$$\int_C d_i\,dW = \int_{t_{i-1}}^{t_i} \left[\int_C V'''(x_{i-1})(\Delta x_i(s))^3\,dW \right] \xi(s)\,ds. \tag{5.10}$$

Now

$$\int_C V'''(x_{i-1})(\Delta x_i(s))^3\,dW$$

$$(5.11)\qquad = (2\pi)^{-n}\int_{R^{2n}} V'''(x_{i-1})(u_i\sqrt{n}(s-t_{i-1})+b_i\sqrt{n}[(s-t_{i-1})(t_i-s)]^{1/2})^3 \cdot \exp\left[\frac{-u_1^2-u_2^2-\cdots-u_n^2-b_1^2-\cdots-b_n^2}{2}\right] du_1\cdots du_n\,db_1\cdots db_n$$

$$=0.$$

Hence, for any $k=k, 3, 5, \cdots$ odd,

$$(5.12)\qquad \sum_{i-1}^{n}\int_C V^{(k)}(x_{i-1})(\Delta x_i(s))^k\,dW=0,\qquad k=1,3,\cdots,$$

and, in particular,

$$(5.13)\qquad \left[E\left|\sum_{i=1}^{n}\int_C d_i\,dW\right|^2\right]^{1/2}=0.$$

So

$$(5.14)\qquad \left[E\left|\int_C\sum_{i=1}^{n}(e_i+d_i)\,dW\right|^2\right]^{1/2}\leqq\int\sum_{i=1}^{n}(Ee_i^2)^{1/2}\,dW=O(n^{-2}).$$

The centering property (5.12) implies that the Wiener integral of first order (in $\Delta x_i(s)$) in (5.3) is zero. This and the estimate (5.14) give (5.1). QED

LEMMA 5.2. *Let* $\Xi_{i-1}=\int_{t_{i-1}}^{t_i}\xi(s)\,ds$. *Then*

$$(5.15)\qquad I^{(11)}=\sum_{i=1}^{n}\Xi_{i-1}\left\{\int_C\left[V(x_{i-1})+\frac{1}{2}V''(x_{i-1})\frac{1}{4}\frac{t^2}{n}(u_i^2+b_i^2)\right]dW\right\}+O(n^{-2}).$$

(Here $\int_C u_i^2\,dW$ is interpreted as a $2n$-fold integral as in (5.11).)

Proof. The first term in the sum in (5.15) is just a change of notation in (5.1). To derive the second, consider

$$(5.16)\qquad (\Delta x_i(s))^2=u_i^2n(s-t_{i-1})^2+b_i^2n[(s-t_{i-1})(t_i-s)] +2u_ib_in(s-t_{i-1})[(s-t_{i-1})(t_i-s)]^{1/2}.$$

That is,

$$(5.17)\qquad \frac{1}{2}\int_{t_{i-1}}^{t_i}V''(x_{i-1})(\Delta x_i(s))^2\xi(s)\,ds =\tfrac{1}{2}V''(x_{i-1})n[u_i^2\hat{\Xi}_{i-1}+b_i^2\tilde{\Xi}_{i-1}+2u_ib_i\hat{\tilde{\Xi}}_{i-1}],$$

where

$$(5.18)\qquad \begin{aligned}\hat{\Xi}_{i-1}&=\int_{t_{i-1}}^{t_i}(s-t_{i-1})^2\xi(s)\,ds=\left(\frac{1}{2}\frac{t}{n}\right)^2\Xi_{i-1}+\hat{e}_{i-1},\\ \tilde{\Xi}_{i-1}&=\int_{t_{i-1}}^{t_i}[(s-t_{i-1})(t_i-s)]\xi(s)\,ds=\left(\frac{1}{2}\frac{t}{n}\right)^2\Xi_{i-1}+\tilde{e}_{i-1},\\ \tilde{\hat{\Xi}}_{i-1}&=\int_{t_{i-1}}^{t_i}(s-t_{i-1})[(s-t_{i-1})(t_{i-s})]^{1/2}\xi(s)\,ds=\left(\frac{1}{2}\frac{t}{n}\right)^2\Xi_{i-1}+\tilde{\hat{e}}_{i-1},\end{aligned}$$

and we claim

$$(E|\hat{e}_{i-1}|^2)^{1/2}+(E|\tilde{e}_{i-1}|^2)^{1/2}+(E|\tilde{\tilde{e}}_{i-1}|^2)^{i/2}=O(n^{-3}). \tag{5.19}$$

To see this, we write out $\hat{e}_{i-1}$,

$$\hat{e}_{i-1}=\int_{t_{i-1}}^{t_i}[(s-t_{i-1})^2-\frac{1}{4}(t_i-t_{i-1})^2]\xi(s)\,ds \tag{5.20}$$

(that is, (5.18) is a kind of midpoint rule). Now

$$|(s-t_{i-1})^2-\frac{1}{4}(t_i-t_{i-1})^2|\leqq\left(\frac{\sqrt{3}}{2}\frac{t}{n}\right)^2, \qquad t_{i-1}\leqq s\leqq t_i=\frac{it}{n}. \tag{5.21}$$

Hence,

$$E|\hat{e}_{i-1}|^2\leqq\left(\frac{\sqrt{3}}{2}\frac{t}{n}\right)^4\frac{t}{n}\int_{t_{i-1}}^{t_i}E\xi^2(s)\,ds=O(n^{-6}). \tag{5.22}$$

The other estimates in (5.19) are similar.

The Wiener integral of the cross term in (5.17) is zero. The estimate in (5.15) follows from the sum of the estimates in (5.19), a total of n terms. QED

It remains to reduce the $2n$-fold integral in (5.15).

LEMMA 5.3.

$$I^{(11)}=\sum_{i=1}^n \Xi_{i-1}\left[(2\pi)^{-n/2}\int_{R^n}\left(V(x_{i-1})+\frac{1}{2}V''(x_{i-1})\frac{1}{2}\frac{t^2}{n}v^2\right)\cdot\exp[(-u_1^2-\cdots-u_{n-1}^2-v^2)/2]\,du_1\cdots du_{n-1}\,dv\right]+O(n^{-2}) \tag{5.23}$$

Proof. Referring to (5.15), consider the term

$$\int_C V''(x_{i-1})(u_i^2+b_i^2)\,dW =(2\pi)^{-n}\int_{R^{2n}}V''(x_{i-1})(u_i^2+b_i^2)\cdot\exp\left[\frac{-u_1^2-\cdots-u_n^2-b_1^2-\cdots-b_n^2}{2}\right]du_1\cdots db_n. \tag{5.24}$$

Now

$$\begin{aligned}\pi^{-1}\int_{R^2}&V''(x_{i-1})(u_i^2+b_i^2)\exp(-u_i^2-b_i^2)\,du_i\,db_i\\ &=\pi^{-1}\int_R\left[\int_R V''(x_{i-1})(u_i^2+b_i^2)\,e^{-b_i^2}\,db_i\right]e^{-u_i^2}\,du_i\\ &=\left(\frac{2}{\pi}\right)\int_R\left[\int_R V''(x_{i-1})v^2e^{-v^2}\,dv\right]e^{-u^2}\,du\\ &=\left(\frac{2}{\sqrt{\pi}}\right)\int_R V''(x_{i-1})v^2e^{-v^2}\,dv.\end{aligned} \tag{5.25}$$

Hence, the $2n$-fold integral in (5.15) may be reduced to an n-fold integral. QED

To complete the proof of the theorem, note that the integrand in (5.23) is an expansion of $V(x_{i-1}+vt/\sqrt{2n})$ around x_{i-1}. The centering property (5.12) cancels odd powers of v, and so, the expansion in (5.23) is accurate to $O(n^{-2})$. This completes the proof of Theorem 1.

5.2. Proof of Theorem 2. Expanding $G(\cdot)$ in a Taylor series, we have

$$\begin{aligned} G\left[\int_0^t V(x(s))\xi(s)\,ds\right] &= G\left[\sum_{i=1}^n \int_{t_{i-1}}^{t_i} V(x_{i-1}+\Delta x_i(s))\xi(s)\,ds\right] \\ &= G(q)+G'(q)\sum_{j=1}^n \Delta q_j+\frac{1}{2}G''(q)\sum_{j,k=1}^n \Delta q_j\Delta q_k \\ &\quad+\frac{1}{6}G'''(q)\sum_{j,k,l=1}^n \Delta q_j\Delta q_k\Delta q_l \\ &\quad+\frac{1}{24}\sum_{j,k,l,m=1}^n G_{jklm}^{(iv)}\Delta q_j\Delta q_k\Delta q_l\Delta q_m, \end{aligned} \tag{5.26}$$

where

$$q=\sum_{i=1}^n q_i,\qquad q_i=V(x_{i-1})\Xi_{i-1},$$

$$\Delta q_j=\int_{t_{j-1}}^{t_j}[V(x_{j-1}+\Delta x_j(s))-V_{j-1})]\xi(s)\,ds, \tag{5.27}$$

$$G_{jklm}^{(iv)}=G''''(q+\theta_j\Delta q_j+\theta_k\Delta q_k+\theta_l\Delta q_l+\theta_m\Delta q_m),\qquad 0\leqq\theta\leqq 1.$$

Our first problem is to estimate the expansion errors

$$\begin{aligned} e_1&=\frac{1}{6}G'''(q)\sum_{j,k,l=1}^n \Delta q_j\Delta q_k\Delta q_l, \\ e_2&=\frac{1}{24}\sum_{k,j,l,m=1}^n G_{jklm}^{(iv)}\Delta q_j\Delta q_k\Delta q_l\Delta q_m. \end{aligned} \tag{5.28}$$

LEMMA 5.4. *Let $E(\cdot)$ be expectation with respect to the distributions of $\{\xi(s), 0\leqq s\leqq t\}$. Then*

$$\left(E\left|\int_C e_1\,dW\right|^2\right)^{1/2}=O(n^{-9/2}). \tag{5.29}$$

Remark. This should be compared with Chorin's estimate [1, p. 8], which is (n^{-2}).

Proof. We introduce the notation (following Chorin)

$$V_i^{(j)}=\begin{cases} V_i=V(x_i), & i<j, \\ V\left(x_i-\dfrac{u_jt}{\sqrt{n}}\right), & i\geqq j, \end{cases} \tag{5.30}$$

which eliminates the jth element in the sum $x_i=t(u_1+\cdots+u_i)/\sqrt{n}$. So $V_i^{(j)}$ does not depend on u_j. Also,

$$V_i^{(jk)} = \begin{cases} V_i, & i<k, \\ V\left(x_i - \frac{u_j t}{\sqrt{n}}\right), & i \geqq j, \quad i<k, \\ V\left(x_i - \frac{u_k t}{\sqrt{n}}\right), & i<j, \quad i \geqq k, \\ V\left(x_i - \frac{(u_j+u_k)t}{\sqrt{n}}\right), & i \geqq k, \end{cases} \tag{5.31}$$

and $V_i^{(jkl)}$ is defined similarly. That is, we write in the superscript the indices of those variables $u_1, \cdots, u_n$ which are set equal to zero in the argument of V. Also,

$$q^{(j)} = \sum_{i=1}^n V_{i-1}^{(j)} \Xi_{i-1}, \qquad q^{(jk)} = \sum_{i=1}^n V_{i-1}^{(jk)} \Xi_{i-1}, \quad \text{etc.} \tag{5.32}$$

Using this notation, we have

$$G'''(q) = G'''(q^{(jkl)}) + G''''\left(\sum_{i=1}^n V(a_{i-1}^{(j)})\Xi_{i-1}\right) \cdot \left(\sum_{i=j+1}^n V'(a_{i-1}^{(j)})\Xi_{i-1}\right)(u_j t\sqrt{n}) + \text{two similar terms in } u_k, u_l. \tag{5.33}$$

Here

$$a_{i-1}^{(j)} = \begin{cases} x_{u-1}, & i \leqq j, \\ \frac{t(u_1 + \cdots + u_{j-1} + \theta_j u_j + u_{j+1} + \cdots + u_{i-1})}{\sqrt{n}}, & i \geqq j+1, \quad 0 \leqq \theta \leqq 1. \end{cases} \tag{5.34}$$

Alternatively,

$$\begin{aligned} G'''(q) = {} & G'''(q^{(jkl)}) + [G''''(q^{(jkl)}) q_{u_j}^{(jkl)} u_j] \\ & + [\text{two similiar terms in } u_k, u_l] \\ & + \left\{\left[G^{(v)}\left(\sum_{i=1}^n V(a_{i-1}^{[jkl]})\Xi_{i-1}\right)(q_{u_j}^{(jkl)}) q_{u_k}^{(jkl)}\right) \right. \\ & \left. + G^{(iv)}\left(\sum_{i=1}^n V(a_{i-1}^{[jkl]})\Xi_{i-1}\right)(q_{u_j u_k}^{(jkl)})\right] u_j u_k\right\} \\ & + \{\text{two similar terms in } (u_j u_l), (u_k u_l)\}, \end{aligned} \tag{5.35}$$

where

$$a_{i-1}^{[jkl]} = \begin{cases} x_{i-1}, & i \leqq j, k, l \\ \frac{t(u_1 + \cdots + u_{j-1} + \theta_j u_j + u_{j+1} + \cdots + u_{i-1})}{\sqrt{n}}, & j < i \leqq k, l \\ \frac{t(u_1 + \cdots + u_{j-1} + \theta_j u_j + \theta_k u_k + \cdots + u_{i-1})}{\sqrt{n}}, & k \neq j < i, \quad k<i, \quad i \leqq l, \end{cases} \tag{5.36}$$

etc., and

$$q_{u_j}^{(jkl)} = \frac{\partial q}{\partial u_j}\bigg|_{u_j,u_k,u_l=0} = \left(\sum_{i=j+1}^{n} V'(x_{i-1}^{(jkl)})\Xi_{i-1}\right)\frac{t}{\sqrt{n}}, \tag{5.37}$$

$$q_{u_ju_k}^{(jkl)} = \frac{\partial^2 q}{\partial u_j \partial u_k}\bigg|_{u_j,u_k,u_l=0} = \left(\sum_{i=\max(j,k,l)+1}^{n} V''(x_{i-1}^{(jkl)})\Xi_{i-1}\right)\frac{t^2}{n}, \tag{5.38}$$

etc. Here $x_{i-1}^{(jkl)} = a_{i-1}^{[jkl]}$ with $\theta_j = \theta_k = \theta_l = 0$ as appropriate. (Note that $\partial x_{i-1}/\partial u_j = 0$ if $i \leqq j$ and $t/\sqrt{n}$ if $i \geqq j+1$.)

Now from Lemma 5.1, specifically the estimate (5.4)–(5.8), we have

$$\begin{aligned} \Delta q_i &= \int_{t_{i-1}}^{t_i} [V(x_{i-1} + \Delta x_i(s)) - V(x_{i-1})]\xi(s)\, ds \\ &= \int_{t_{i-1}}^{t_i} [V'(x_{i-1})\Delta x_i(s) + \tfrac{1}{2}V''(x_{i-1})(\Delta x_i(s))^2]\xi(s)\, ds + \alpha_i, \end{aligned} \tag{5.39}$$

where

$$\left[E\left|\int_C \alpha_i\, dW\right|^2\right]^{1/2} \leqq O(n^{-3}) \tag{5.40}$$

To define $\Delta q_i^{(jkl)}$, we use the notation $x_{i-1}^{(j)}$, which means that u_j has been set equal to zero in the expression for x_{i-1}, and the expression

$$V'(x_{i-1}) = \begin{cases} V'(x_{i-1}^{(j)}), & i \leqq j, \\ V'(x_{i-1}^{(j)} + V''(x_{i-1}^{(j)})u_j t/\sqrt{n} + O(n^{-1}), & i > j, \end{cases} \tag{5.41}$$

and similarly for V''. Hence, for $i > j$,

$$\begin{aligned} \Delta q_i &= \int_{t_{i-1}}^{t_i} [V'(x_{i-1}^{(j)}\Delta x_i(s) + \tfrac{1}{2}V''(x_{i-1}^{(j)})(\Delta x_i(s))^2]\xi(s)\, ds \\ &\quad + \left(\frac{u_j t}{\sqrt{n}}\right)\int_{t_{i-1}}^{t_i} [V''(x_{i-1}^{(j)})\Delta x_i(s) + \tfrac{1}{2}V'''(x_{i-1}^{(j)})(\Delta x_i(s))^2]\xi(s)\, ds + \tilde{\alpha}_i, \qquad i > j, \end{aligned} \tag{5.42}$$

where $(E|\int_C \tilde{\alpha}_i\, dW|^2)^{1/2} \leqq O(n^{-3})$. The second term in (5.42) is linear in u_j and is (pointwise in u_j, u_k, b_k, $k \leqq i$, $k \neq j$) of order n^{-2}. Lumping this term with $\tilde{\alpha}_i$ to give $\hat{\alpha}_i$, we have

$$\Delta q_i = \Delta q_i^{(j)} + \hat{\alpha}_i, \qquad \left(E\left|\int_C \hat{\alpha}_i\, dW\right|^2\right)^{1/2} \leqq O(n^{-2}). \tag{5.43}$$

In a similar way we can define $\Delta q_i^{(jk)}$, $\Delta q_i^{(jkl)}$, etc., with the obvious change in (5.42), and

$$\Delta q_i = \Delta q_i^{(jk)} + \bar{\alpha}_i, \qquad \bar{\alpha}_i = O(n^{-2}). \tag{5.44}$$

Using this notation, we consider the integral

$$
\begin{aligned}
I_{jkl} &\triangleq \int_C G'''(q)\Delta q_j \Delta q_k \Delta q_l \, dW \\
&= \int_C G'''(q^{(jkl)})\Delta q_j \Delta q_k \Delta q_l \, dW \\
&\quad + \int_C \left(\left[G''''(q^{(jkl)}) \cdot \sum_{i=j+1}^{n} V'(x_{i-1}^{(jkl)}) \Xi_{i-1} \frac{u_j t}{\sqrt{n}} \right] \Delta q_j \Delta q_k \Delta q_l \right) dW \\
&\quad + \text{two similar terms in } u_k,\ u_l \\
&\quad + \int_C \left\{ \left[(G^{v}(q^{(jkl)}) \sum_{i=j+1}^{n} \sum_{p=k+1}^{n} V'(x_{i-1}^{(jkl)}) V'(x_{p-1}^{(jkl)}) \Xi_{i-1} \Xi_{p-1} \right.\right. \\
&\qquad \left.\left. + G''''(q^{(jkl)}) \sum_{i=\max(k,j)+1}^{n} V''(x_{i-1}^{(jkl)}) \Xi_{i-1}) \frac{u_j u_k y^2}{n} \right] \Delta q_j \Delta q_k \Delta q_l \right\} dW \\
&\quad + \text{two similar terms in } (u_j u_l),\ (u_k u_l) \\
&\quad + O(n^{-17/2}).
\end{aligned}
\tag{5.45}
$$

The first term on the right of (5.45) is zero by the centering property (5.12). The second term may be written as

$$
\begin{aligned}
T_2 = \int_C \Bigg\{ & \left[G''''(q^{(jkl)}) \cdot \sum_{i=j+1}^{n} V'(x_{i-1}^{(jkl)}) \Xi_{i-1} \frac{u_j t}{\sqrt{n}} \right] \\
& \cdot (\Delta q_j^{(kl)} + \bar{\alpha}_j)(\Delta q_k^{(jl)} + \bar{\alpha}_k)(\Delta q_l^{(jk)} + \bar{\alpha}_l) \Bigg\} dW,
\end{aligned}
\tag{5.46}
$$

where $(E|\int_C \alpha \, dW|^2)^{1/2} \leqq O(n^{-2})$ in each case. Then, using the same parenthetical structure as in (5.46), we have

$$
\begin{aligned}
T_2 &= (2\pi)^{-n} \int_{R^{2n}} [G''''—](\Delta q^{(kl)}—)(—)(—) \\
&\qquad \cdot \exp\left(\frac{-u_1^2 - \cdots - u_n^2 - b_1^2 - \cdots - b_n^2}{2} \right) du_1 \cdots db_n \\
&= (2\pi)^{-(n-3)} \int_{R^{n-3}} db_1 \overset{(jkl)}{\cdots} db_n \exp\left(\frac{-b_1^2 - \overset{(jkl)}{\cdots} - b_n^2}{2} \right) \\
&\quad \cdot \Bigg\{ (2\pi)^{-(n-3)} \int_{R^{n-3}} du_1 \overset{(jkl)}{\cdots} du_n \exp\left(\frac{-u_1^2 - \overset{(jkl)}{\cdots} - u_n^2}{2} \right) \\
&\quad \cdot \Bigg\{ \left[G''''(q^{(jkl)}) \left(\sum_{i=j+1}^{n} V'(x_{i-1}^{(jkl)}) \Xi_{i-1} \right) \right] \\
&\quad \cdot \left[\frac{1}{\sqrt{n}} (2\pi)^{-1} \int_{R^2} \exp\left(\frac{-u_j^2 - b_j^2}{2} \right) u_j \Delta q_j^{(kl)} \, du_j \, db_j \right] \\
&\quad \cdot \left[(2\pi)^{-1} \int_{R^2} \exp\left(\frac{-u_k^2 - b_k^2}{2} \right) \Delta q_k^{(jl)} \, du_k \, db_k \right] \\
&\qquad \cdot \left[(2\pi)^{-1} \int_{R^3} \exp\left(\frac{-u_l^2 - b_l^2}{2} \right) \Delta q_l^{(kj)} \, du_l \, db_l \right] \Bigg\} \Bigg\} + A(\bar{\alpha}_j, \bar{\alpha}_k, \bar{\alpha}_l).
\end{aligned}
\tag{5.47}
$$

Here the notation $du_1 \overset{(jkl)}{\cdots} du_n$ means that the u_j, u_k, u_l terms are omitted from the expression. Also, $A(\bar{\alpha}_j, \bar{\alpha}_k, \bar{\alpha}_l)$ contains products of the $\bar{\alpha}$'s. Evidently, $A(\bar{\alpha}_j, \bar{\alpha}_k, \bar{\alpha}_l)$ is smaller than $O(n^{-5/2})$. Using (5.38), (5.39), we have

$$m_k = (2\pi)^{-1} \int_{R^2} \exp\left(\frac{-u_k^2 - b_k^2}{2}\right) \Delta q_k^{(jl)}\, du_k\, db_k$$

$$= (2\pi)^{-1} \int_{R^2} \exp\left(\frac{-u_k^2 - b_k^2}{2}\right) \left[\int_{t_{k-1}}^{t_k} (V'(x_{k-1}^{(jl)}) \Delta x_k(s) + \frac{1}{2} V''(x_{k-1}^{(jl)}) \cdot (\Delta x_k(s)^2) \xi(s)\, ds + \bar{\alpha}_k \right] du_k\, db_k$$

$$= 0 + \int_{t_{k-1}}^{t_k} \frac{1}{2} V''(x_{k-1}^{(jl)}) (\overline{\Delta x_k^2(s)}) \xi(s)\, ds + O(n^{-3})),$$

where

$$\overline{\Delta x^2(s)} = n[\bar{u}^2 (s - t_{k-1})^2 + \bar{b}^2 (s - t_{k-1})(t_k - s)] \tag{5.49}$$
$$= \frac{t}{n}(s - t_{k-1}).$$

Now

$$\sup \overline{\Delta x^2(s)} = \frac{t^2}{n}, \tag{5.50}$$

and it follows that

$$(Em_k^2)^{1/2} \leqq O(n^{-2}) + O(n^{-3}) = O(n^{-2}). \tag{5.51}$$

The $O(n^{-3})$ term is the integral of $\bar{\alpha}$, and $O(n^{-2})$ follows from (5.50). By a similar argument

$$\left(E \left| \frac{t}{\sqrt{n}} (2\pi)^{-1} \int_{R^2} \exp\left(\frac{-u_j^2 - b_j^2}{2}\right) u_j \Delta q_j^{(kl)}\, du_j\, db_j \right|^2 \right)^{1/2} = O(n^{-5/2}). \tag{5.52}$$

Combining these estimates in (5.47), we have

$$(ET_2^2)^{1/2} \leqq K \cdot O(n^{-1}) O(n^{-5/2}) O(n^{-2}) O(n^{-2}) + (EA^2)^{1/2}. \tag{5.53}$$

Using similar arguments and the estimate (5.44) for α, we can show that

$$(EA^2(\alpha_j, \alpha_k, \alpha_l))^{1/2} = O(n^{-15/2}), \tag{5.54}$$

and so that

$$(ET_2^2)^{1/2} = O(n^{-15/2}). \tag{5.55}$$

A similar analysis of the third integral (T_3) in (5.45) yields the bound

$$(ET_3^2)^{1/2} = O(n^{-8}) \tag{5.56}$$

and the error estimate in (5.45). Combining these results, we see that

$$(E|I_{jkl}|2)^{1/2} = O(n^{-15/2}), \tag{5.57}$$

and so, that

$$(E|e_1|^2)^{1/2} = \left(E \left| \sum_{j,k,l=1}^{n} I_{jkl} \right|^2 \right)^{1/2} = O(n^{-9/2}), \tag{5.58}$$

as desired. QED

LEMMA 5.5. *The term e_2 in* (5.28) *satisfies*

$$\left(E\left|\int_C e_2\,dW\right|^2\right)^{1/2} = O(n^{-2}). \tag{5.59}$$

Proof. From (5.28),

$$\int_C e_2\,dW = \sum_{j,k,l,m=1}^{n} I_{jklm}, \tag{5.60a}$$

$$I_{jklm} = \int G^{(iv)}_{jklm}\Delta q_j \Delta q_k \Delta q_l \Delta q_m\,dW. \tag{5.60b}$$

Now

$$(E|I_{jklm}|^2)^{1/2} \leqq L\left(\int_C E\Delta q_j^{16}\,dW\right)^{1/16} \cdots \left(\int E\Delta q_m^{1/16}\,dW\right)^{1/16}, \tag{5.61}$$

where $L=[\int_C E(G^{(iv)}_{jklm})^4\,dW]^{1/4}$, and we assume it to be bounded. With probability one,

$$\Delta q_j = \int_{t_{j-1}}^{t_j} [V'(x_{j-1})\Delta x_j(s) + \tfrac{1}{2}V''(x_{j-1}+\theta\Delta x_j(s))(\Delta x_j(s))^2]\xi(s)\,ds \tag{5.62}$$

and so, defining $v_j(s) = [V'(x_{j-1}) + \frac{1}{2}V''(x_{j-1}+\theta\Delta x_j(s))\Delta x_j(s)]\Delta x_j(s)$, we have

$$\begin{aligned} E(\Delta q_j)^{16} &\leqq \left[\int_{t_{j-1}}^{t_j} v_j^2(s)\,ds\right]^8 E\left[\int_{t_{j-1}}^{t_j} \xi^2(s)\,ds\right]^8 \\ &\leqq \left[\int_{t_{j-1}}^{t_j} v_j^2(s)\,ds\right]^8 \left(\frac{t}{n}\right)^7 \int_{t_{j-1}}^{t_j} E\xi^{16}(s)\,ds. \end{aligned} \tag{5.63}$$

Now

$$\sup_{t_{j-1}\leqq s\leqq t_j} |v_j(s)|^2 = c(x_{j-1}, u_j, b_j)/n \quad \text{and} \quad E\xi^{16}(s) < \infty$$

for every $s\in[0,t]$. Hence, for some K,

$$E(\Delta q_j)^{16} \leqq Kn^{-16}n^{-8} \tag{5.64}$$

and so,

$$\left(\int E\Delta q_j^{16}\,dW\right)^{1/16} \leqq Kn^{-3/2}. \tag{5.65}$$

It follows that

$$(E|I_{jklm}|^2)^{1/2} \leqq Kn^{-6}. \tag{5.66}$$

There are n^4 such terms in (5.60a); the estimate (5.59) follows. QED

Remark. If we attempt to use the simple estimates like (5.61)–(5.63) in the estimate of e_1 in Lemma 5.4, we would have

$$\begin{aligned} (E|I_{jkl}|^2)^{1/2} &\leqq \left[\int E\Delta q_j^2 \Delta q_k^2 \Delta q_l^2\,dW\right]^{1/2} \\ &\leqq K\left(\int E\Delta q_j^{16}\,dW\right)^{1/16}\left(\int E\Delta q_k^{16}\,dW\right)^{1/16}\left(\int E\Delta q_l^{16}\,dW\right)^{1/16} \\ &= O(n^{-9/2}). \end{aligned} \tag{5.67}$$

Since there are n^3 such terms in the expression for e_1, we would have $(E|\int e_1\, dW|^2)^{1/2} \leqq O(n^{-3/2})$. The elaborate argument used in the proof of Lemma 5.4 is (apparently) necessary to improve this estimate and to obtain the sharp estimate in Theorem 2.

Continuing the proof of Theorem 2, we can use an argument similar to that in Lemma 5.4 to prove

$$\left(E\left|\int_C G''(q) \sum_{j,k=1}^{n} \Delta q_j \Delta q_k\, dW\right|^2\right)^{1/2} = O(n^{-2}). \tag{5.68}$$

(Once again simple estimates like (5.63) are inadequate for this purpose.)

At this point we have shown that

$$\begin{aligned} I^{(12)} &= \int_C \left\{ G\left[\int_0^T V(x(s))\xi(s)\, ds\right]\right\} dW \\ &= \int_C \left\{ G\left(\sum_{i=1}^{n} q_i\right) + G'\left(\sum_{j=1}^{n} q_i\right) \sum_{j=1}^{n} \Delta q_j \right\} dW + O(n^{-2}) \\ &\triangleq I + O(n^{-2}). \end{aligned} \tag{5.69}$$

Using $\Delta s_j = s - t_{j-1}$ and an obvious shorthand, we have

$$\begin{aligned} I = (2\pi)^{-n} \int_{R^{2n}} &\left\{ G\left(\sum_{i=1}^{n} V(x_{i-1})\Xi_{i-1}\right) + G'\left(\sum_{i=1}^{n} V(x_{i-1})\Xi_{i-1}\right)\right. \\ &\cdot \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} [V(x_{j-1} + u_j\sqrt{n}\Delta s_j + b_j\sqrt{n}(\Delta s_j(t/n - \Delta s_j))^{1/2}) - V(x_{j-1})]\xi(s)\, ds \\ &\left. \cdot \exp\left[\frac{-u_1^2 - \cdots - u_n^2 - b_1^2 - \cdots - b_n^2}{2}\right]\right\} du_1 \cdots db_n \end{aligned} \tag{5.70}$$

$$\begin{aligned} &= (2\pi)^{-n} \int_{R^{2n}} \{G(\text{—})\} \exp(\text{—})\, du_1 \cdots db_n \\ &\quad + \pi^{-n} \int_{R^{2n}} \left\{ G'(\text{—}) \cdot \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} [\text{—}]\xi(s)\, ds \cdot \exp(\text{—})\right\} du_1 \cdots db_n \\ &\triangleq I_1 + I_2. \end{aligned}$$

Since $V(x_{i-1})$, $i = 1, 2, \cdots, n$ is independent of $u_n = v$, and

$$\int_{R^n} \exp\left[\frac{-b_1^2 - \cdots - b_n^2}{2}\right] db_1 \cdots db_n = (2\pi)^{n/2}, \tag{5.71}$$

the $2n$-fold integral I_1 reduces to

$$\begin{aligned} I_1 = (2\pi)^{-n/2} \int_{R^n} &\left\{ G\left(\sum_{i=1}^{n} V(x_{i-1})\Xi_{i-1}\right)\right. \\ &\left. \cdot \exp\left[\frac{-u_1^2 - \cdots - u_{n-1}^2 - v^2}{2}\right]\right\} du_1 \cdots du_{n-1}\, dv. \end{aligned} \tag{5.72}$$

To reduce I_2, we regard it as a sum $I_2 = \sum_{j=1}^{n} I_{2j}$ where (using an obvious shorthand)

$$I_{2j} = (2\pi)^{-n} \int_{R^{2n}} \left\{ G'\left(\sum_{i=1}^{n} V(x_{i-1})\Xi_{i-1}\right) \cdot \int_{t_{j-1}}^{t_j} [V(x_{j-1}+\Delta x_j(s)) - V(x_{j-1})]\xi(s)\, ds \right\} \cdot \exp\left[\frac{-u_1^2 - \cdots - b_n^2}{2}\right] du_1 \cdots db_n$$

(5.73)
$$= (2\pi)^{-n} \int_{R^{2n}} \left\{ \left[G'\left(\sum_{i=1}^{n} V_{i-1}^{(j)}\, \Xi_{i-1}\right) + \frac{1}{2} G''\left(\sum_{i=1}^{n} V_{i-1}^{(j)}\, \Xi_{i-1}\right) \cdot \sum_{i>j}^{n} V_{i-1}'^{(j)}\, \Xi_{i-1} u_j t\sqrt{n} + O(n^{-2}) \right] \cdot \left[V'_{j-1} \frac{t(u_j+b_j)}{2\sqrt{n}} + \frac{1}{2} V''_{j-1} t^2 \frac{u_j^2 + b_j^2 + 2u_j b_j}{4n} + O(n^{-2}))\Xi_{j-1} \right] \right\} \cdot \exp\left[\frac{-u_1^2 + \cdots - b_n^2}{2}\right] du_1 \cdots db_n$$

The first estimate on the right is from Taylor's formula; the second is from (5.18) (and its modification for the first order integrals $\int \Delta s_j \xi(s)\, ds$, $\int (\Delta s_j(t_j - s))^{1/2} \xi(s)\, ds$). We omit these terms in subsequent expressions. Substituting $u_j = v$, and using superscript $^{(j)}$ to indicate omission of the jth component, we have

(5.74)
$$I_{2j} = (2\pi)^{-n} \int_{R^{n-1}} \left\{ \left[G'\left(\sum_{i=1}^{n} V_{i-1}^{(j)}\, \Xi_{i-1}\right) \exp\left[\frac{-u_1^2 - \overset{(j)}{\cdots} - u_n^2}{2}\right]\right] \cdot \left[\int_{R^{n+1}} \left\{ \left(V'_{j-1} \frac{t(v+b_j)}{2\sqrt{n}} + \frac{1}{2} V''_{j-1} t^2 \frac{v^2 + b_j^2 + 2vb_j}{4n} \right) \cdot \left[\Xi_{j-1} \exp\left[\frac{-v^2 - b_1^2 - \cdots - b_n^2}{2}\right] db_1 \cdots db_n\, dv \right\} \right] \right\} du_1 \overset{(j)}{\cdots} du_n$$
$$+ (2\pi)^{-n} \int_{R^{2n}} \left\{ \left[\frac{1}{2} G''\left(\sum_{i=1}^{n} V_{i-1}^{(j)}\, \Xi_{i-1}\right) \cdot \sum_{i>j}^{n} V'_{i-1} \frac{vt}{\sqrt{n}} \right] \cdot \left[V'_{j-1} t \frac{v+b_j}{2\sqrt{n}} + \frac{1}{2} V''_{j-1} t^2 \frac{v^2 + b_j^2 + 2vb_j}{4n} \right] \cdot \Xi_{j-1} \exp\left[\frac{-u_1^2 - \overset{(j)}{\cdots} - u_n^2 - b_1^2 - \cdots - b_n^2 - v^2}{2}\right] \right\} \cdot du_1 \overset{(j)}{\cdots} du_n db_1 \cdots db_n dv + O(n^{-3})$$

$$\triangleq S_{1j} + S_{2j} + O(n^{-3}).$$

Note that $\int_R v^k e^{-v^2/2}\, dv = 0$ when $k = 1, 3, 5, \cdots$ (odd), and that

$$\int_R (v^2+b_j^2)\frac{e^{(-v^2-b_j^2)}}{2}\, dv\, db_j = (2\pi)^{1/2}\int v^2 e^{-v^2/2}\, dv. \tag{5.75}$$

It follows that

$$\begin{aligned} S_{1j} = (2\pi)^{-n/2}\int_{R^n}\Big\{G'\Big(\sum_{i=1}^n V_{i-1}^{(j)}\,\Xi_{i-1}\Big)\Big[\frac{1}{2}V_{j-1}''t^2\frac{v^2}{4n}\Big] \\ \cdot\,\Xi_{j-1}\exp\Big[\frac{-v^2-u_1^2-\overset{(j)}{\cdots}-u_n^2}{2}\Big]\Big\}\, du_1\overset{(j)}{\cdots}du_n\, dv, \end{aligned} \tag{5.76}$$

and that

$$\begin{aligned} S_{2j} = (2\pi)^{n/2}\int_{R^n}\Big\{\Big[\frac{1}{2}G''\Big(\sum_{i=1}^n V_{i-1}^{(j)}\,\Xi_{i-1}\Big)\cdot\sum_{i>j}^n V_{i-1}'^{(j)}\,\Xi_{i-1}\frac{t}{\sqrt{n}}\Big] \\ \cdot\Big[V_{j-1}'\frac{t}{2\sqrt{n}}\Big]\Xi_{j-1}v^2\exp\Big[\frac{-v^2-u_1^2-\overset{(j)}{\cdots}-u_n^2}{2}\Big]\Big\}\, du_1\overset{(j)}{\cdots}du_n\, dv. \end{aligned} \tag{5.77}$$

Therefore

$$\begin{aligned} I_2 &= \sum_{j=1}^n (S_{1j}+S_{2j}) \\ &= (2\pi)^{-n/2}\int_{R^n}\Big\{\sum_{j=1}^n\Big(\Big[G'\Big(\sum_{i=1}^n V_{i-1}^{(j)}\,\Xi_{i-1}\Big)V_{j-1}''\,\Xi_{j-1}\Big] \\ &\qquad + \Big[G''\Big(\sum_{i=1}^n V_{i-1}^{(j)}\,\Xi_{i-1}\Big)\Big(\sum_{i>j}^n V_{i-1}'^{(j)}\,\Xi_{i-1}\Big)V_{j-1}'\,\Xi_{j-1}\Big]\Big)\frac{t^2v^2}{4n} \\ &\qquad \cdot\exp\Big[\frac{-v^2-u_1^2-\overset{(j)}{\cdots}-u_n^2}{2}\Big]\Big\}\, du_1\overset{(j)}{\cdots}du_n\, dv + O(n^{-2}) \\ &= (2\pi)^{-n/2}\int_{R^n}\Big\{\sum_{j=1}^n\Big[G'\Big(\sum_{i=1}^n V_{i-1}^{(j)}\,\Xi_{i-1}\Big)V_{j-1}''\,\Xi_{j-1}\Big]\frac{t^2v^2}{4n} \\ &\qquad + \sum_{i,j=1}^n\Big[G''\Big(\sum_{k=1}^n V_{k-1}^{(j)}\,\Xi_{k-1}\Big)V_{i-1}'^{(j)}\,\Xi_{i-1}V_{j-1}'^{(j)}\,\Xi_{j-1}\Big]\frac{t^2v^2}{4n}\Big\} \\ &\qquad \cdot\exp\Big[\frac{-v^2-u_1^2-\overset{(j)}{\cdots}-u_n^2}{2}\Big]\, du_1\overset{(j)}{\cdots}du_n\, dv \\ &\quad -(2\pi)^{-n/2}\int_{R^n}\Big\{\sum_{i=j=1}^n\Big[G''\Big(\sum_{k=1}^n V_{k-1}^{(j)}\,\Xi_{k-1}\Big)V_{i-1}'^{(j)}\,\Xi_{i-1}V_{j-1}^{(j)}\,\Xi_{j-1}\Big]\frac{t^2v^2}{4n}\Big\} \\ &\qquad \cdot\exp\Big[\frac{-v^2-u_1^2-\overset{(j)}{\cdots}-u_n^2}{2}\Big]\, du_1\overset{(j)}{\cdots}du_n\, dv. \end{aligned} \tag{5.78}$$

A simple estimate shows that the last integral is $O(n^{-2})$ in the same sense used above.

Combining these results, we have

$$
\begin{aligned}
I^{(12)} &= I_1 + I_2 \\
&= I_1 + \sum_{j=1}^{n} I_{2j} \\
&= (2\pi)^{-n/2} \int_{R^n} \left\{ G\Big(\sum_{i=1}^{n} V(x_{i-1}) \Big) \exp\left[\frac{-u_1^2 - \cdots - u_{n-1}^2 - v^2}{2} \right] \right\} du_1 \cdots du_{n-1}\, dv \\
&\quad + (2\pi)^{-n/2} \int_{R^n} \left\{ \frac{1}{2} \sum_{j=1}^{n} \left[G'\Big(\sum_{i=1}^{n} V_{i-1}^{(j)} \Xi_{i-1} \Big) V''_{j-1} \Xi_{j-1} \right] \right. \\
&\qquad \left. + \frac{1}{2} \sum_{i_1 j=1}^{n} \left[G''\Big(\sum_{k=1}^{n} V_{k-1}^{(j)} \Xi_{k-1} \Big) V_{i-1}'^{(j)} \Xi_{i-1} V_{j-1}'^{(j)} \Xi_{j-1} \right] \right\} \\
&\qquad \cdot \frac{t^2 v^2}{2n} \exp\left[\frac{-u_1^2 - \overset{(j)}{\cdots} - u_n^2 - v^2}{2} \right] du_1 \overset{(j)}{\cdots} du_n\, dv + O(n^{-2}).
\end{aligned}
\tag{5.79}
$$

A change of variables $u_j \to u_{n-1}$ in each of the $j = 1, 2, \cdots, n$ integrals in the second part of the expression gives

$$
\begin{aligned}
I^{(12)} &= (2\pi)^{-n/2} \int_{R^n} \left\{ G\Big(\sum_{k=1}^{n} V_{k-1} \Xi_{k-1} \Big) \right. \\
&\qquad + \frac{1}{2} \sum_{j=1}^{n} \left[G'\Big(\sum_{k=1}^{n} V_{k-1} \Xi_{k-1} \Big) V''_{j-1} \Xi_{j-1} \frac{t^2 v^2}{2n} \right] \\
&\qquad \left. + \frac{1}{2} \sum_{i,j=1}^{n} \left[G''\Big(\sum_{k=1}^{n} V_{k-1} \Xi_{k-1} \Big) V'_{i-1} \Xi_{i-1} \cdot V'_{j-1} \Xi_{j-1} \frac{t^2 v^2}{2n} \right] \right\} \\
&\qquad \cdot \exp\left[\frac{-u_1^2 - \cdots - u_n^2 - v^2}{2} \right] du_1 \cdots du_{n-1}\, dv + O(n^{-2}) \\
&= (2\pi)^{-n/2} \int_{R^n} \left\{ G\Big[\sum_{i=1}^{n} V(x_{i-1} + vt/(2n)^{1/2}) \Xi_{i-1} \Big] \right\} \\
&\qquad \cdot \exp\left[\frac{-u_1^2 - \cdots - u_{n-1}^2 - v^2}{2} \right] du_1 \cdots du_{n-1}\, dv + O(n^{-2}).
\end{aligned}
\tag{5.80}
$$

The latter is the desired integration formula. This completes the proof of Theorem 2. QED

6. Proofs of Theorems 3 and 4. The basic plan of the proofs is identical to that used for Theorems 1 and 2. The major difference is the incorporation of the estimate of Δy into the process. Since $y(t)$ in (1.9) is an Ito process, we have

$$E\left|y\left(t+\frac{1}{n}\right)-y(t)\right|^2 = E\left|\int_t^{t+1/n} f(s)\,ds\right|^2 + \int_t^{t+1/n} Eg^2(s)\,ds$$

$$\leqq n^{-1}\int_t^{t+1/n} Ef^2(s)\,ds + \int_t^{t+1/n} Eg^2(s)\,ds \tag{6.1}$$

$$= O(n^{-2}) + O(n^{-1}) = O(n^{-1}).$$

Therefore, with $\Delta y(t) = y(t+1/n) - y(t)$ we have

$$(E|\Delta y(t)|^2)^{1/2} = O(n^{-1/2}). \tag{6.2}$$

This basic estimate makes the error term in the approximations in Theorems 3 and 4 larger than those in Theorems 1 and 2.

6.1. Proof of Theorem 3.

LEMMA 6.1. *Let V and y satisfy the hypotheses of the theorem. Then*

$$I^{(21)} = \int_C \left(\sum_{i=1}^n \left[V(x_{i-1})\Delta y_{i-1} + \frac{1}{2}\int_{t_{i-1}}^{t_i} V''(x_{i-1})(\Delta x_i(s))^2\,dy(s)\right]\right) dW + O(n^{-3/2}). \tag{6.3}$$

Proof. This differs only in detail from Lemma 5.1. Consider the expansion (5.3) with $\xi(s)\,ds$ replaced by $dy(s)$. Let e_i be that last term in that expansion, i.e., (without the $\frac{1}{24}$)

$$e_i = \int_{t_{i-1}}^{t_i} V''''(x_{i-1} + \theta\Delta x_i(s))(\Delta x_i(s))^4\,dy(s). \tag{6.4}$$

Omitting some arguments, we have

$$Ee_i^2 = E\left(\int_{t_{i-1}}^{t_i} V''''(\text{—})(\Delta x_i(s))^4\,dy(s)\right)^2$$

$$= E\left(\int_{t_{i-1}}^{t_i} V''''(\text{—})(\Delta x_i(s))^4 f(s)\,ds\right)^2 \tag{6.5}$$

$$+ \int_{t_{i-1}}^{t_i} [V''''(\text{—})(\Delta x_i(s))^4]^2 Eg^2(s)\,ds.$$

Using (5.6) and the assumption that $Ef^2(s) < \infty$, $Eg^2(s) < \infty$, uniformly in s, and assumption (A2), we have

$$Ee_i^2 \leqq C(|u_i| + |b_i|)^8$$

$$\cdot\, t^8 n^{-4}\left[\int_{t_{i-1}}^{t_i} |V''''(\text{—})|^2 (Ef^2(s)t/n + Eg^2(s))\,ds\right] \tag{6.6}$$

$$= O(n^{-5}).$$

Hence, $(Ee_i^2)^{1/2} = O(n^{-5/2})$, which compares with (5.8). This changes the estimate in (5.14) to $O(n^{-3/2})$. The remainder of the proof is identical to Lemma 5.1. QED

The next step in the proof is to prove the analogue of formula (5.15), i.e.,

$$I^{(21)} = \sum_{i=1}^n \Delta y_{i-1}\left\{\int_C \left[V(x_{i-1}) + \frac{1}{2}V''(x_{i-1})\frac{t^2}{4n}(u_i^2 + b_i^2)\right] dW\right\} + O(n^{-3/2}). \tag{6.7}$$

Following the reasoning in the proof of Lemma 5.2, we write

$$\frac{1}{2}\int_{t_{i-1}}^{t_i} V''(x_{i-1})(\Delta x_i(s))^2\,dy(s) = \frac{1}{2}V''(x_{i-1})n[u_i^2\hat{Y}_{i-1}+b_i^2\tilde{Y}_{i-1}+2u_ib_i\hat{\tilde{Y}}_{i-1}], \tag{6.8}$$

where

$$\hat{Y}_{i-1}=\int_{t_{i-1}}^{t_i}(s-t_{i-1})^2\,dy(s)=\left(\frac{t}{2n}\right)^2\Delta y_{i-1}+\hat{e}_{i-1} \tag{6.9}$$

and the other terms are clear from (5.18), (6.8). Now

$$\hat{e}_{i-1}=\int_{t_{i-1}}^{t_i}[(s-t_{i-1})^2-\frac{1}{4}(t_i-t_{i-1})^2]\,dy(s), \tag{6.10}$$

and it is a simple matter to show that

$$(E|\hat{e}_{i-1}|^2)^{1/2}=O(n^{-5/2}) \tag{6.11}$$

(compare (5.22)). The n terms of this form give the $O(n^{-3/2})$ estimate in (6.7). The remainder of the argument follows the proof of Lemma 5.2.

The remainder of the proof of Theorem 3 is identical to that of Theorem 2, as embodied in Lemma 5.3, with the obvious modifications in notation and the estimate.

6.2. Proof of Theorem 4. The main line of the argument is similar to that which we used to prove Theorem 2. However, the property (6.2) of $dy(s)$ causes some basic difficulties in estimating the approximation error. We will highlight the changes these problems impose.

Again consider the expansion

$$G\left(\int_0^t V(x(s))\right)dy(s)=G\left(\sum_{i=1}^n\int_{t_{i-1}}^{t_i}V(x_{i-1}+\Delta x_i(s))\,dy(s)\right) = G(q)+G'(q)\sum_{j=1}^n\Delta q_i+\frac{1}{2}G''(q)\sum_{j,k=1}^n\Delta q_j\Delta q_k+e_1+e_2, \tag{6.12}$$

where

$$q=\sum_{i=1}^n q_i,\qquad q_i=V(x_{i-1})\Delta y_{i-1},$$

$$\Delta q_j=\int_{t_{j-1}}^{t_j}[V(x_{j-1}+\Delta x_j(s))-V(x_{j-1})]\,dy(s),$$

$$e_1=\frac{1}{6}G'''(q)\sum_{j,k,l=1}^n\Delta q_j\Delta q_k\Delta q_l, \tag{6.13}$$

$$e_2=\frac{1}{24}\sum_{j,k,l,m=1}^n G_{jklm}^{(iv)}\Delta q_j\Delta q_k\Delta q_l\Delta q_m.$$

LEMMA 6.2. *Let $E(\cdot)$ be expectation with respect to the distribution of $\{y(s), 0\leqq s\leqq T\}$. Then*

$$\left(E\left|\int_C e_1\,dW\right|^2\right)^{1/2}=O(n^{-3}) \tag{6.14}$$

(compare this with (5.29)).

Proof. The long argument used to prove Lemma 5.4 suffices here as well. Property (6.2) causes the estimates to be larger. For example, (5.40) is replaced by $O(n^{-5/2})$, and (5.43), (5.44) are replaced by $O(n^{-3/2})$. The estimate (5.57) is replaced by $O(n^{-6})$, and this with the formula (5.58) yields (6.14). Once again simple estimates based on the Schwartz–Hölder inequalities badly over estimate the integral of e_1, and so, the long argument is necessary to achieve the desired accuracy. QED

LEMMA 6.3. *The term* e_2 *in* (6.12) *satisfies*

$$\left(E\left|\int_C e_2\, dW\right|^2\right)^{1/2} = O(n^{-2})$$

in the case when $G(\cdot) = \exp(\cdot)$.

Proof. The simple argument used to prove Lemma 5.5 (and used by Chorin) fails here. It appears to be necessary to evaluate more explicit expressions for moments. This is the reason that we must add the restriction $G = \exp$.

From (6.13) and the interpolation formula (5.2) we have

$$\begin{aligned} q &= \sum_{i=1}^{n} V(x_{i-1})\Delta y_{i-1} \\ &= \sum_{i=1}^{n} V_{i-1}^{(j)}\, \Delta y_{i-1} + \left(\sum_{i=j+1}^{n} V'^{(j)}_{i=j+1}\, \Delta y_{i-1}\right)\frac{t}{\sqrt{n}} u_j + \varepsilon_j \\ &\triangleq q^{(j)} + \delta q_j + \varepsilon_j \end{aligned} \tag{6.15}$$

where

$$\left(E\left|\int_C \varepsilon_j\, dW\right|^2\right)^{1/2} = O(n^{-1/2}) \tag{6.16}$$

and we have used the notation (5.30). Similarly, for $k \neq j$, $q = q^{(jk)} + \delta q_k + O(n^{-1/2})$, etc. Also,

$$\begin{aligned} \Delta q_j &= \left[\sqrt{n} V'_{j-1} \int_{t_{j-1}}^{t_j} \alpha_j(s)\, dy(s)\right] u_j + \left[\sqrt{n} V'_{j-1} \int_{t_{j-1}}^{t_j} \beta_j(s)\, dy(s)\right] b_j + O(n^{-3/2}) \\ &\triangleq a_{11} u_j + a_{12} b_j + O(n^{-3/2}), \end{aligned} \tag{6.17}$$

where

$$\alpha_j(s) = s - t_{j-1}, \qquad \beta_j(s) = [(s - t_{j-1})(t_j - s)]^{1/2}, \qquad t_{j-1} \leqq s \leqq t_j. \tag{6.18}$$

Now consider $G(x) = e^x$ and the generic term in e_2 in (6.13),

$$\begin{aligned} I_{jklm} &= \int_C G^{(iv)}_{jklm} \Delta q_j \cdots \Delta q_m\, dW \\ &= \int_C [\exp(q + \theta_j \Delta q_j + \cdots + \theta_m \Delta q_m)] \Delta q_j \cdots \Delta q_m\, dW, \qquad j, k, l, m = 1, 2, \cdots, n. \end{aligned} \tag{6.19}$$

Case 1. $j = k = l = m$, in which case $I_{jklm} = O(n^{-4})$. That is ($\theta = \theta_j = \cdots = \theta$),

$$\begin{aligned} I_{jklm} &= \int_C [e^q e^{(4\theta \Delta q_j)} (\Delta q_j)^4]\, dW, \qquad 0 \leqq \theta \leqq 1 \\ &= \int_C e^{q^{(j)}} \exp(\delta q_j + \varepsilon_j + 4\theta \Delta q_j)(\Delta q_j)^4\, dW \\ &= \int_{C^{(j)}} e^{q^{(j)}} \left[\int_{C_j} \exp(\delta q_j + 4\theta \Delta q_j + \varepsilon_j)(\Delta q_j)^4\, dW_j\right] dW^{(j)} \end{aligned} \tag{6.20}$$

Here we use the notation

$$(6.21)\quad \int_{C_j} f(x)\,dW_j(x) \triangleq (2\pi)^{-1}\int_{R^2} f(u_j, b_j; u_1, \cdots, b_n)\exp\left(\frac{-u_j^2-b_j^2}{2}\right) du_j\, db_j$$

and $\int_{C^{(j)}} dW^{(j)}$ is the remainder of the $2n$-fold integral $\int_C dW$. Using this and (6.17),

$$H_j = \int_{C_j} \exp(\delta q_j + 4\theta\Delta q_j)(\Delta q_j)^4\, dW_j$$

$$(6.22)\quad = (2\pi)^{-1}\int_{R^2}\exp\left(\frac{-u_j^2-b_j^2}{2}\right)\exp[a'_{11}u_j + 4\theta a_{12}b_j + O(n^{-1/2})] \cdot [a_{11}u_j + a_{12}b_j + O(n^{-3/2})]^4\, du_j\, db_j$$

where a_{11}, a_{12} are from (6.17) and

$$(6.23)\quad a'_{11} = 4\theta a_{11} + \left(\sum_{i=j+1}^{n} V'^{(j)}_{i-1}\Delta y_{i-1}\frac{t}{\sqrt{n}}\right).$$

Note that

$$a_{11} = V'_{j-1}\sqrt{n}\int_{t_{j-1}}^{t_j}\alpha_j(s)\,dy(s) = O(n^{-1}),$$

$$(6.24)\quad a_{12} = O(n^{-1}),$$

$$a'_{11} = O(1) \qquad \text{(by a simple Schwartz estimate)}^7$$

in the sense $(Ea_{11}^2)^{1/2} = O(n^{-1})$, etc. Now, for some constants C_l,

$$(6.25)\quad [a_{11}u_j + a_{12}b_j + O(n^{-3/2})]^4 = \sum_{l=0}^{4} C_l(a_{11}u_j)^{4-l}(a_{12}b_j)^l + O(n^{-9/2}),$$

and the sum is $O(n^{-4})$. Consider

$$\tilde{H}_j = (2\pi)^{-1}\int_{R^2}\exp\left(\frac{u_j^2-b_j^2}{2}\right)\exp(a'_{11}u_j + 4\theta a_{12}b_j) \cdot\left[\sum_{l=0}^{4} C_l(a_{11}u_j)^{4-l}(a_{12}b_j)^l\right] du_j\, db_j$$

$$(6.26)\quad = \sum_{l=0}^{4} C_l\left[(2\pi)^{-1}a_{11}^{4-l}\int_{-\infty}^{\infty}\exp\left(\frac{-u^2}{2}\right)e^{a'_{11}u}u^{4-l}\,du\right] \cdot\left[(2\pi)^{-1/2}a_{12}^{l}\int_{-\infty}^{\infty}\exp\left(\frac{-b^2}{2}\right)e^{4\theta a_{12}b}b^l\,db\right].$$

To evaluate this, we use the formulas

$$F_k(a) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp\left(\frac{-x^2}{2}\right)e^{ax}x^k\,dx, \qquad k = 0, 1, 2, 3, 4,$$

$$(6.27)\quad = \exp\left(\frac{a^2}{4}\right), \qquad k = 0,$$

$$= a\exp\left(\frac{a^2}{2}\right), \qquad k = 1,$$

$$= \tfrac{1}{2}aF_{k-1} + \tfrac{1}{2}(k-1)F_{k-2}, \qquad k = 2, 3, 4.$$

---

[7] In fact, $a'_{11} = O(n^{-1/2})$, see the analysis after (6.40).

From these expressions we can see that the term $l=0$ is the dominant one (since $a_{12}=O(n^{-1})$ in the sum (6.26)). It is $O(n^{-4})$ and so, therefore, is the sum $\hat{H}_j$.

Therefore

$$H_j+\tilde{H}_j+\tilde{\varepsilon}_j=O(n^{-4})+\tilde{\varepsilon}_j, \tag{6.28}$$

where

$$\begin{aligned}\tilde{\varepsilon}_j=(2\pi)^{-1}\int_{R^2}\exp\left(\frac{-u_j^2-b_j^2}{2}\right)&\exp[a'_{11}u_j+4\theta a_{12}b_j]\\ &\cdot[a_{11}u_j+a_{12}b_j]^4[e^{\varepsilon_i}-1]\,du_j\,db_j\\ +4(2\pi)^{-1}\int_{R^2}\exp\left(\frac{-u_j^2-b_j^2}{2}\right)&\exp[a'_{11}u_j+4\theta a_{12}b_j]\\ &\cdot[a_{11}u_j+a_{12}b_j]^3O(n^{-3/2})e^{\varepsilon_i}\,du_j\,db_j\\ +\text{smaller terms}\qquad&(\text{integrand}\leqq O(n^{-8})).\end{aligned} \tag{6.29}$$

Evidently, the first term on the right in (6.29) is $O(n^{-4})\cdot O(n^{-1/2})=O(n^{-9/2})$. For n large enough $\exp(O(n^{-1/2}))\approx 1+O(n^{-1/2})$. (One can write out the explicit expression for the exponent and confirm this.) In this instance the second term on the right in (6.29) behaves like

$$\int_{R^2}\exp\left(\frac{-u_j^2-b_j^2}{2}\right)\exp[a'_{11}u_j+4\theta a_{12}b_j][a_{11}u_j+a_{12}b_j]^3O(n^{-3/2})\,du_j\,db_j. \tag{6.30}$$

From (6.17) the $O(n^{-3/2})$ term is actually

$$O(n^{-3/2})=\int_{t_{j-1}}^{t_j}\frac{1}{2}V''[x_{j-1}+\theta\Delta x_j(s)]\Delta x_j^2(s)\,dy(s),\qquad 0\leqq\theta\leqq 1. \tag{6.31}$$

Substituting this in (6.30) and computing the integrals explicitly (using (6.27) and the interpolation formula for $\Delta x(s)$), one finds that (6.30) is $O(n^{-9/2})$. It follows that $\tilde{\varepsilon}_j=O(n^{-9/2})$, and so, from (6.28)

$$H_j=O(n^{-4}). \tag{6.32}$$

Referring back to (6.22) and (6.20), we see that $I_{jklm}=O(n^{-4})$ when $G(x)=\exp(x)$ and $j=k=l=m$. In the expression (6.13) for e_2 there are n such terms whose total contribution is $O(n^{-3})$, (when G is exp).

Case 2. $j\neq k\neq l\neq m$ (mutually) in which case $I_{jklm}=O(n^{-6})$. That is,

$$I_{jklm}=\int_C[\exp(q+\theta_j\Delta q_j+\cdots+\theta_m\Delta q_m)]\Delta q_j\cdots\Delta q_m\,dW. \tag{6.33}$$

Since the increments Δx_j, Δx_k, etc., $j\neq k$, are independent with respect to the measure dW, the variables Δq_j, Δq_k are also independent (for each sample $y(\cdot)$). We use this property to evaluate I_{jklm}. As in Case 1 (6.15), we write

$$\begin{aligned}q&=q^{(j)}+\delta q_j+\varepsilon_j\\ &=q^{(jklm)}+\delta q_j+\cdots+\delta q_m+\varepsilon_j+\cdots+\varepsilon_m\\ &=q^{(jklm)}+\delta q_j^{(klm)}+\cdots+\delta q_m^{(jkl)}+\hat{\varepsilon}\end{aligned} \tag{6.34}$$

where (using (6.15)) the typical term is

$$\delta q_j^{(klm)} = \left(\sum_{i=j+1}^{n} V_{i-1}^{\prime(jklm)} \Delta y_{i-1}\right) \frac{u_j t}{\sqrt{n}} \tag{6.35}$$

and $\hat{\varepsilon} = O(n^{-1/2})$. Using this notation, consider

$$\begin{aligned} \tilde{I}_{jklm} = \int_{C^{(jklm)}} \exp\,(q^{(jklm)}) \left[\int_{C_j} \exp\,(\delta q_j^{(klm)} + \theta_j \Delta q_j) \Delta q_j \, dW_j\right] \\ \cdots \left[\int_{C_m} \exp\,(\delta q_m^{(jkl)} + \theta_m \Delta q_m) \Delta q_m \, dW_m\right] dW^{(jkl)}. \end{aligned} \tag{6.36}$$

The terminology in (6.21) has been extended to encompass (6.36) in an obvious way. We proceed by evaluating $\tilde{I}_{jklm}$ and then the difference $I_{jklm} - \tilde{I}_{jklm}$.

Using (6.22), we can evaluate

$$\begin{aligned} \hat{H}_j &= \int_{C_j} \exp\,(\delta q_j^{(klm)} + \theta_j \Delta q_j) \Delta q_j \, dW_j \\ &= (2\pi)^{-1} \int_{R^2} \exp\left(\frac{-u_j^2 - b_j^2}{2}\right) \exp\,[a_{11}'' u_j + \theta_j a_{12} b_j + O(n^{-3/2})] \\ &\qquad \cdot [a_{11} u_j + a_{12} b_j + O(n^{-3/2})] \, du_j \, db_j. \end{aligned} \tag{6.37}$$

Here we have used (6.17) for Δq_j, and

$$a_{11}'' = \theta a_{11} + \left(\sum_{i=j+1}^{n} V_{i-1}^{\prime(jkl)} \Delta y_{i-1}\right) \frac{t}{\sqrt{n}}. \tag{6.38}$$

Again $a_{12} = O(n^{-1}) = a_{11}$. We may ignore the $O(n^{-3/2})$ exponential term in (6.37). Using (6.27) for $k = 0, 1$, we have

$$\hat{H}_j = (a_{11} a_{11}'' + \theta_j a_{12}^2) \exp\,[(a_{11}'')^2 + \theta_j^2 a_{12}^2] + O(n^{-3/2}). \tag{6.39}$$

Claim. $\hat{H}_j = O(n^{-3/2})$. To see this, note that the sum $S_n = \sum_{i=j+1}^{n} V_{i-1}^{\prime(jklm)} \Delta y_{i-1}$ is, for j, k, l, m fixed, a "retarded" sum which converges in mean square to an Ito stochastic integral, i.e.,

$$S_n = \sum_{i=j+1}^{n} V_{i-1}^{\prime(jklm)} [y(t_i) - y(t_{i-1})] \xrightarrow[n \to \infty]{(\text{m.s.})} \int_0^t v(s) \, dy(s) = r(t) \tag{6.40}$$

for some sure function $v(s)$. It follows that $\sqrt{n} a_{11}'' \to \text{tr}\,(t)$ in mean square as $n \to \infty$; that is, a_{11}'' is $O(n^{-1/2})$ A similar, but more complex, analysis shows that $a_{11} a_{11}''$ is $O(n^{-3/2})$. This, together with an analysis like that in (6.29) establishes the claim.

Now $\hat{H}_j = O(n^{-3/2})$ implies

$$\tilde{I}_{jklm} = O(n^{-6}). \tag{6.41}$$

From (6.33)–(6.36) we see that I_{jklm} and $\tilde{I}_{jklm}$ differ by the $O(n^{-1/2})$ term $\hat{\varepsilon}$ (in (6.34)) which appears in the exponent in (6.33) (implicitly) and which is (explicitly) omitted in (6.36). Let $\hat{q} = q^{(jklm)} + \delta q_j^{(klm)} + \cdots + \delta q_m^{(jkl)} + \theta_j \Delta q_j + \cdots + \theta_m \Delta q_m$, and $\Delta \hat{q} = \Delta q_j \cdots \Delta q_m$. Then

$$I_{jklm} - \tilde{I}_{jklm} = \int_C e^{\hat{q}} [e^{\hat{\varepsilon}} - 1] \Delta \hat{q} \, dW = \int_C e^{\hat{q}} \Delta \hat{q} \hat{\varepsilon} \, dW + R_n. \tag{6.42}$$

The first term on the right is strictly smaller than $\tilde{I}_{jklm}$, which differs from it by the absence of $\hat{\varepsilon}$ in the integrand. The remainder term is $O(n^{-1/2})$ smaller than this. (A complete analysis of (6.42) is very tedious and is omitted here.)

It follows that $I_{jklm} = O(n^{-6})$ for $j \neq k \neq l \neq m$, mutually. In the expression (6.13) for e_2 there are $n(n-1)(n-2)(n-3) \approx O(n^4)$ such terms whose total contribution is $O(n^{-2})$.

The other cases $j = k \neq l \neq m$ (piecewise) and $j = k = l \neq m$ are treated similarly. They produce smaller net contributions to e_2 than does Case 2, $j \neq k \neq l \neq m$. This dominant term produces the estimate $e_2 = O(n^{-2})$ which completes the proof of Lemma 6.3. QED

Referring to (6.12), we have shown that

$$\int_C G\left[\int_0^T V(x(s))\,dy(s)\right]dW$$

$$(6.43)\qquad = \int_C \left\{G(q) + G'(q)\sum_{j=1}^n \Delta q_j + \tfrac{1}{2}G''(q)\sum_{j=1}^n \Delta q_1^2\right\} dW$$

$$+ \int_C \frac{1}{2} G''(q) \sum_{j\neq k=1}^n \Delta q_j \Delta q_k\, dW + O(n^{-2}).$$

We must estimate the second integral and show that the first is equal to the expression (2.9).

To carry out the first step, we follow the argument used in the proof of Lemma 5.4. Let

$$I_{jk} = \int_C G''(q)\Delta q_j \Delta q_k\, dW, \qquad k \neq j$$

$$= \int_C G''(q^{(jk)})\Delta q_j \Delta q_k\, dW$$

$$+ \int_C \left[G'''(q^{(jk)}) \cdot \sum_{i=j+1}^n V'(x_{i-1}^{(jk)})\Delta y_{i-1}\frac{u_j t}{\sqrt{n}}\right]\Delta q_j \Delta q_k\, dW$$

$$(6.44)\qquad + \text{a similar term in } u_k$$

$$+ \int_C \left\{\left[G''''(q^{(jk)}) \sum_{i=j+1}^n \sum_{p=k+1}^n V'(x_{i-1}^{(jk)})V'(x_{p-1}^{(jk)})\Delta y_{i-1}\Delta y_{p-1}\right]\right.$$

$$\left. + G'''(q^{(jk)}) \sum_{i=\max(j,k)+1}^n \left([V''(x_{i-1}^{(jk)})\Delta y_{i-1}]u_j u_k t^2/n\right)\Delta q_j \Delta q_k\right\} dW + R_n.$$

Since $j \neq k$ and Δq_j has zero mean relative to dW, the first term is zero. The second one is zero since Δq_k has zero mean and Δq_j, Δq_k are independent under dW. An analysis similar to the proof of Lemma 6.3 shows that the third integral (written out) on the right in (6.44) is $O(n^{-4})$. The remainder R_n is smaller by at least $O(n^{-1/2})$ (since its dominant terms involve odd powers of u_j or u_k). Hence, I_{jk} is $O(n^{-4})$ for $j \neq k$. Since there are $n(n-1)$ terms in the sum in (6.43), their net contribution is $O(n^{-2})$. This term may be included in the $O(n^{-2})$ error in (6.43).

To complete the proof of Theorem 4, we must show that the first integral on the right in (6.43) is equal to the right side of (2.9). This process differs little from the final arguments (from (5.69) on) in the proof of Theorem 2. The main difference is the need to account for the properties of the Ito process $dy(t) = f(t)\,dt + g(t)\,dv(t)$. Namely, $dy^2(t) \approx g^2(t)\,dt$.

Thus far we have shown that

$$I^{(22)} = \int_C \left\{ G(q) + G'(q) \sum_{j=1}^{n} \Delta q_i + \tfrac{1}{2} G''(q) \sum_{j=1}^{n} \Delta q_i^2 \right\} dW + O(n^{-2}), \tag{6.45}$$

when $G(\cdot) = \exp(\cdot)$. Consider the last term in the integrand. Simple modifications of a standard argument for Ito integrals [24, p. 89], shows that

$$\begin{aligned} \sum_{j=1}^{n} \Delta q_j^2 &= \sum_{j=1}^{n} \left(\int_{t_{j-1}}^{t_j} [V(x_{j-1} + \Delta x_j(s)) - V(x_{j-1})] \, dy(s) \right)^2 \\ &\underset{n\to\infty}{\longrightarrow} \int_0^t v(s) g^2(s) \, ds \qquad \text{(in mean square)} \end{aligned} \tag{6.46}$$

for some $v(s)$. From this argument we have the result

$$\sum_{j=1}^{n} \Delta q_j^2 = \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} [V(x_{j-1} + \Delta x_j(s)) - V(x_{j-1})]^2 g^2(t_{j-1}) \, ds + r^n, \tag{6.47}$$

where $r^n \approx O(n^{-2})$. Using this, we have

$$\begin{aligned} I^{(22)} &= \int_C \left\{ G(q) + G'(q) \sum_{j=1}^{n} \Delta q_j \right\} dW \\ &\quad + \int_C \left\{ \frac{1}{2} G''(q) \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} [V(x_{j-1} + \Delta x_j(s)) - V(x_{j-1})]^2 g^2(t_{j-1}) \, ds \right\} dW \\ &\quad + O(n^{-2}). \end{aligned} \tag{6.48}$$

This constitutes an expansion of $G[\int_0^t V(x(s)) \, dy(s)]$ about $q = \sum_{i=1}^{n} V(x_{i-1}) \Delta y_{i-1}$ taking into acount the Ito–Stratonovich correction term (the second integral in (6.48)).

The reduction of the $(2n)$-fold integrals in (6.48) to the n-fold integral in (2.9) follows the argument in the proof of Lemma 5.4 and is omitted for brevity. QED

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