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Chandrasekhar Algorithms for Linear Time Varying Distributed Systems

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ABSTRACT

Generalized Chandrasekhar type algorithms are obtained that are applicable to linear time varying as well as time invariant distributed systems. Utilizing the framework of J. L. Lions for quadratic control and the framework of A. Bensoussan for filtering of linear distributed systems, we analyze the implications of these algorithms.

1. INTRODUCTION

Chandrasekhar type algorithms have recently been developed for filtering in lumped systems [1–4, 15–20]. These algorithms were initially developed for linear time invariant systems with finite dimensional state spaces. They represent an alternative to the traditional Riccati equation approach to linear filtering, with significant numerical advantages [2, 4].

Lindquist's derivation [4] of new differential equations, which can be used to compute the gains needed in the implementation of the Kalman filter, established that similar algorithms could be developed for linear time invariant distributed systems. The filtering theory for linear distributed parameter systems has been rigorously established and is well understood, (cf. Bensoussan [7], Curtain and Falb [8], Curtain [9], Vinter and Mitter [10], Balakrishnan [11–12]). Excellent accounts of quadratic optimal control theory for distributed parameter systems can be found in the book by Lions [13] and the forthcoming monograph by Bensoussan, Delfour, and Mitter [14]. The applicability of these theories is severely hindered by the task of solving an operator Riccati

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equation [9, 14]. The development of alternative methods is therefore very significant for the useful application of the available theories. The new Chandrasekhar type algorithms hold promise for significant reduction in the computational complexity of these problems, particularly in the distributed parameter case, as was pointed out by Kailath in [2]. Very briefly, these new algorithms replace the problem of solving a doubly infinite matrix differential equation with the problem of solving a matrix differential equation which is infinite in one dimension only. Or equivalently, they replace the problem of solving a system of partial differential equations in two space variables with the problem of solving a system of partial differential equations in one space variable. These new algorithms have been recently developed for linear time invariant distributed systems by Casti and Ljung [5].

Recently in [15]–[20] generalized Chandrasekhar algorithms were obtained that incorporate time varying linear finite dimensional systems. In the present paper we study a nonsymmetric operator Riccati equation with time varying operators, and derive various alternative equations for the solution. As a result we obtain generalizations of the results of [15–20] in linear time varying distributed systems and Chandrasekhar type algorithms.

2. THE NONSYMMETRIC OPERATOR RICCATI EQUATION

We follow the notation and framework of [14] (see also [7, 13]) which is standard for distributed parameter systems. Let V, X be two Hilbert spaces, V dense in X with continuous injection i. Denoting by V', X' the duals of V and X, and identifying X with X', we obtain the following triple of Hilbert spaces, where each is dense and continuously injected into the next space:

$$V \xrightarrow{i} X = X' \xrightarrow{i^*} V'. \tag{1}$$

We denote by $(\cdot,\cdot)$ the inner product in X and by $\langle\cdot,\cdot\rangle$ the natural pairing between V and V'. $\mathcal{L}(X,Y)$ denotes the Banach space of all continuous linear maps between the Hilbert spaces X and Y, endowed with the natural uniform operator topology. For convenience we let also

$$\Delta(a,b) = \{(t,s) | a \leqslant s \leqslant b,\ s \leqslant t \leqslant b\}. \tag{2}$$

An *evolution operator* on X is a map $\Lambda : \Delta(0,\infty) \to \mathcal{L}(X)$ with the properties:

$$\Lambda(t,r) = \Lambda(t,s)\Lambda(s,r), \qquad t \geqslant s \geqslant r \geqslant 0, \tag{3a}$$

$$\Lambda(t,t) = I(\text{identity in } \mathcal{L}(X), \qquad t \geqslant 0, \tag{3b}$$

$$\forall x \in X, \qquad (t,s) \to \Lambda(t,s)x \text{ is continuous as a function from } \Delta(0,\infty) \to X. \tag{3c}$$

We will need the following spaces, where D denotes the operation of taking distributional derivatives:

$$W(a,b;X,V')=\{y\in L^{\infty}(a,b;X)|Dy\in L^{2}(a,b;V')\}, \tag{4}$$

$$W(a,b;V,X)=\{y\in L^{2}(a,b;V)|Dy\in L^{2}(a,b;X)\}. \tag{5}$$

Throughout the paper we will make the following assumptions about evolution operators

A.1. There exists $A:[0,T]\to\mathcal{L}(V,X)$ which is strongly measurable and bounded so that for all k in X and all s in $(0,T]$ the map

$$t\to\Lambda(t,s)^{*}k$$

(where * indicates adjoint) is the unique solution in $W(s,T;X,V')$ of

$$\frac{dy(t)}{dt}=A(t)^{*}y(t) \qquad \text{in } (s,T),$$

$$y(s)=k. \tag{6}$$

A.2. For all h in V and s in $[0,t)$, the map

$$s\to\Lambda(t,s)ih$$

is the unique solution in $W(0,t;V,X)$ of

$$\frac{dx(s)}{ds}=-A(s)x(s) \qquad \text{in } (0,t),$$

$$x(t)=ih. \tag{7}$$

(A is the same as in A.1.)

These assumptions are satisfied by wide classes of evolution operators arising in time varying distributed systems [14]. We shall say that the evolution operator $\Lambda(t,s)$ *is generated* by $A(t)$. Then A.1 alone implies that

$$i^{*}\Lambda(t,s)^{*}k=i^{*}k+\int_{s}^{t}A(r)^{*}\Lambda(r,s)^{*}k\,dr \tag{8}$$

and that for all h in V and s in $(0,t]$

$$\Lambda(t,s)ih=ih+\int_{s}^{t}\Lambda(r,s)A(r)h\,dr, \tag{9}$$

and the map $t \to \Lambda(t,s)ih$ is continuous with a distributional derivative in $L^\infty(s,T;X)$. On the other hand A.2 implies that

$$\Lambda(t,s)ih = ih + \int_s^t A(r)\Lambda(t,r)ih\,dr, \qquad (10)$$

where we view $\Lambda(t,r)ih$ as an element of V.

We consider now the following nonsymmetric operator Riccati equation in the space of operator valued functions

$\mathcal{W}(0,T;X,V') = \{K:[0,T] \to \mathcal{L}(X) | \forall h \in X$, the X-valued function $K(t)h$ belongs to $L^\infty(0,T;X)$ and $\exists DK:[0,T] \to \mathcal{L}(V,V')$ so that $\forall h \in V$, $DK(\cdot)h$ belongs to $L^\infty(0,T;V')$ and is the distributional derivative of $K(\cdot)ih\}$, (11)

$$\left.\begin{aligned} \frac{d}{dt}N(t,\tau) &= A^*(t)N(t,\tau)i + i^*N(t,\tau)B(t) \\ &\quad + i^*[C(t) - N(t,\tau)D(t)N(t,\tau)]i \\ &\qquad \text{in } (\tau,T), \end{aligned}\right\} \qquad (12)$$

$$N(\tau,\tau) = \Pi,$$

where $A,B:[0,T] \to \mathcal{L}(V,X)$ and are strongly measurable and bounded; $C,D:[0,T] \to \mathcal{L}(X)$ and are strongly measurable, bounded; $C(t) = C(t)^*$, $D(t) = D(t)^*$, $C(t) \geqslant 0$ and $D(t) \geqslant 0$; $\Pi = \Pi^*$, $\Pi \geqslant 0$.

It is straightforward to establish that under the hypotheses made, Eq. (12) has a unique solution in $\mathcal{W}(\tau,T;X,V')$, which is self-adjoint and positive semidefinite. Let now $\Phi(t,\tau)$ and $\Psi(t,\tau)$ be the evolution operators generated by $A(t)$ and $B(t)$, respectively (see A.1 and A.2). We define the operators $\Phi_N(t,\tau)$ and $\Psi_N(t,\tau)$ on $\Delta(0,T)$ as follows:

$\forall h$ in X and τ in $[0,T)$, $\Phi_N^*(t,\tau)h$ is the unique solution of the equation

$$y(t,\tau) = \Phi^*(t,\tau)h - \int_\tau^t \Phi^*(t,\sigma)N(\sigma,\tau)D(\sigma)y(\sigma,\tau)\,d\sigma, \qquad \tau \leqslant t \leqslant T, \quad (13)$$

and $\Psi_N^*(t,\tau)h$ is the unique solution of

$$z(t,\tau) = \Psi^*(t,\tau)h - \int_\tau^t \Psi^*(t,\sigma)N(\sigma,\tau)D(\sigma)z(\sigma,\tau)\,d\sigma, \qquad \tau \leqslant t \leqslant T, \quad (14)$$

in the space of continuous functions $C(\tau,T;X)$. It is easy [14] to establish that $\Phi_N(t,\tau)$, $\Psi_N(t,\tau)$, defined above, satisfy the properties of evolution operators

(3), and that for every (t,τ) in $\Delta(0,T)$ and k in X,

$$\Phi_N(t,\tau)k=\Phi(t,\tau)k-\int_\tau^t \Phi_N(\sigma,\tau)D(\sigma)N(\sigma,\tau)\Phi(t,\sigma)k\,d\sigma \tag{15}$$

$$\Psi_N(t,\tau)k=\Psi(t,\tau)k-\int_\tau^t \Psi_N(\sigma,\tau)D(\sigma)N(\sigma,\tau)\Psi(t,\sigma)k\,d\sigma \tag{16}$$

$$\int_\tau^t \Phi_N(\sigma,\tau)D(\sigma)N(\sigma,\tau)\Phi(t,\sigma)k\,d\sigma=\int_\tau^t \Phi(\sigma,\tau)D(\sigma)N(\sigma,\tau)\Phi_N(t,\sigma)k\,d\sigma, \tag{17}$$

$$\int_\tau^t \Psi_N(\sigma,\tau)D(\sigma)N(\sigma,\tau)\Psi(t,\sigma)k\,d\sigma=\int_\tau^t \Psi(\sigma,\tau)D(\sigma)N(\sigma,\tau)\Psi_N(t,\sigma)k\,d\sigma. \tag{18}$$

As a consequence now of assumptions A.1, A.2, and Eqs. (9) and (10), we have also that:

For all k in X, s in $(0,T]$, f in $L^2(0,T;X)$, the map

$$t\to\Phi_N^*(t,\tau)k+\int_\tau^t \Phi_N^*(t,\sigma)f(\sigma)\,d\sigma$$

is the unique solution in $W(\tau,T;X,V')$ of

$$\begin{aligned}\frac{dz}{dt}^{(t)}&=[A(t)^*-i^*N(t,\tau)D(t)]z(t)+i^*f(t) \qquad \text{in } (\tau,T),\\ z(\tau)&=k,\end{aligned} \tag{19}$$

and for all h in V and τ in $(0,t]$,

$$\Phi_N(t,\tau)ih=ih+\int_\tau^t \Phi_N(\sigma,\tau)[A(\sigma)-D(\sigma)N(\sigma,\tau)i]h\,d\sigma. \tag{20}$$

Moreover, for all h in V and τ in $[0,t)$ the map

$$\tau\to\Phi_N(t,\tau)ih$$

is the unique solution in $W(0,t;V,X)$ of

$$\begin{aligned} \frac{dx(\tau)}{d\tau} &= -[A(\tau)-D(\tau)N(\tau,\tau)]x(\tau) \qquad \text{in} \quad (0,t), \\ x(t) &= ih, \end{aligned} \tag{21}$$

and

$$\Phi_N(t,\tau)ih = ih + \int_\tau^t [A(\sigma)-D(\sigma)N(\sigma,\tau)]\Phi_N(t,\sigma)ih\,d\sigma. \tag{22}$$

Equations (19)–(22) are also valid for $\Psi_N(t,\tau)$, provided we change $A(\cdot)$ to $B(\cdot)$.

Considering now the nonsymmetric operator Riccati equation (12), we conclude that equivalently $N(t,\tau)$ is the unique solution of the equation:

$$\begin{aligned} \frac{d}{dt}\big(N(t,\tau)ih, i\bar{h}\big) &= \big(ih, N(t,\tau)A(t)\bar{h}\big) + \big(N(t,\tau)B(t)h, i\bar{h}\big) \\ &\quad + \big(C(t)ih, i\bar{h}\big) - \big(N(t,\tau)D(t)N(t,\tau)ih, i\bar{h}\big) \qquad (23) \\ N(\tau,\tau) &= \Pi, \qquad \text{for all } h, \bar{h} \text{ in } V \text{ and } t \text{ in } (\tau, T), \end{aligned}$$

which implies that for all $h,\bar{h}$ in V

$$\begin{aligned} \big(N(t,\tau)ih, i\bar{h}\big) &= (\Pi ih, i\bar{h}) \\ &\quad + \int_\tau^t \Big\{ \big(N(\sigma,\tau)ih, [A(\sigma)-D(\sigma)N(\sigma,\tau)i]\bar{h}\big) \\ &\quad + \big(N(\sigma,\tau)[B(\sigma)-D(\sigma)N(\sigma,\tau)i]h, i\bar{h}\big)\Big\}\,d\sigma \\ &\quad + \int_\tau^t \big([C(\sigma)+N(\sigma,\tau)D(\sigma)N(\sigma,\tau)]ih, i\bar{h}\big)\,d\sigma. \end{aligned} \tag{24}$$

Utilizing the properties of the evolution operators $\Phi_N(t,\tau)$, and $\Psi_N(t,\tau)$ [(15–22)], we obtain from (24)

$$\begin{aligned} \big(N(t,\tau)ih, i\bar{h}\big) &= \big(\Pi\Psi_N(t,\tau)ih, \Phi_N(t,\tau)i\bar{h}\big) \\ &\quad + \int_\tau^t \big([C(\sigma)+N(\sigma,\tau)D(\sigma)N(\sigma,\tau)]\Psi_N(t,\sigma)ih, \Phi_N(t,\sigma)i\bar{h}\big)\,d\sigma. \end{aligned} \tag{25}$$

Therefore we conclude that $N(t,\tau)$ is the unique solution in the space of operator valued functions $C(0,T;\ L(X)$ with the weak operator topology) of the following system:

$$\text{Eqs. (15) and (16) together with}$$

$$\begin{aligned}\big(N(t,\tau)h,\bar{h}\big) &= \big(\Pi\Psi_N(t,\tau)h, \Phi_N(t,\tau)\bar{h}\big)\\ &\quad + \int_\tau^t ([C(\sigma)+N(\sigma,\tau)D(\sigma)N(\sigma,\tau)]\\ &\quad \Psi_N(t,\sigma)h, \Phi_N(t,\sigma)\bar{h}\big)d\sigma,\\ &\quad \forall t \text{ in } [\tau,T], \quad h \text{ and } \bar{h} \text{ in } X. \end{aligned} \tag{26}$$

Conversely one can show (see [14]) that under our hypotheses A.1 and A.2 if $N(t,\tau)$ satisfies (26) then it also solves the nonsymmetric operator Riccati equation (12). From (24) it follows that $\forall h,\bar{h}$ in V,

$$\begin{aligned}\frac{d}{d\tau}\big(N(t,\tau)ih,i\bar{h}\big)\Big|_{t=\tau} &= -\big(\Pi ih,[A(\tau)-D(\tau)\Pi i]\bar{h}\big)\\ &\quad -\big(\Pi[B(\tau)-D(\tau)\Pi i]h,i\bar{h}\big)\\ &\quad -\big([C(\tau)+\Pi D(\tau)\Pi]ih,i\bar{h}\big)\\ &= -\big(ih,\Pi A(\tau)\bar{h}\big)-\big(\Pi B(\tau)h,i\bar{h}\big)\\ &\quad -\big(C(\tau)ih,i\bar{h}\big)+\big(\Pi D(\tau)\Pi ih,i\bar{h}\big)\\ &= -\frac{d}{dt}\big(N(t,\tau)ih,i\bar{h}\big)\Big|_{t=\tau}, \end{aligned} \tag{27}$$

using (23). So $\frac{d}{d\tau}N(t,\tau)\Big|_{t=\tau}$ is an operator in $\mathcal{L}(V,V')$ and

$$-\frac{d}{dt}N(t,\tau)\Big|_{t=\tau} = \frac{d}{d\tau}N(t,\tau)\Big|_{t=\tau}. \tag{28}$$

From (12)

$$\begin{aligned}N_t(\tau,\tau) \triangleq \frac{d}{dt}N(t,\tau)\Big|_{t=\tau} &= A^*(\tau)N(\tau,\tau)i + i^*N(\tau,\tau)B(\tau)\\ &\quad + i^*[C(\tau)-N(\tau,\tau)D(\tau)N(\tau,\tau)]i. \end{aligned} \tag{29}$$

Utilizing (25), (21), we have that for every $h, \bar{h} \in V$

$$\begin{aligned}\frac{d}{d\tau}\left(N(t,\tau)ih, i\bar{h}\right) = &-\left(N(\tau,\tau)B(\tau)\Psi_N(t,\tau)ih, \Phi_N(t,\tau)i\bar{h}\right)\\ &-\left(\Psi_N(t,\tau)ih, N(\tau,\tau)A(\tau)\Phi_N(t,\tau)i\bar{h}\right)\\ &-\left(C(\tau)\Psi_N(t,\tau)ih, \Phi_N(t,\tau)i\bar{h}\right)\\ &+\left(N(\tau,\tau)D(\tau)N(\tau,\tau)\Psi_N(t,\tau)ih, \Phi_N(t,\tau)i\bar{h}\right), \qquad (30)\end{aligned}$$

and from (26), $N(t,t)=\Pi$. Therefore

$$\begin{aligned}&\frac{d}{d\tau}N(t,\tau) = -\Phi_N^*(t,\tau)N_t(\tau,\tau)\Psi_N(t,\tau), \qquad 0 \leqslant \tau \leqslant t,\\ &N(t,t)=\Pi,\end{aligned} \qquad (31)$$

utilizing (28) and (29). We summarize these results in:

THEOREM 1. *Under the assumptions of this section* (*i.e.*, A.1, A.2), *the solution* $N(t,\tau)$ *of the nonsymmetric operator Riccati equation* (12) *is also the solution of the following equations*:

$$\frac{d}{d\tau}N(t,\tau) = -\Phi_N^*(t,\tau)N_t(\tau,\tau)\Psi_N(t,\tau), \qquad N(t,t)=\Pi,$$

$$\frac{d}{dt}\Phi_N^*(t,\tau) = [A(t) - D(t)N(t,\tau)i]^*\Phi_N^*(t,\tau), \qquad \Phi_N^*(\tau,\tau) = I,$$

$$\frac{d}{dt}\Psi_N^*(t,\tau) = [B(t) - D(t)N(t,\tau)i]^*\Psi_N^*(\tau,\tau) \qquad \Psi_N^*(\tau,\tau) = I,$$

where the interpretation of these operator equations is as in (19) *and* (30), *and* $N_t(\tau,\tau)$ *is given by* (29).

We note that the result of Theorem 1 constitutes a generalization of the Chandrasekhar equations to linear time varying distributed systems. This will be further demonstrated in the subsequent section. A special case of particular interest is when $N(\tau,\tau)=0$ (i.e. zero initial condition) [2, 5, 6, 15].

COROLLARY 1. *If* $N(\tau,\tau)=0$, *then the solution* $N(t,\tau)$ *of the operator Riccati equation is also the solution of the following system of equations*:

$$\frac{d}{d\tau}N(t,\tau)=-\Phi^*{}_N(t,\tau)C(\tau)\Psi_N(t,\tau)$$

and the last two equations of Theorem 1.

Proof. Immediate, since in this case (29) gives $N_t(\tau,\tau)=C(\tau)$.

3. APPLICATIONS

A general quadratic optimal control problem for linear time varying distributed systems is described below (see [14] for details). The state $x(t;s,x_0,u)$ of the system at time $t\geqslant s$ with initial datum x_0 and control $u\in L^2_{\mathrm{loc}}(0,\infty;U)$ is given by

$$x(t;s,x_0,u)=\Phi(t,s)x_0+\int_s^t\Phi(t,r)G(r)u(r)\,\mathrm{dr}, \tag{32}$$

where G belongs to $L^\infty_{\mathrm{loc}}(0,\infty,\mathcal{L}(U,X))$. Since the evolution operator $\Phi(t,s)$ satisfies hypotheses A.1 and A.2, the state is the unique solution in $W(s,\tau;V,X)$ of

$$\begin{aligned}&\frac{dx(t)}{dt}=A(t)x(t)+G(t)u(t),\\&x(\tau)=x_0.\end{aligned}\tag{33}$$

The output of the system is described by

$$y(t)=C(t)x(t), \tag{34}$$

where C belongs to $L^\infty_{\mathrm{loc}}(0,\infty,\mathcal{L}(X,Y))$. The problem is to choose u to minimize

$$\begin{aligned}J(u)=(x(\tau),Lx(\tau))+\int_0^\tau\{&(x(t),C^*(t)C(t)x(t))\\&+(u(t),N(t)u(t))_U\}\,dt,\end{aligned}\tag{35}$$

where $L\in L(X)$ and $N:[0,\tau]\rightarrow\mathcal{L}(U)$ is strongly measurable and bounded; $L=L^*$, $N(t)=N(t)^*$ and $N(t)>0$. Then it is well known (see for example [14]) that an optimal control exits and has the expression

$$u^*(t) = -N(t)^{-1} G^*(t) S(t,\tau) x(t) = K(t,\tau) x(t) \quad \text{a.e. in } [0,\tau], \tag{36}$$

where the operator $S(t,\tau)$ is the unique solution in $\mathcal{W}(0,\tau;X,V')$ of the symmetric backward operator Riccati differential equation

$$\begin{aligned} \frac{d}{dt} S(t,\tau) &= -A(t)^* S(t,\tau) i - i^* S(t,\tau) A(t) \\ &\quad - i^* \big[C^*(t) C(t) \\ &\quad - S(t,\tau) G(t) N(t)^{-1} G^*(t) S(t,\tau) \big] i, \\ S(\tau,\tau) &= L \qquad \text{in } 0 \leqslant t \leqslant \tau. \end{aligned} \tag{37}$$

Then we have the following immediate corollaries:

COROLLARY 2. *$S(t,\tau)$ is also the solution of the system of equations*

$$\frac{d}{d\tau} S(t,\tau) = \Phi_S^*(t,\tau) S_t(\tau,\tau) \Phi_S(t,\tau) \qquad S(t,t) = L,$$

$$\frac{d}{dt} \Phi_S(t,\tau) = \Phi_S(t,\tau) \big[A(t) - G(t) N(t)^{-1} G^*(t) S(t,\tau) \big],$$

$$\Phi_S(\tau,\tau) = I,$$

$$S_t(\tau,\tau) = A^*(\tau) S(\tau,\tau) i + i^* S(\tau,\tau) A(\tau) + C^*(\tau) C(\tau) - S(\tau,\tau) G(\tau) N^{-1}(\tau) G^*(\tau) S(\tau,\tau).$$

COROLLARY 3. *When $S(\tau,\tau) = L = 0$, the first equation of Corollary 2 must be replaced by*

$$\frac{d}{d\tau} S(t,\tau) = \Phi_S^*(t,\tau) C^*(\tau) C(\tau) \Phi_S(t,\tau).$$

Note that the case described in Corollary 3 corresponds to zero final cost.

Suppose in addition that the distributed system is time invariant, [i.e., $A(t) = A$, $G(t) = G$, $C(t) = C$] and that $N(t) = N$. Then we have

COROLLARY 4. *In the time invariant case $S(t,\tau)$ is the solution to the system of equations*

$$\frac{d}{dt}S(t,\tau) = -\Phi_S^*(t,\tau)S_t(\tau,\tau)\Phi_S(t,\tau), \qquad S(\tau,\tau) = L,$$

and

$$\frac{d}{dt}\Phi_S(t,\tau) = -\Phi_S(t,\tau)\left[A - GN^{-1}G^*S(t,\tau)\right]$$

$$\Phi_S(\tau,\tau) = I,$$

$$S_t(\tau,\tau) = A^*S(\tau,\tau)i + i^*S(\tau,\tau)A + C^*C - S(\tau,\tau)GN^{-1}G^*S(\tau,\tau).$$

If in addition the final cost is zero [*i.e.*, $S(\tau,\tau) = 0$], *then one computes directly the optimal gain* $K(t,\tau)$ *via*

$$\frac{d}{dt}K(t,\tau) = N^{-1}G^*L(t,\tau)^*L(t,\tau),$$

$$\frac{d}{dt}L(t,\tau) = -L(t,\tau)[A + GK(t,\tau)],$$

$$K(\tau,\tau) = 0, \qquad L(\tau,\tau) = C,$$

where $K(t,\tau)\in\mathcal{L}(X,U)$, $L(t,\tau)\in\mathcal{L}(X,Y)$.

Note that the final equations of the above corollary are those obtained by Casti and Ljung [5]. This special case presents the most important numerical savings, since in typical applications U and Y are finite dimensional.

We turn now to distributed parameter filtering or estimation problems. We follow Bensoussan's approach [7], and we consider again the distributed system

$$\frac{d}{dt}x(t) = A(t)x(t) + G(t)\xi(t),$$

$$y(t) = C(t)x(t) + n(t), \tag{38}$$

$$x(0) = x_0,$$

where $\xi(t)$ and $n(t)$ represent input and output noise. The triple (x_0,ξ,n) is modeled as a Gaussian linear random functional on the Hilbert $X \times L^2(0,T;U) \times L^2(0,T;Y)$ with zero mean and covariance operator

$$\begin{bmatrix} P_0 & 0 & 0 \\ 0 & Q(t) & 0 \\ 0 & 0 & R(t) \end{bmatrix}.$$

$G(t)$ and $C(t)$ as in (33) and (34). P_0, $Q(t)$ and $R(t)$ are self-adjoint, positive definite, and invertible. Then $x(t)$ is interpreted as a Gaussian linear random functional on X (for any t). The filtering estimate of $x(t)$ is denoted by $\hat{x}(t)$ and is given by

$$\begin{aligned} \frac{d\hat{x}(t)}{dt} &= A(t)\hat{x}(t) + P(t,\tau)C^*(t)R(t)^{-1}[z(t) - C(t)\hat{x}(t)], \\ \hat{x}(0) &= x_0, \end{aligned} \tag{39}$$

where $\hat{x}(t)$ is to be interpreted as a linear random functional and $P(t,\tau)$ satisfies the symmetric forward operator Riccati equation:

$$\begin{aligned} \frac{d}{dt}P(t,\tau) = A(t)P(t,\tau) &+ P(t,\tau)A^*(t) \\ &+ G(t)Q(t)G^*(t) \\ &- P(t,\tau)C^*(t)R^{-1}(t)C(t)P(t,\tau), \\ P(\tau,\tau) = P_0 \qquad & t \geqslant \tau. \end{aligned} \tag{40}$$

Then we have the following immediate corollaries:

COROLLARY 5. *$P(t,\tau)$ solves the following system of equations*:

$$\begin{aligned} \frac{d}{d\tau}P(t,\tau) &= -\Phi_P(t,\tau)P_t(\tau,\tau)\Phi_P^*(t,\tau), \\ P(t,t) &= P_0, \\ \frac{d}{dt}\Phi_P(t,\tau) &= \left[A(t) - P(t,\tau)C^*(t)R^{-1}(t)C(t)\right]\Phi_P(t,\tau), \\ \Phi_P(\tau,\tau) &= I, \end{aligned}$$

where

$$P_t(\tau,\tau)=A(\tau)P(\tau,\tau)i+i^*P(\tau,\tau)A^*(\tau)$$
$$+G(\tau)Q(\tau)G^*(\tau)$$
$$-P(\tau,\tau)C^*(\tau)R^{-1}(\tau)C(\tau)P(\tau,\tau).$$

COROLLARY 6. *When* $P(\tau,\tau)=0$ *the first equation of Corollary* 5 *must be replaced by*

$$\frac{d}{d\tau}P(t,\tau)=-\Phi_P(t,\tau)G(\tau)Q(\tau)G^*(\tau)\Phi_P^*(t,\tau).$$

In the time invariant case, i.e., $A(t)=A$, $G(t)=G$, $C(t)=C$, $Q(t)=Q$, $R(t)=R$, we have:

COROLLARY 7. *In the time invariant case* $P(t,\tau)$ *is the solution to the system of equations*

$$\frac{d}{dt}P(t,\tau)=\Phi_P(t,\tau)P_t(\tau,\tau)\Phi_P^*(t,\tau)$$
$$P(\tau,\tau)=P_0,$$
$$\frac{d}{dt}\Phi_P(t,\tau)=\left[A-P(t,\tau)C^*R^{-1}C\right]\Phi_P(t,\tau)$$
$$\Phi_P(\tau,\tau)=I,$$

where

$$P_t(\tau,\tau)=A\,P(\tau,\tau)+P(\tau,\tau)A^*(\tau)$$
$$+G\,Q\,G^*-P(\tau,\tau)C^*R^{-1}C\,P(\tau,\tau).$$

If in addition the initial condition is zero, one can compute directly the Kalman filter gain operator $P(t,\tau)C^*R^{-1}=K(t,\tau)$ *from the equations*

$$\frac{d}{dt}K(t,\tau)=L(t,\tau)Q\,L^*(t,\tau)C^*R^{-1},$$
$$\frac{d}{dt}L(t,\tau)=[A-K(t,\tau)C]L(t,\tau),$$

where

$$K(\tau,\tau)=0,$$

$$L(\tau,\tau)=G,$$

and

$$K(t,\tau)\in\mathcal{L}(Y,X), \qquad L(t,\tau)\in\mathcal{L}(U,X).$$

We note again that this special case provides significant numerical savings for typical applications where U, Y are finite dimensional.

Our results are directly applicable to linear hereditary systems, as developed by Delfour and Mitter [14]. Similar results can be easily derived for other classes of distributed parameter systems with specific properties of the evolution operators, using similar arguments. These, together with specific applications, will be given elsewhere. We would like to note that in the case of 0 final cost (or initial condition) the results of Corollary 3 (Corollary 6) provide similar numerical savings with the time invariant case. For example, the optimal gain for the regulator can be directly computed from

$$\frac{d}{d\tau}K(t,\tau)=-N(t)^{-1}G^*(t)L^*(t,\tau)L(t,\tau) \qquad t\leqslant\tau,$$

$$\frac{d}{dt}L(t,\tau)=-L(t,\tau)[A(t)+G(t)K(t,\tau)],$$

with

$$K(t,t)=0,$$

$$L(\tau,\tau)=C(\tau).$$

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