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H^2-FUNCTIONS AND INFINITE-DIMENSIONAL REALIZATION THEORY*

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Abstract. In this paper the realization question for infinite-dimensional linear systems is examined for both bounded and unbounded operators. In addition to obtaining realizability criteria covering the basic cases, we discuss the relationship between canonical realizations of the same system. What one finds is that the set of transfer functions which are realizable by triples (A, b, c) with A bounded is related in a close way to the space of complex functions analytic and square integrable on the disk $|s| < 1$, and that the set of transfer functions which are realizable by triples (A, b, c) with A unbounded but generating a strongly continuous semigroup is related in a close way to functions analytic and square integrable on a half-plane. This relation makes possible a deeper study between the transfer function and the models which realize it. Some examples illustrate the results and their applications.

1. Preliminaries and notation. In this paper we study realization theory for a class of infinite-dimensional linear systems. On one hand our motivation comes from a desire to understand engineering problems involving transmission lines, elastic deformations, moving fluids, and related matters, where the assumption of finite-dimensionality is too restrictive; on the other hand, we want to see the finite-dimensional results themselves as part of a larger picture.

For the sake of definiteness we work in the most basic Hilbert space,

$$l_2(\mathbb{Z}^+) = \{\{a_i\}, i = 1, 2, 3, \cdots, \text{such that } \{a_i\} \text{ is a square summable sequence}\}.$$

This makes possible a fairly direct comparison with many well-known results concerning the finite-dimensional case. The problem is to express a given real function T defined on $[0, \infty)$ as $T(t) = c[e^{At}b]$, or to express its Laplace transform $\tilde{T}(s)$ as $c[(Is - A)^{-1}b]$ in some appropriately defined region of the complex plane.

We consider several distinct, but related cases. The first centers around the existence of realizations (A, b, c) with A a bounded operator on $l_2(\mathbb{Z}^+)$, b an element of $l_2(\mathbb{Z}^+)$ and c a bounded linear functional on $l_2(\mathbb{Z}^+)$. We call such triples *bounded realizations.* We call a triple (A, b, c) *a regular realization* if A is the infinitesimal generator of a strongly continuous semigroup of bounded operators $\{e^{At}\}$ on $l_2(\mathbb{Z}^+)$, b is an element of $l_2(\mathbb{Z}^+)$ and c is a bounded linear functional on $l_2(\mathbb{Z}^+)$. In both cases above the output can also be expressed, as is well known, as the inner product of $x(t)$ with some element of $l_2(\mathbb{Z}^+)$ which is uniquely determined by the functional c, and which we denote also by c; i.e., we shall write $y(t) = c[x(t)] = \langle c, x(t)\rangle$.

We also consider cases where A is the infinitesimal generator of a strongly continuous semigroup of bounded operators on $l_2(\mathbb{Z}^+)$, b is restricted to belong to the domain of A (written $\mathscr{D}_0(A)$) but c is a linear functional defined on $\mathscr{D}_0(A)$ and such that $|c(x)| \leq k(\|Ax\| + \|x\|)$ for all $x \in \mathscr{D}_0(A)$ and some constant k. Such

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realizations will be called *balanced realizations*. They have important properties not shared by regular realizations.

The triple (A, b, c) is called a realization for the weighting pattern $T(t)$ if and only if $T(t) = c[e^{At}b]$. The system theoretic interpretation of this equation in terms of "external" and "internal" descriptions of time-invariant linear systems with scalar input, scalar output is considered to be well known. The fact that we are using $l_2(\mathbb{Z}^+)$ as our state space is not very restrictive, since any separable Hilbert space is isometrically isomorphic to $l_2(\mathbb{Z}^+)$.

In order to describe what realizations realize what weighting patterns, we need to introduce some notation. The open disk of radius ρ is denoted by $\mathbb{D}_\rho = \{s \mid |s| < \rho\}$. We write $\mathbb{D}$ for $\mathbb{D}_1$. The boundary of $\mathbb{D}$, the unit circle, is denoted by $\mathbb{T}$. By $H^2(\mathbb{D})$ we mean the set of complex-valued functions which are holomorphic in $\mathbb{D}$ and have a Taylor series about zero with square summable coefficients. The space $H^2(\mathbb{D}_\rho)$ is defined by saying that $\psi(s)$ belongs to $H^2(\mathbb{D}_\rho)$ if and only if $\psi(s/\rho)$ belongs to $H^2(\mathbb{D})$. By $L^2(\mathbb{T})$ we mean the set of complex-valued functions which are defined and square integrable, in the Lebesgue sense, on the unit circle. By $H^2(\mathbb{T})$ we mean the subspace of $L^2(\mathbb{T})$ of functions with vanishing negative Fourier coefficients. $H^2(\mathbb{D})$ and $H^2(\mathbb{T})$ are related by the fact that for any function in $H^2(\mathbb{D})$ the radial limits from within the disk $\lim_{r\to 1} \psi(re^{i\theta}) = \phi(\theta)$ exist for almost all θ and give an element ϕ of $H^2(\mathbb{T})$. This correspondence is, moreover, one-to-one and onto so that $H^2(\mathbb{D})$ and $H^2(\mathbb{T})$ are closely related indeed. In fact, the Fourier coefficients of ϕ are the Taylor coefficients of ψ. In addition, $H^2(\mathbb{D})$ is a Hilbert space with the inner product

$$\langle \psi_1, \psi_2 \rangle = \sum_{n=0}^{\infty} \bar{\alpha}_n \beta_n,$$

where $\psi_1(s) = \sum_{n=0}^{\infty} \alpha_n s^n$ and $\psi_2(s) = \sum_{n=0}^{\infty} \beta_n s^n$. This makes $H^2(\mathbb{D})$, $H^2(\mathbb{T})$ and $l_2(\mathbb{Z}^+)$ isomorphic as Hilbert spaces with the isomorphisms defined by

$$(a_0, a_1, a_2, \cdots) \leftrightarrow \sum_{i=0}^{\infty} a_i s^i \leftrightarrow \sum_{n=0}^{\infty} a_n e^{in\theta}.$$

We denote by $\prod_\rho^+$ the half-plane Re $s > \rho$. We understand by $H^2(\prod_\rho^+)$ the space of functions which are analytic in $\prod_\rho^+$ and square integrable along vertical lines in $\prod_\rho^+$ such that

$$\sup_{x>\rho} \int_{-\infty}^{+\infty} |\psi(x + iy)|^2 \, dy \leqq M < \infty.$$

The relationship between $H^2(\mathbb{D})$ and $H^2(\prod^+)$ is this: $\phi(\cdot) \in H^2(\prod^+)$ if and only if ψ defined by

$$\psi(s) = \frac{1}{s-1} \phi\left(\frac{s+1}{s-1}\right)$$

belongs to $H^2(\mathbb{D})$. (See Hoffman [7, p. 130].)

We would like to recall some of the facts from Fourier transform theory that involve $H^2(\prod^+)$ and especially the Paley–Wiener theorem. We denote by $\mathbb{I}$ the imaginary axis in the complex plane. It is well known that the Fourier transform

$$g(t) \overset{\mathscr{F}}{\mapsto} \int_{-\infty}^{\infty} e^{-i\omega t} g(t) \, dt = G(i\omega)$$

is a unitary map between $L_2(-\infty, \infty)$ and $L_2(\mathbb{I}, d\omega/2\pi)$. Consider $L_2(0, \infty)$ as the subspace of $L_2(-\infty, \infty)$ of functions which vanish on $(-\infty, 0)$, and $L_2(-\infty, 0)$ as the subspace of $L_2(-\infty, \infty)$ of functions which vanish on $(0, \infty)$. Then obviously $L_2(-\infty, 0) = L_2(0, \infty)^\perp$ in $L_2(-\infty, \infty)$. Moreover, if we let $H^2(\mathbb{I}) = \mathscr{F}L_2(0, \infty)$ and $\tilde{H}^2(\mathbb{I}) = \mathscr{F}L_2(-\infty, 0)$, we see that $H^2(\mathbb{I})^\perp = \tilde{H}^2(\mathbb{I})$. $H^2(\mathbb{I})$ consists exactly of the boundary values of the elements of $H^2(\prod^+)$ (which exist for almost all ω). Moreover, if $\mathscr{L}$ denotes the Laplace transform

$$g(t) \overset{\mathscr{L}}{\mapsto} \int_0^\infty e^{-st} g(t)\, dt = G(s) \qquad \text{for } g \in L_2(0, \infty),$$

$$f(t) \overset{\mathscr{L}}{\mapsto} \int_{-\infty}^0 e^{-st} f(t)\, dt = F(s) \quad \text{for } f \in L_2(-\infty, 0),$$

the Paley–Wiener theorem says that $H^2(\prod^+) = \mathscr{L}L_2(0, \infty)$. If we let $\prod^-$ denote the half-plane $\operatorname{Re} s < 0$, then also $H^2(\prod^-) = \mathscr{L}L_2(-\infty, 0)$. Moreover, $\tilde{H}^2(\mathbb{I})$ consists exactly of the boundary values of the elements of $H^2(\prod^-)$. The relation between $H^2(\prod^+)$ and $H^2(\prod^-)$ is simple. A function $f(s)$ belongs to $H^2(\prod^+)$ if and only if $\overline{f(-\bar{s})}$ belongs to $H^2(\prod^-)$. Then obviously we see that $\tilde{H}^2(\mathbb{I}) = \overline{H^2(\mathbb{I})}$.

As we were preparing the original version of this paper we received from Paul Fuhrmann a manuscript [13] which analyzes the bounded case and obtains a number of the results described here with certain small changes due to the fact that he works with discrete time systems. Helton [14] also investigates some questions of this type but emphasizes a different class of ideas. A result similar to our Theorem 4 appears in Balakrishnan [23].

2. Realizability criteria, bounded case. In this section we characterize the class of weighting patterns which admit bounded realizations.

Let $T:[0, \infty) \to \mathbb{R}^1$ be a continuous function of time. When can it be written as

$$T(\mathrm{t}) = \langle c, e^{At} b \rangle,$$

where b, $c \in l_2(\mathbb{Z}^+)$ and $A: l_2(\mathbb{Z}^+) \to l_2(\mathbb{Z}^+)$ is bounded? As is well known such a representation is possible for T with $[A, b, c]$ all finite-dimensional if and only if T is of the exponential order and its transform

$$\tilde{T}(s) = \int_0^\infty e^{-st} T(t)\, dt, \qquad \operatorname{Re} s > \sigma_0,$$

is rational. In the present case A is bounded; $\{e^{At}\}$ defines a uniformly continuous semigroup of operators (see [1, p. 626]), and since b and c belong to $l_2(\mathbb{Z}^+)$ we have

$$\langle c, e^{At} b \rangle \leqq \|b\| \cdot \|c\| \cdot M \cdot e^{\|A\|\,|t|}, \qquad t \in \mathbb{R}^1,$$

where $\|e^{At}\| \leqq M e^{\|A\|\,|t|}$, and the norms are $l_2(\mathbb{Z}^+)$ and induced $l_2(\mathbb{Z}^+)$ respectively. Thus the class we are looking for includes only functions of exponential order. Moreover, since A is bounded, $\langle c, e^{At} b \rangle$ is an entire function.

The following two theorems characterize in the time and frequency domain the set of realizable input–output maps.

THEOREM 1. *T: $[0, \infty) \to \mathbb{R}^1$ has a bounded realization if and only if T is an entire function of exponential order.*[1]

[1] This is a standard engineering term; in the mathematical literature this is called "exponential type".

Proof. The necessity follows from the above. For the sufficiency, since T is entire it has a power series expansion

$$T(t) = \sum_{n=0}^{\infty} c_n t^n$$

valid in the finite complex plane. Let σ_0 be the exponential order of $T(\cdot)$. Then $\overline{\lim}_{n\to\infty} (n!|c_n|)^{1/n} = \sigma_0$ (see [20, p. 95]). So for $k > \sigma_0$ we have

$$n!|c_n|/k^n \leqq (\sigma_0/k)^n$$

and consequently the sequence $\{n!|c_n|/k^n\}_{n=0}^{\infty} \in l_2(\mathbb{Z}^+)$. Now take

$$A = k\begin{bmatrix} 0 & & & & & \\ 1 & 0 & & & & \\ 0 & 1 & 0 & & & \\ & 0 & 1 & 0 & & \\ & & 0 & 1 & 0 & \\ & & & 0 & \ddots & \ddots \end{bmatrix},$$

$$b = \{1, 0, 0, \cdots\},$$
$$c = \{c_0, c_1/k, \cdots, n!c_n/k^n, \cdots\}$$

and this completes the proof.

Now using Laplace transforms in the complex domain we pass from the equation $T(t) = \langle c, e^{At}b\rangle$ to the equation $\tilde{T}(s) = \langle c, (Is - A)^{-1}b\rangle$ for $\operatorname{Re} s > \|A\|$. Since A is bounded using an elementary analytic continuation argument we see that $\tilde{T}(s)$ is analytic for $|s| > \|A\|$ and also that $\tilde{T}(\infty) = 0$. Hence $\tilde{T}(s) = \langle c, b\rangle s^{-1} + \langle c, Ab\rangle s^{-2} + \langle c, A^2b\rangle s^{-3} + \cdots$ for $|s| > \|A\|$.

THEOREM 2. *The function $T:[0, \infty) \to \mathbb{R}^1$ has a bounded realization if and only if the Laplace transform $\tilde{T}(\cdot)$ of $T(\cdot)$ is analytic at infinity and vanishes there.*

Proof. The necessity follows clearly from the above. For the sufficiency since $\tilde{T}(s)$ is analytic at infinity and vanishes there, it has a power series expansion

$$\tilde{T}(s) = \sum_{i=0}^{\infty} a_i s^{-(i+1)} \quad \text{for} \quad |s| > c$$

for some finite c. Then for $k > c$ we have that the sequence $\{|a_i|/k^i\} \in l_2(\mathbb{Z}^+)$. So again take

$$A = k\begin{bmatrix} 0 & & & & \\ 1 & 0 & & & \\ 0 & 1 & 0 & & \\ & 0 & 1 & 0 & \\ & & 0 & \ddots & \ddots \end{bmatrix},$$

$$b = \{1, 0, 0, \cdots\},$$
$$c = \{a_0, a_1/k, a_2/k^2, \cdots\},$$

and this completes the proof.

Remark 2.1. It is clear from the above that the singularities of $\tilde{T}(\cdot)$ must be contained in a compact set of the complex plane. Hence if the singularities include branch points, in order for $\tilde{T}(\cdot)$ to have a bounded realization it must be possible to consider the branch cuts in the finite plane (since any $\tilde{T}(\cdot)$ which has a bounded realization is analytic at infinity). For example, a transfer function with a single branch point does not have a bounded realization. On the other hand, there are many cases of complex-valued functions with branch points for which the branch cuts can be taken either in the finite plane or through infinity (e.g., $1/\sqrt{s^2+1}$ is typical).

Suppose we have a physical system (and perhaps a model based on the underlying physical theories) with a transfer function which has this property, i.e., we can consider the branch cuts either in the finite plane or through infinity. We consider the question of whether or not this particular transfer function has a bounded realization. Now in most physical problems the position of the branch cuts is implied by physical or asymptotic conditions (see the first few pages of Noble's book [25] on the Wiener–Hopf technique for an example). Therefore if these conditions exclude the possibility of considering the branch cuts in the finite plane, then one concludes immediately that there is no bounded realization for this transfer function.

Remark 2.2. It is apparent from the above that if $T(\cdot)$ has any bounded realization, it can be realized by a multiple of the unilateral shift in $l_2(\mathbb{Z}^+)$ or by a multiple of the bilateral shift in $l_2(\mathbb{Z})$. To see the latter take

$$A = k\begin{bmatrix} \ddots & & & & & & \\ \ddots\ 0 & & & & & & \\ \ddots\ 1 & 0 & & & & & \\ \ddots\ 0 & 1 & 0 & & & & \\ & 0 & 1 & 0 & & & \\ & & 0 & 1 & 0 & \ddots & \\ & & & 0 & 1 & \ddots & \\ & & & & 0 & \ddots & \end{bmatrix},$$

$$b = \{\cdots, 0, 1, 0, 0, \cdots\},$$
$$c = \{\cdots, 0, 0, c_0, c_1/k, \cdots, n!c_n/k^n\}$$

(with $\{c_i\}$ and k as in Theorem 1). This is not surprising in view of the fact that the shift can be considered as a "universal model" for bounded operators in Hilbert space (cf. [9]).

We would like to discuss in more detail the relation $\tilde{T}(s) = \langle c, (Is - A)^{-1}b\rangle$ for Re $s > \gamma$, where γ is large enough, and draw some conclusions from it. Since A here is a bounded operator the point at infinity is in the resolvent set of A, denoted $\rho(A)$. We denote by $\rho_0(A)$ the connected component of $\rho(A)$ containing the point at infinity, by $\sigma(A)$ the spectrum of A and by $\sigma_0(A)$ the complement in $\mathbb{C}$ (the complex plane) of $\rho_0(A)$. The function $\langle c, (Is - A)^{-1}b\rangle$ is obviously analytic for $|s| > \|A\|$ and thus for $\operatorname{Re} s > \|A\|$. $T(\cdot)$ is of exponential order, say $|T(t)| \leqq Me^{\sigma_0 t}$; then $\tilde{T}(s)$ is analytic for $\operatorname{Re} s > \sigma_0$ (and also for $|s| > \sigma_0$) (see [20, p. 95]). Since $T(t) = \langle c, e^{At}b\rangle$ we see that $\sigma_0 \leqq \|A\|$. So from the equation $\tilde{T}(s) = \langle c, (Is - A)^{-1}b\rangle$ valid for $\operatorname{Re} s > \|A\|$ we deduce by analytic continuation that $\tilde{T}(\cdot)$ is analytic for all $s \in \rho_0(A)$.

If we let $\sigma(\tilde{T}) = \{s \in \mathbb{C} | \tilde{T}(\cdot) \text{ is not analytic at } s\}$ we deduce that for any bounded realization (A, b, c) of $T(\cdot)$ we must have

$$\sigma(\tilde{T}) \subseteq \sigma_0(A).$$

This relation will be referred in the sequel as the *spectral inclusion property*. For example, the function $T(t) = e^{t/2}$ can obviously be realized by the unilateral shift as above. Then for A being the unilateral shift we have $\sigma(A) = \mathbb{D} \cup \mathbb{T}$ (i.e., the closed disk). But $\tilde{T}(\cdot)$ has just a pole at $s = \frac{1}{2}$. Consider now

$$T(t) = \sum_{n=0}^{\infty} \frac{t^{2^n - 1}}{(2^n - 1)!}.$$

Then we know that $\tilde{T}(s) = \sum_{n=0}^{\infty} s^{-2^n}$ has $\mathbb{T}$ as its natural boundary. Obviously we can realize T using the construction of Theorems 1 or 2 with any $k > 1$.

Remark 2.3. The realization constructed in Theorems 1 and 2 uses the operator kU, where U is the unilateral shift on $l_2(\mathbb{Z}^+)$. The spectrum of kU is the closed disk of radius k, and hence its resolvent set is connected. The value of k we used is big enough so that the singularities of $\tilde{T}(\cdot)$ are included in the disk of radius k. In other words, we used an operator with spectrum big enough to include all the singularities of $\tilde{T}(\cdot)$. It is therefore of interest to know how small k can be taken for a given realizable weighting pattern $T(\cdot)$. It follows from a theorem in Widder [20, p. 95], that if σ_0 is the exponential order (or "type" in Widder's terminology) of the entire function $T(\cdot)$, then $\tilde{T}(\cdot)$ will be analytic for $|s| > \sigma_0$ and will vanish at infinity, and conversely. Hence k in Theorems 1 and 2 must satisfy $k \geqq \sigma_0$.

The connectedness of the resolvent set of the infinitesimal generator A has important implications as far as the relationship to frequency response methods for system identification is concerned. The values of $\tilde{T}$ for s purely imaginary are often empirically determined by letting $u(t) = \sin \omega t$ and looking at the periodic solution which results. If the periodic component of the response is $M(\omega) \sin(\omega t + \phi(\omega))$, then

$$\tilde{T}(i\omega) = M(\omega)\, e^{i\phi(\omega)}.$$

However, if the domain of analyticity of $\tilde{T}$ is such that the entire imaginary axis does not belong to a single component, then there is no way that experimental data taken in different components can be pieced together and we must regard the system as consisting of several unrelated parts.

Remark 2.4. There are two typical sets of examples of systems with uniformly continuous state-transition operators, i.e., examples for which the operator A is bounded. The first comes from systems governed by parabolic and certain hyperbolic partial differential equations with constant coefficients, where the spatial domain is infinite or semi-infinite, after semidiscretization with uniform spatial mesh (see Birkhoff and Varga [24] and Brockett and Willems [17]).

The second comes from systems governed by certain particular classes of partial differential equations. The ideas involved are best illustrated by the following example. Consider the system

$$\frac{\partial}{\partial t}\frac{\partial}{\partial z}x(t,z) + \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial z}\right)x(t,z) = b(z)u(t),$$

$$y(t) = x(t,1),$$

where $x(t,\cdot) \in L_2(0,1)$ is absolutely continuous and $x(t,0) = 0$; $x(\cdot,z)$ and $(\partial/\partial z)x(\cdot,z)$ are differentiable; $b(\cdot) \in L_2(0,1)$. The domain of $-\partial/\partial z$ is $\mathscr{D}_0(-\partial/\partial z) = \{h \in L_2(0,1)$ such that h is absolutely continuous and $h(0) = 0\}$. If we let $\xi(t,z) = (\partial/\partial z + I)x(t,z)$, we obtain the system

$$\frac{\partial}{\partial t}\xi(t,z) = \frac{\partial}{\partial z}\left(\frac{\partial}{\partial z} + I\right)^{-1}\xi(t,z) + b(z)u(t),$$

$$y(t) = \left(\frac{\partial}{\partial z} + I\right)^{-1}\xi(t,z)\rfloor_{z=1}.$$

We can study the first system via this second one and in the latter the operator $(\partial/\partial z)(\partial/\partial z + I)^{-1}$ is a bounded operator on $L_2(0,1)$ (indeed is given by $\xi(t,z) \mapsto \xi(t,z) - \int_0^z e^{-(z-\sigma)}\xi(t,\sigma)\,d\sigma$).

Other examples can be found in problems of infinite queues and in systems governed by certain classes of integro-differential equations (compare with previous example).

3. Realizability criteria, general case. In this section we investigate the realizability problem when A is the infinitesimal generator of a C_0 semigroup of bounded operators on a Hilbert space $\mathscr{H}$. By the Hille–Yosida theorem [18] a necessary and sufficient condition that a closed linear operator A with domain $\mathscr{D}_0(A)$ dense be the infinitesimal generator of a C_0 semigroup, is that there exist positive real numbers M and β such that for every real $\lambda > \beta$, λ is in the resolvent set of A and

$$\|(I\lambda - A)^{-n}\| \leqq \frac{M}{(\lambda - \beta)^n}, \qquad n = 1, 2, \cdots.$$

If these conditions hold for all $\lambda > \beta$, then $(Is - A)^{-1}$ exists for all complex s with $\operatorname{Re} s > \beta$ and is given by $(Is - A)^{-1}x = \int_0^\infty e^{-st}e^{At}x\,dt$ for all $x \in \mathscr{H}$, $\|(Is - A)^{-n}\| \leqq M/(\operatorname{Re} s - \beta)^n$ for $\operatorname{Re} s > \beta$, and $\|e^{At}\| \leqq Me^{\beta t}$.

In case of a *regular realization*, $b \in \mathscr{H}$ and c is a bounded linear functional on $\mathscr{H}$. Then the observation procedure (i.e., $y(t) = c[x(t)]$) is somehow restricted since we cannot have point evaluations, or point evaluations of derivatives as $c(\cdot)$. Moreover, since b is just an element of $\mathscr{H}$ we can regard in general the equation

$$\frac{d}{dt}x(t) = Ax(t) + bu(t) \tag{1}$$

only in the weak sense (i.e., $x(\cdot)$ satisfies the integral equation given by the variation of constants formula, but not the differential equation). On the other hand, in a regular realization the properties of b and c are symmetric, a fact which has some implications for the desired *duality* in systems theory.

In case of a *balanced realization*, $b \in \mathscr{D}_0(A)$ and c is a linear functional defined on $\mathscr{D}_0(A)$ and such that $|c(x)| \leqq k(\|Ax\| + \|x\|)$ for all $x \in \mathscr{D}_0(A)$ and some constant k. Here we can regard equation (1) in the strong sense. Moreover, we can allow point evaluations, or point evaluations of derivatives as $c(\cdot)$; (for example, with A being $\partial^2/\partial z^2$ on $L_2[0, \infty)$ and $c(\cdot)$ being $\lfloor \partial/\partial z(\cdot)\rfloor_0$, or with A being $\partial/\partial z$ on $L_2[0, \infty)$ and c being evaluated at 0). However, in this case c and b do not have symmetric properties.

Remark 3.1. If c is a closed linear functional on $\mathscr{H}$ with $\mathscr{D}(c) \supseteq \mathscr{D}_0(A)$, then c satisfies the conditions stated above in the case of a balanced realization. To see this we have that $\mathscr{D}_0(A)$ with the norm $\|x\|_1 = \|Ax\| + \|x\|$ becomes a Banach space since A is closed. Then the restriction of c to $\mathscr{D}_0(A)$ is a closed linear operator, defined everywhere, and hence by the closed graph theorem is bounded. Hence there exists a k such that

$$|c(x)| \leqq k\|x\|_1 = k(\|Ax\| + \|x\|) \quad \text{for all } x \in \mathscr{D}_0(A).$$

The following theorem proves that in our setting (more specifically when the state space is a Hilbert space) the class of weighting patterns which admit balanced realizations is identical with the class of weighting patterns which admit regular realizations.

THEOREM 3. *A weighting pattern $T(\cdot)$ has a balanced realization if and only if it has a regular one. Moreover, the infinitesimal generators in the two cases can be taken to be the same.*

Proof. Suppose $T(\cdot)$ has a regular realization. Then there exist c_1, b_1, elements of $\mathscr{H}$, and a linear operator A generating a C_0 semigroup e^{At} on $\mathscr{H}$ such that

$$T(t) = \langle c_1, e^{At}b_1\rangle.$$

By the Hille–Yosida theorem there exists a positive real number β such that for every real $\lambda > \beta$, λ is in the resolvent set of A. Choose such a $\lambda > 1$. Then $(\lambda I - A)^{-1}$ is an everywhere defined bounded operator, and it maps the whole $\mathscr{H}$ onto $\mathscr{D}_0(A)$ since A is closed (see [21, p. 209]). Let

$$b = (\lambda I - A)^{-1}b_1.$$

Then $b \in \mathscr{D}_0(A)$ and $b_1 = (\lambda I - A)b$. Hence

$$T(t) = \langle c_1, e^{At}(\lambda I - A)b\rangle = \langle c_1, (\lambda I - A)e^{At}b\rangle.$$

Define the linear functional $c(\cdot)$ via

$$c(x) = \langle c_1, (\lambda I - A)x\rangle \quad \text{for} \quad x \in \mathscr{D}_0(A).$$

Then

$$|c(x)| \leqq \|c_1\|\,\|\lambda x - Ax\| \leqq \|c_1\|(\lambda\|x\| + \|Ax\|)$$
$$\leqq \lambda\|c_1\|(\|Ax\| + \|x\|).$$

Therefore $T(t) = c[e^{At}b]$ and this is a balanced realization.

Conversely, assume that $T(\cdot)$ has a balanced realization. Then there exist A, b, c as in the definition of a balanced realization so that $T(t) = c[e^{At}b]$. Consider $\mathscr{D}_0(A)$ with the inner product $\langle x, y\rangle_A = \langle Ax, Ay\rangle + \langle x, y\rangle$ for $x, y \in \mathscr{D}_0(A)$. This inner product induces the norm $\|x\|_A = (\|Ax\|^2 + \|x\|^2)^{1/2}$. Since A is closed,

$\mathscr{D}_0(A)$ is complete under the norm $\|\cdot\|_A$ and hence it is a Hilbert space with the inner product $\langle\cdot,\cdot\rangle_A$. For $x \in \mathscr{D}_0(A)$ we have

$$|c(x)| \leqq k(\|Ax\| + \|x\|) \leqq 2k(\|Ax\|^2 + \|x\|^2)^{1/2} = 2k\|x\|_A.$$

Thus $c(\cdot)$ is a bounded linear functional on the Hilbert space $\mathscr{D}_0(A)$ (with the above inner product). Hence by the Riesz representation theorem there exists $d \in \mathscr{D}_0(A)$ such that

$$c(x) = \langle d, x\rangle_A \quad \text{for all } x \in \mathscr{D}_0(A).$$

Hence $T(t) = \langle Ad, Ae^{At}b\rangle + \langle d, e^{At}b\rangle$.

Since the space we are working with is a Hilbert space, A^* also generates a C_0 semigroup which is exactly $(e^{At})^*$. Hence if we choose a real $\lambda > \beta$ (β from the Hille–Yosida theorem), then both $(\lambda I - A)$ and $(\lambda I - A^*)^{-1}$ are everywhere defined bounded operators.

We then have

$$\begin{aligned} T(t) &= \langle Ad, (A - \lambda I)e^{At}b\rangle + \lambda\langle Ad, e^{At}b\rangle + \langle d, e^{At}b\rangle \\ &= \langle Ad, e^{At}(A - \lambda I)b\rangle + \langle \lambda Ad + d, e^{At}b\rangle. \end{aligned}$$

If we let $(A - \lambda I)b = b_1 \in \mathscr{H}$, then $b = (A - \lambda I)^{-1}b_1$. Therefore

$$\begin{aligned} T(t) &= \langle Ad, e^{At}b_1\rangle + \langle \lambda Ad + d, e^{At}(A - \lambda I)-{}^1b_1\rangle \\ &= \langle Ad + (A^* - \lambda I)^{-1}(\lambda Ad + d), e^{At}b_1\rangle. \end{aligned}$$

Let $c_1 = Ad + (A^* - \lambda I)^{-1}(\lambda Ad + d)$. Then

$$T(t) = \langle c_1, e^{At}b_1\rangle$$

and obviously A, b_1, c_1 is a regular realization for $T(\cdot)$.

The last statement in the theorem is obvious from the above construction.

This theorem motivates the following definition.

DEFINITION. A weighting pattern $T(\cdot)$ is *realizable* if and only if it has a balanced realization.

We now give a preliminary description of the realizable weighting patterns.

THEOREM 4. *A necessary condition for $T(\cdot)$ to be realizable is that it be continuous and of exponential order. A sufficient condition is that it be locally absolutely continuous (i.e., absolutely continuous, on each bounded closed interval) and that $\dot{T}(t)$ (which then exists as an a.e. defined function) be of exponential order (i.e.,* ess sup $|\dot{T}(t)| \leqq Ke^{\alpha t}$ *for some positive K, α).*

Proof. Necessity. Since $T(\cdot)$ has a balanced realization, and hence by Theorem 3 it has a regular one, $T(t) = \langle c, e^{At}b\rangle$. Since e^{At} is strongly continuous we get that $T(\cdot)$ is continuous. Since $\|e^{At}\| \leqq Me^{\beta t}$, by the Hille–Yosida theorem we get that $T(\cdot)$ is of exponential order.

Sufficiency. Let $T(\cdot)$ be as in the hypothesis. Then for large enough σ, $e^{-\sigma t}\dot{T}(t) \in L_2(0, \infty)$. Hence the function $e^{-\sigma t}T(t)$ is in $L_2(0, \infty)$, it is locally absolutely continuous and its derivative belongs to $L_2[0, \infty)$. Take as b the function $e^{-\sigma t}T(t)$, and as Hilbert space $\mathscr{H}$ the space $L_2(0, \infty)$. The differentiation operator $A = \partial/\partial z$ on $L_2(0, \infty)$ is a closed operator with domain dense, generates the semigroup of left translations (restricted to $[0, \infty)$ of course) and its spectrum is the closed left half-plane (i.e., $\sigma(A) = \{s \in \mathbb{C} | \operatorname{Re} s \leqq 0\}$) (see [18]). Its domain consists of elements of $L_2(0, \infty)$ which are locally absolutely continuous and whose derivatives also belong to $L_2(0, \infty)$. Consider as c the linear functional whose action on a function f is described by

$$c[f] = f(0) \quad \text{(i.e., evaluation at 0)}.$$

Then c is defined on $\mathscr{D}_0(A)$. Moreover, for $x \in \mathscr{D}_0(A)$ we have

$$|c(x)|^2 = |x(0)|^2 \leqq \int_0^\infty 2|x(z)|\,|\dot{x}(z)|\,dz$$

$$\leqq \int_0^\infty |x(z)|^2\,dz + \int_0^\infty |\dot{x}(z)|^2\,dz.$$

So $|c(x)| \leqq (\|Ax\| + \|x\|)$. Hence b, c satisfy our requirements. Now $c[e^{At}b] = c[e^{-\sigma(t+z)}T(t+z)] = e^{-\sigma t}T(t)$ and therefore $T(t) = c[e^{(A+\sigma I)t}b]$. This is a balanced realization.

From the equation $T(t) = \langle c, e^{At}b\rangle$ we get via Laplace transform the equation

$$\tilde{T}(s) = \langle c, (Is - A)^{-1}b\rangle \quad \text{for} \quad \operatorname{Re} s > \beta,$$

where the β comes from the Hille–Yosida theorem. On the other hand, since $T(\cdot)$ is realizable we know it is of exponential order, say σ_0. Hence $\tilde{T}(\cdot)$ is analytic in $\operatorname{Re} s > \sigma_0$. Moreover, from Theorem 4 we have that $\sigma_0 \leqq \beta$ and by the Hille–Yosida theorem the function $\langle c, (Is - A)^{-1}b\rangle$ is analytic in $\operatorname{Re} s > \beta$. Let $\rho_0(A)$ be the connected component of $\rho(A)$ which contains the half-plane $\operatorname{Re} s > \beta$. Then by analytic continuation we see that $\tilde{T}(\cdot)$ is analytic for all $s \in \rho_0(A)$. So again (as in the bounded case) we arrive at the conclusion, that for any realization A, b, c of $T(\cdot)$ we must have the *spectral inclusion property*

$$\sigma(\tilde{T}) \subseteq \sigma_0(A),$$

where $\sigma_0(A)$ is the complement of $\rho_0(A)$ in $\mathbb{C}$.

The corresponding (to Theorem 4) conditions in the complex domain are described below.

THEOREM 5. *A necessary condition for $T(\cdot)$ to be realizable is that its Laplace transform $\tilde{T}(s)$ belong to $H^2(\prod_\rho^+) \cap H^\infty(\prod_\rho^+)$ for some $\rho > 0$. A sufficient condition is that $\tilde{T}(s) \in H^2(\prod_\rho^+)$ and $(s\tilde{T}(s) - T(0)) \in H^2(\prod_\rho^+)$ for some $\rho > 0$.*

Proof. This is an immediate consequence of Theorem 4, the Paley–Wiener theorem [7] and the Hille–Yosida theorem.

Example. The delayed step whose transform is e^{-s}/s is not realizable, whereas the delayed ramp e^{-s}/s^2 is realizable.

Remark 5.1. Suppose $T(\cdot)$ is continuous and of exponential order. Let $b \in L_2(0, \infty)$ be the function $e^{-\sigma t}T(t)$, where σ is large enough. Let $c_\lambda \in L_2(0, \infty)$ be the function $c_\lambda(z) = (2/\pi)\lambda/(\lambda^2 + z^2)$ and A be the differentiation operator. Then by Theorem 9.9 in [19] we have

$$e^{-\sigma t}T(t) = \lim_{\lambda \to 0} \langle c_\lambda, e^{At}b\rangle = \lim_{\lambda \to 0} \int_0^\infty e^{-\sigma(t+z)}T(t+z)c_\lambda(z)\,dz.$$

Hence $T(t) = \lim_{\lambda \to 0} \langle c_\lambda, e^{(A+\sigma I)t} b \rangle$. Therefore $T(\cdot)$ is the pointwise limit of a one-parameter family of realizable functions.

In order to give some better sufficient conditions for realizability we need the following well-known result [22] from the theory of $H^p(\prod^+)$ functions: If $f \in H^p(\prod^+)$, $1 \leqq p < \infty$, then it is represented by the proper Cauchy integral of its boundary values. That is, for $\operatorname{Re} s > 0$ we have the representation

$$f(s) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{f(i\omega)}{s - i\omega} \, d(i\omega).$$

THEOREM 6. *Let $T \in L_2(0, \infty)$ and be continuous. If $\tilde{T}(i\omega) = \overline{F_1(i\omega)} F_2(i\omega)$, where F_1, F_2 belong to $H^2(\mathbb{I})$, then T is realizable.*

Proof. Certainly $\tilde{T}(i\omega) \in L_1(\mathbb{I}; d\omega/2\pi)$. Hence since $T \in L_2(0, \infty)$ we have that

$$T(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{T}(i\omega) e^{i\omega t} \, d\omega \quad \text{a.e.}$$

But since both sides are continuous the equality holds everywhere. Thus

$$T(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \overline{F_1(i\omega)} e^{i\omega t} F_2(i\omega) \, d\omega.$$

But this equality says that if we take as Hilbert space $H^2(\mathbb{I})$, as b the function F_2, as c the function F_1 and as A the operator induced on $H^2(\mathbb{I})$, by multiplication by $i\omega$ we have

$$T(t) = \langle c, e^{At} b \rangle$$

(where the inner product is that of $L^2(\mathbb{I}, d\omega/2\pi)$). Hence A, b, c is a regular realization for T, and by Theorem 3, T is realizable.

Let us note that since the Fourier transform is a unitary map between $L_2(0, \infty)$ and $H^2(\mathbb{I})$, and multiplication by $e^{i\omega t}$ on $H^2(\mathbb{I})$ corresponds to left translation on $L_2(0, \infty)$, we can give also a realization of T in $L_2(0, \infty)$ by the left translation semigroup. Indeed, if we let

$$f_1 = \mathscr{F}^{-1}(F_1), \qquad f_2 = \mathscr{F}^{-1}(F_2)$$

and e^{At} = left translation semigroup restricted to $L_2(0, \infty)$, we have $T(t) = \langle f_1, e^{At} f_2 \rangle$. Moreover, we can give a realization in terms of the right translation semigroup on $L_2(0, \infty)$ since we also have

$$T(t) = \langle f_2, e^{A^* t} f_1 \rangle$$

with A, f_1, f_2 as above. (Note that $L_2(0, \infty)$ is invariant under right translations.)

Note. If T satisfies the conditions of Theorem 6, then by the Paley–Wiener theorem $\tilde{T} \in H^2(\prod^+)$. Hence

$$\tilde{T}(s) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{\tilde{T}(i\omega) \, d(i\omega)}{s - i\omega} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\overline{F_1(i\omega)} F_2(i\omega) \, d\omega}{s - i\omega}$$

and we could have used this approach in the proof.

COROLLARY 6.1. *Suppose T is continuous and of exponential order. If for some α the function $T_1(t) = e^{-\alpha t} T(t)$ satisfies the conditions of Theorem 6, then T is realizable.*

Proof. Of course if α is bigger than the exponential order of T then $e^{-\alpha t}T(t)$ belongs to $L_2(0, \infty)$. So we really have to check for the factorization only. Now, if T_1 satisfies Theorem 6, then $T_1(t) = \langle c, e^{At}b\rangle$. So $T(t) = \langle c, e^{(A+\alpha I)t}b\rangle$.

Remark 6.2. We see from the spectral inclusion property and from the Hille–Yosida theorem that the singularities of $\tilde{T}(\cdot)$, for any realizable T, are in some left half-plane. In all our constructions we used operators with spectrum large enough (in fact with spectrum some left half-plane) to include the singularities of a large class of realizable functions.

Remark 6.3. The conditions of Corollary 6.1 are weaker than those of Theorem 4. To see this observe first of all that continuity is required in both. Theorem 4 implies that for large enough σ the function $e^{-\sigma t}T(t) = T_1(t)$ belongs to $L_2(0, \infty)$, is locally absolutely continuous and its derivative belongs to $L_2(0, \infty)$. Hence $\tilde{T}_1(i\omega)$ and $i\omega\tilde{T}_1(i\omega)$ belong to $H^2(\mathbb{I})$ by the Paley–Wiener theorem. But $(1 - i\omega)\tilde{T}_1(i\omega) = g(i\omega)$ also belongs to $H^2(\mathbb{I})$. Hence

$$\tilde{T}_1(i\omega) = \frac{1}{1 - i\omega}\, g(i\omega) = \overline{\frac{1}{1 + i\omega}}\, g(i\omega)$$

and since $1/(1 + i\omega)$ belongs to $H^2(\mathbb{I})$ we see that T satisfies the conditions of Corollary 6.1.

We give another sufficient condition for realizability.

THEOREM 7. *If $T \in L_2(0, \infty)$ is continuous and $\tilde{T} \in H^1(\prod^+)$, then T is realizable.*

Proof. Since T is continuous, belongs to $L_2(0, \infty)$ and $\tilde{T}(i\omega) \in L^1(\mathbb{I};\, d\omega/2\pi)$ we have that

$$T(t) = \frac{1}{2\pi}\int_{-\omega}^{\omega} \tilde{T}(i\omega)e^{i\omega t}\, d\omega,$$

the equality holding everywhere. We know that $f \in H^1(\prod^+)$ if and only if $f = f_1 f_2$, where $f_1, f_2 \in H^2(\prod^+)$ (see [7, p. 134]). Hence there exist $F_3, F_2 \in H^2(\prod^+)$ such that $\tilde{T}(i\omega) = F_3(i\omega)F_2(i\omega) = \overline{F_1(i\omega)}F_2(i\omega)$, where $F_1 = \bar{F}_3 \in H^2(\prod^-)$. Hence

$$T(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{F_1(i\omega)e^{i\omega t}}F_2(i\omega)\, d\omega.$$

Thus by taking as Hilbert space $\mathscr{H}$ the space $L_2(\mathbb{I}; d\omega/2\pi)$, as A multiplication by $i\omega$; as c the function F_1 and as b the function F_2, we get

$$T(t) = \langle c, e^{At}b\rangle$$

and T is realizable.

Again using Fourier transforms we can give in the above case a realization of T on $L_2(-\infty, \infty)$ using the left translation semigroup or the right translation semigroup.

COROLLARY 7.1. *Let T be continuous and of exponential order. If for some α, $\tilde{T}(\cdot) \in H^1(\prod_\alpha^+)$, then T is realizable.*

Proof. By taking σ big enough, then, we can make the function $e^{-\sigma t}T(t)$ satisfy the conditions of Theorem 7.

Remark 7.2. The realization constructed in Theorem 7 has as infinitesimal generator the differentiation operator on $L_2(-\infty, \infty)$, whose spectrum is just the imaginary axis. So in this model we do not have a connected resolvent set.

On the other hand, in the realization of Theorem 6 we do have a connected resolvent set, but instead the spectrum becomes very large.

This last theorem indicates some other classes of realizable functions. We need first a standard definition for fractional derivatives in the L_2-sense, or equivalently for the Sobolev spaces of fractional order.

DEFINITION. *Let* $0 \leqq \gamma \leqq 1$. Then $f \in L_2(0, \infty)$ has an L_2-derivative of fractional order γ if and only if there exists $g \in L_2(0, \infty)$ such that $s^\gamma \tilde{f}(s) = \tilde{g}(s)$, where we always choose the branch of s^γ so that $\operatorname{Re} s^\gamma > 0$ for $\operatorname{Re} s > 0$. The space of all those f is usually denoted by H^2_γ.

COROLLARY 7.3. *If T is continuous and belongs to H^2_γ for $\frac{1}{2} < \gamma \leqq 1$, then T is realizable.*

Proof. We have that $\tilde{T}(s)$ and $s^\gamma \tilde{T}(s) = g(s) \in H^2(\prod^+)$. Hence $\tilde{T}(s) = (1/s^\gamma)g(s)$. Since for $\frac{1}{2} < \gamma \leqq 1$ and for all $\alpha > 0$ we have trivially that $1/(s + \alpha)^\gamma \in H^2(\prod^+)$, we get finally that for all $\alpha > 0$, $\tilde{T} \in H^1(\prod^+_\alpha)$ and the result follows from Corollary 7.1.

Finally we have the obvious generalization of Corollary 7.3.

COROLLARY 7.4. *If T is continuous and for some $\alpha > 0$, $e^{-\alpha t}T(t) \in H^2_\gamma$ with $\frac{1}{2} < \gamma \leqq 1$, then T is realizable.*

4. Canonical realizations. In the rest of this paper we shall restrict our study to weighting patterns T with bounded realizations. Moreover, we assume that T is realizable by the unilateral shift itself (i.e., we can take $k = 1$ in the construction of Theorems 1 or 2), or equivalently that $(1/s)\tilde{T}(1/s) \in H^2(\mathbb{D})$. This does not harm the generality of the discussion, since we can reduce the general case by a simple change of variable, to the above case. Indeed, if we define

$$\tilde{T}_k(s) = \sum_{i=0}^{\infty} \frac{a_i}{k^i} s^{-(i+1)} = k\tilde{T}(ks) = \mathscr{L}\left[T\left(\frac{t}{k}\right)\right],$$

where $\mathscr{L}$ denotes Laplace transform, then $\tilde{T}_k$ satisfies the above for some finite $k > 0$.

It is obvious that if a weighting pattern has one realization it has many. An element ϕ of a separable Hilbert space $\mathscr{H}$ *is called a cyclic vector* for a bounded operator A if and only if the linear span of ϕ, $A\phi$, $A^2\phi$, $A^3\phi, \cdots$ is dense in $\mathscr{H}$. One calls a realization (A, b, c) *canonical* whenever b is a cyclic vector for A, and c is a cyclic vector for A^*. However, we avoid the term "minimal" because many of the implications of this term are absent in the present setting. (Some authors prefer to call such a realization "controllable and observable" or "ε-controllable and ε-observable" or "weakly controllable and weakly observable".) If a weighting pattern T has a finite-dimensional realization, it has one with minimal dimension of the state space, which is called minimal. As is well known (see [2, pp. 105–115]) a finite-dimensional system is minimal if and only if it is controllable and observable (canonical). Moreover, any two minimal realizations differ by a change of basis in the state space, and the spectral properties of A in any minimal realization are *uniquely* determined by the weighting pattern. Here the situation is much more complicated. It happens that a canonical realization is much more loosely specified by the weighting pattern.

We start with a construction of a canonical realization starting from a given one.

THEOREM 8. *Let* $T(t) = \langle c, e^{At}b \rangle$. *Let* M *be the closed linear span of* c, A^*c, $A^{*2}c, \cdots$ *in* $\mathscr{H}$ *(a separable Hilbert space). Let* P_M *be the orthogonal projection on* M. *Then* (i) $T(t) = \langle c, e^{P_MAP_Mt}P_Mb \rangle$.

Now let N *be the closed linear span of* $P_Mb, \cdots, (P_MAP_M)^iP_Mb, \cdots$ *in* M *and let* P_N *be the orthogonal projection on* N. *Then* (ii) $T(t) = \langle P_Nc, e^{(P_NAP_N)t}P_Mb \rangle$. *Moreover,* N *is the closed linear span of* $P_Mb, \cdots, (P_NAP_N)^iP_Mb, \cdots$ *and the closed linear span of* $P_Nc, \cdots, (P_NAP_N)^{*i}P_Nc$.

Proof. It is obvious that M is the smallest closed subspace of $\mathscr{H}$ which contains c and is invariant under A^*. Hence $M^\perp$ is invariant under A. Hence $A(I - P_M)x \in M^\perp$ for all $x \in \mathscr{H}$. So $P_MA(I - P_M)x = 0$ for all $x \in \mathscr{H}$. Hence

$$P_MA = P_MAP_M. \tag{2}$$

Using (2) we get $(P_MAP_M)^iP_Mb = P_MA^ib$.

Hence $\langle c, e^{P_MAP_Mt}P_Mb \rangle = \langle c, P_Me^{At}b \rangle = \langle c, e^{At}b \rangle = T(t)$ and this proves (i). Similarly, N is the smallest closed subspace of M which contains P_Mb and is invariant under P_MAP_M. Then for every $x \in \mathscr{H}$, $(I - P_N)P_MAP_MP_Nx = 0$. Thus

$$P_NAP_N = P_NP_MAP_MP_N = P_MAP_MP_N = P_MAP_N. \tag{3}$$

Using (2) and (3) we obtain

$$\begin{aligned} (P_NAP_N)^iP_Mb &= P_MAP_N(P_NAP_N)^{i-1}P_Mb = P_MAP_NP_NAP_N(P_NAP_N)^{i-2}P_Mb \\ &= P_MAP_MAP_N(P_NAP_N)^{i-2}P_Mb = P_MA^2P_N(P_NAP_N)^{i-2}P_Mb \\ &= P_MA^iP_NP_Mb = P_MA^iP_Mb = (P_MAP_M)^iP_Mb. \end{aligned} \tag{4}$$

Thus

$$\begin{aligned} \langle P_Nc, e^{(P_NAP_N)t}P_Mb \rangle &= \langle c, P_Ne^{(P_NAP_N)t}P_Mb \rangle = \langle c, e^{(P_MAP_N)t}P_Mb \rangle \\ &= \langle c, P_Me^{At}b \rangle = \langle c, e^{At}b \rangle = T(t), \end{aligned}$$

and this proves (ii).

From (3) we get

$$(P_NAP_N)^{*i}P_Nc = P_NA^{*i}c. \tag{5}$$

The first assertion in the last statement is proved by (4), i.e., by the fact that $(P_NAP_N)^iP_Mb = (P_MAP_M)^iP_Mb$ for all i. The second is an easy consequence of (5) and of the cyclicity of c for A^*.

Here, as we assumed in the beginning of this section, if $T(\cdot)$ is realizable, it can be realized by the shift (unilateral or bilateral). Some important questions which arise naturally are the following. It is obvious that the realization given by Theorems 1 and 2 is controllable. Also we know that the spectrum of the unilateral shift is the closed unit disk. Given a weighting pattern T, how simple can the spectrum of the infinitesimal generator A of a realization be? How small can the spectrum be? If we take a canonical realization (A, b, c), is the spectrum of A uniquely determined by T? How are all canonical realizations of a given T related to each other? When can we make the resolvent set of the infinitesimal generator A connected?

An immediate observation, which gives, however, some indication of the interplay of the notions described in these questions is the following: We can realize any such T by the bilateral shift. Such a realization is obviously noncanonical. On the other hand, since the spectrum of the bilateral shift is just $\mathbb{T}$, the spectrum can be considered as "simple". However, the resolvent set is not connected.

Given a weighting pattern T we have the *shift realization* as described in Theorems 1 and 2:

$$\frac{d}{dx} x(t) = Ux(t) + bu(t),$$

$$y(t) = \langle c, x(t) \rangle,$$

where $x(t) \in l_2(\mathbb{Z}^+)$ for all t, $b = \{1, 0, 0, \cdots\}$, U is the unilateral shift and $c = \{T(0), T^{(1)}(0), T^{(2)}(0), \cdots\}$. Here b is obviously a cyclic vector for U. It is immediately seen as a consequence of Theorem 8, that if we let M be the closed linear span of c, $U^*c, \cdots, U^{*i}c, \cdots$ and P_M the projection on M, then $(P_M U P_M, P_M b, c)$ is a canonical realization of T, with state space M. We can write the "shift realization" in terms of H^2 functions as follows:

$$\frac{d}{dt} x(t, s) = sx(t, s) + u(t),$$

$$y(t) = \int_{\mathbb{T}} s\tilde{T}(s) x(t, s)\, d\mu(s),$$

where $x(t, \cdot) \in H^2$ ($H^2(\mathbb{D})$ or $H^2(\mathbb{T})$). (Compare with [17] where similar equations are used.) Under the isomorphism between $l_2(\mathbb{Z}^+)$ and $H^2(\mathbb{D})$, c corresponds to $(1/s)\tilde{T}(1/s)$ which equals $\overline{s\tilde{T}(s)}$ *on* $\mathbb{T}$ (since $\tilde{T}(\cdot)$ has real Taylor coefficients). U corresponds to multiplication by s, U^* corresponds to the mapping:

$$f(s) \mapsto \frac{f(s) - f(0)}{s} \quad \text{on } H^2(\mathbb{D}).$$

We need a few well-known facts from the theory of H^2-functions and Toeplitz operators. The reader is referred to [7], [8] and [10] for further details. A function $f \in H^2(\mathbb{D})$ is called *inner* if $|f(e^{i\theta})| = 1$ a.e. A function $f \in H^2(\mathbb{D})$ is called *outer* if it is a cyclic vector for the shift in $H^2(\mathbb{D})$ (i.e., the linear span of the functions f, sf, $s^2 f, \cdots$ is dense in $H^2(\mathbb{D})$). A *Blaschke product* is a function of the form

$$B(s) = s^k \prod_{j=1}^{\infty} \frac{a_j - s}{1 - \bar{a}_j s} \frac{\bar{a}_j}{|a_j|},$$

where k is a nonnegative integer and the a_j are complex numbers (not necessarily distinct) such that $0 < |a_j| < 1$, $\sum_{j=1}^{\infty} (1 - |a_j|) < \infty$. A *singular function* is a function of the form

$$S(s) = \exp\left(- \int \frac{e^{i\theta} + s}{e^{i\theta} - s}\, d\mu(\theta) \right),$$

where μ is any positive finite measure on $[0, 2\pi]$ which is singular with respect to the normalized Lebesgue measure. Every $f \in H^2(\mathbb{D})$ has a factorization $f = \phi \cdot h$, where ϕ is *inner* and h is *outer*. The factors are unique up to constant factors of modulus one. Any inner function has a factorization $\phi = cBS$, where c is a constant of modulus one, B is a Blachke product and S is a singular function. An *inner* function is *normalized* if we choose $c = 1$, or equivalently if we require the first nonzero Taylor coefficient to be real and positive. Beurling showed that to every closed subspace M of $H^2(\mathbb{D})$ which is invariant under the shift (i.e., under multiplication by s) there corresponds a unique normalized inner function ϕ such that $M = \phi H^2(\mathbb{D})$ and conversely. We also have the corresponding facts for $H^2(\mathbb{T})$.

A Laurent operator on $l_2(\mathbb{Z})$ has a matrix representative which is constant on diagonals (i.e., $\alpha_{ij} = a_{i-j}$) and corresponds to multiplication by $\phi(s) = \sum_{i=-\infty}^{\infty} a_i s^i$ on $L^2(\mathbb{T})$ (where $a_i = \alpha_{i+k,k}$). A Toeplitz operator A on $l_2(\mathbb{Z}^+)$ has a similar matrix representative (which is infinite in only one direction). If $P: L^2(\mathbb{T}) \to H^2(\mathbb{T})$ is the associated projection, then for all $f \in H^2(\mathbb{T})$ we have

$$Af = P(\phi \cdot f).$$

The only way the "shift realization" can be canonical is if c is a cyclic vector for U^* (i.e., for the backward shift) or equivalently if $(1/s)\tilde{T}(1/s)$ is a cyclic vector for the backward shift on $H^2(\mathbb{D})$. (See also Fuhrmann [13, Thm. 2.6].) In [5] the authors studied cyclic vectors of the backward shift very extensively. We are going to use some of their results and we refer to [5] for further details. There exist many cyclic vectors for the backward shift on $H^2(\mathbb{D})$, as well as noncyclic ones. The rational functions are noncyclic. The authors give several ways of constructing cyclic vectors. Any H^2-function with isolated branch points on $\mathbb{T}$ is a cyclic vector and any function with lacunary Taylor series and square summable Taylor coefficients is also a cyclic vector. Since $f(s) \in H^2(\mathbb{D})$ is a cyclic vector for the backward shift if and only if $sf(s)$ is one, we have two cases to consider: namely, the case when $\tilde{T}(1/s)$ is a cyclic vector for the backward shift and the case when $\tilde{T}(1/s)$ is noncyclic.

We would like to close this section with some important remarks about the cyclic and noncyclic case. Let Q be the subset of the realizable transfer functions, for elements of which the set of real numbers k, such that $\tilde{T}(k/s)$ is in $H^2(\mathbb{D})$, is open. Let G be the complement of G in the set of realizable functions. Theorem 2.2.4 in [5] reads as follows: *If f is holomorphic in $|s| < R$ for some $R > 1$, then f is either cyclic or a rational function.* Since $k > \sigma_0$ for elements of Q, an immediate consequence of the above theorem is that the elements of Q are either cyclic or rational functions. Then in G we have either cyclic or noncyclic but not rational functions, as illustrated in Fig. 1.

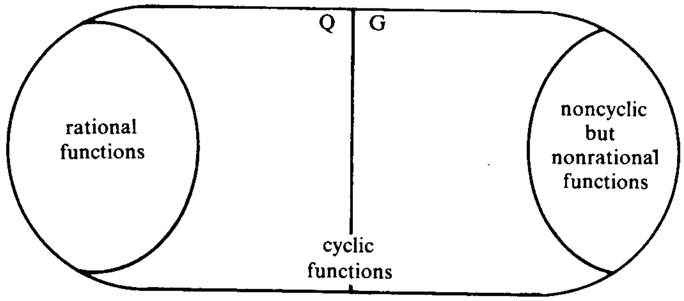

FIG. 1

Also from [5] we have that the set of cyclic vectors is dense in $H^2(\mathbb{D})$ as is the set of noncyclic vectors. However, the set of noncyclic vectors is a set of the first category, whereas the set of cyclic vectors is not. Hence the noncyclic vectors are somehow much more rare than the cyclic ones. Moreover, an element of $H^2(\mathbb{D})$ is noncyclic if and only if there exists a sequence of rational functions (satisfying special conditions [5, Thm. 4.1.1]) which converges to it in the $L^2(\mathbb{T})$-norm, a fact which indicates that the noncyclic case is very much like the rational functions whereas the cyclic situation is new, harder, and potentially more interesting.

5. The noncyclic case. Now consider the case where $\tilde{T}(1/s)$ is not a cyclic vector for the backward shift. This case is treated by Fuhrmann [13] in detail; however, there are some additional facts given here about the spectrum of A. (H^2 stands for $H^2(\mathbb{D})$ or $H^2(\mathbb{T})$.)

To proceed we need the following theorem from [5, p. 56].

THEOREM 9 ([5]). *$f \in H^2(\mathbb{D})$ is noncyclic if and only if there exist $g \in H^2(\mathbb{D})$ and an inner function ϕ such that $f(e^{i\theta}) = e^{-i\theta}\overline{g(e^{i\theta})}\phi(e^{i\theta})$ a.e. on $\mathbb{T}$. Moreover, if we require that ϕ be normalized and relatively prime to the inner factor of g, then ϕ and g are uniquely determined. In this case the closed subspace generated by $U^{*n}f$, $n = 0, \cdots, \infty$, is precisely $(\phi H^2(\mathbb{D}))^\perp$.*

The normalized inner function ϕ thus uniquely associated with each noncyclic (for the backward shift) vector f is called the *associated inner function* of f.

We see immediately that the subspace M of $l^2(\mathbb{Z}^+)$ which is the state space for the canonical realization $(P_M U P_M, P_M b, c)$ derived from the "shift realization" corresponds to the closed subspace of H^2 generated by $\{U^{*n}(1/s)\tilde{T}(1/s)\}_{n=0}^{\infty}$ which we also call M. Hence applying Theorem 9 we get that

$$M = (\phi H^2)^\perp,$$

where $\phi(e^{i\theta})\overline{g(e^{i\theta})} = \tilde{T}(e^{-i\theta}) = \overline{\tilde{T}(e^{i\theta})}$ a.e. on $\mathbb{T}$ (since $\tilde{T}(e^{i\theta})$ has real Fourier coefficients), and ϕ and g are uniquely determined by Theorem 9.

We need another theorem now from [6].

THEOREM 10 [6]. *Let $K = \phi H^2$, i.e., K is a closed subspace of H^2 invariant under the shift U. Let $M = (\phi H^2)^\perp$. Then the spectrum of U restricted on M is the set s_ϕ which consists of*

(i) *all the points in $\mathbb{C}$ with $|\lambda| < 1$, where $\phi(\lambda) = 0$,*

(ii) *all the points in $\mathbb{C}$ with $|\lambda| = 1$, where $\phi(\cdot)$ is* not *continuable analytically across the boundary $\mathbb{T}$ of $\mathbb{D}$ at λ.*

Using Theorems 9 and 10 we see that the spectrum of the infinitesimal generator of the canonical realization $(P_M U P_M, P_M b, c)$ *is uniquely determined by T.* Namely, the spectrum consists of the zeros of ϕ in $\mathbb{D}$ (which coincide with the zeros of the *Blaschke product* part of ϕ) and the points of $\mathbb{T}$ through which ϕ is not continuable analytically outside the unit circle (which coincide with the union of the support of the measure of $\mathbb{T}$ which is associated with the *singular* part of ϕ and the set of points of $\mathbb{T}$ which are accumulation points of the sequence of zeros of ϕ (see [7, pp. 68–69])).

Recall now that $\tilde{T}(e^{i\theta}) = \bar{\phi}(e^{i\theta})g(e^{i\theta})$ a.e. on $\mathbb{T}$, where $g \in H^2(\mathbb{T})$. Since ϕ is inner we get $\tilde{T}(e^{i\theta}) = g(e^{i\theta})/\phi(e^{i\theta})$ a.e. on $\mathbb{T}$. When $\tilde{T}$ has a meromorphic continuation in $\mathbb{D}$ we have $\tilde{T}(s) = g(s)/\phi(s)$ in $\mathbb{D}$. Since g, ϕ are analytic on $\mathbb{D}$ and ϕ is relatively prime to the inner factor of g, it follows that the singularities of $\tilde{T}(\cdot)$ in $\mathbb{D}$ are exactly the zeros of $\phi(\cdot)$ in $\mathbb{D}$ (with the same multiplicity as well). On the other hand, since $e^{-i\theta}\tilde{T}(e^{-i\theta}) \in (\phi H^2(\mathbb{T}))^\perp$ and $e^{-i\theta}\tilde{T}(e^{-i\theta})$ is a *noncyclic vector* for the backward shift by assumption, we know [5, pp. 58–59, Cors. 3.1.8 and 3.1.10] that the set of points of $\mathbb{T}$, through which ϕ is analytically continuable, coincides with the set of points of $\mathbb{T}$, through which $(1/s)\tilde{T}(1/s)$ is analytically continuable. Hence the set of points of $\mathbb{T}$, through which ϕ is analytically continuable, coincides with the set of points of $\mathbb{T}$, through which $\tilde{T}(1/s)$ is analytically continuable, which is the same as the set of points of $\mathbb{T}$, through which $\tilde{T}$ is analytically continuable (in the reverse direction). So in this case we arrived at the conclusion that the spectrum of $P_M U P_M$ consists of the set of points of $\mathbb{D}$ which are singularities of $\tilde{T}$ and of points of $\mathbb{T}$ through which $\tilde{T}$ cannot be continued analytically. Obviously the last set is what we have defined as $\sigma(\tilde{T})$. Hence we obtain

$$\sigma(\tilde{T}) = \sigma(P_M U P_M).$$

We have thus proved the following theorem.

THEOREM 11. *Let T be given weighting pattern with $(1/s)\tilde{T}(1/s) \in H^2(\mathbb{D})$, such that $\tilde{T}(1/s)$ is not a cyclic vector for the backward shift on $H^2(\mathbb{D})$. Then there exists a canonical realization of T with the spectrum of the infinitesimal generator of the realization being exactly s_ϕ, where ϕ is the associated inner factor of $(1/s)\tilde{T}(1/s)$. If $M = (\phi H^2)^\perp$, this realization is contructed by taking as c the function $(1/s)\tilde{T}(1/s)$, as b the projection of 1 on M and as A the restriction of the forward shift on M. Moreover, if T has a meromorphic continuation in $\mathbb{D}$, this spectrum is just $\sigma(\tilde{T})$.*

We see that in the above case the "spectral inclusion property" becomes in fact an equality, i.e., the spectrum of the infinitesimal generator of the realization described in Theorem 11 is *minimal*. This motivates the following definition.

DEFINITION. A canonical realization (A, b, c) of a weighting pattern T is called *S-minimal* (S from spectrum) if $\sigma(A) = \sigma(\tilde{T})$ (multiplicities counted whenever possible).

These considerations lead us to a trivial corollary of Theorem 11.

COROLLARY 11.1. *Any T which has the "shift realization" and is such that $\tilde{T}(1/s)$ is not a cyclic vector of the backward shift on H^2, and where $\tilde{T}$ has a meromorphic continuation in $\mathbb{D}$, has an* S-minimal *realization, with A having a connected resolvent set.*

We do not have a complete picture for the relation between canonical (resp. S-minimal) realizations of the same weighting pattern T, in this case. However, a partial analysis indicates that the noncyclic case is very similar to the rational case.

6. The cyclic case. The cyclic case is very interesting since it reflects a number of physically interesting phenomena; for example, transfer functions with branch points and branch cuts. Transfer functions like these arise in systems governed by partial differential equations. Hence an understanding of the cyclic case should undoubtedly shed some light towards the realization problem for distributed systems.

This case is more difficult, since the associated inner factor of $\tilde{T}$ which proves so crucial in the noncyclic case is now trivial. That is, the shift realization for cyclic transfer functions is already canonical. However, the spectrum of this realization is far from being equal to $\sigma(\tilde{T})$, unless we have a pathological transfer function with branch points on a dense subset of $\mathbb{T}$. *Hence canonical by no means implies S-minimal.* (Again compare with Fuhrmann [13, Cor. 2.7] who observes the nonuniqueness of the spectrum.)

It is apparent from the spectral inclusion property that all the points on the branch cuts (if the transfer function has branch points) are included in the spectrum of any infinitesimal generator A with connected resolvent set which realizes the transfer function. However, branch cuts are not uniquely defined (e.g., for $1/\sqrt{s^2+1}$ any curve between i and $-i$ can be a branch cut provided it has no self intersection). *Hence the set $\sigma(\tilde{T})$ is not uniquely determined* and consequently there is not a unique "minimal spectrum" for the infinitesimal generators of the realizations. A reasonable expectation is that the spectrum of an S-minimal realization (provided one exists) will be unique if there are no branch points and otherwise will be unique modulo the branch cuts.

We conclude this section with an example of a realization for the Bessel function of zeroth order $\mathcal{J}_0(\cdot)$ which is S-minimal. It is easy to verify that

$$\mathcal{J}_0(t) = \sum_{m=0}^{\infty} \frac{(-1)^m (t/2)^{2m}}{(m!)^2}$$

satisfies our realizability criteria. $\tilde{\mathcal{J}}_0(s) = 1/\sqrt{s^2+1}$ has branch points on $\pm i$, hence $\tilde{\mathcal{J}}_0(\cdot)$ is a cyclic vector. We must take the branch cut in the finite plane. We are after a canonical realization whose infinitesimal generator has spectrum exactly the line between i and $-i$. Recalling that $\exp(\frac{1}{2}(s-1/s)t)$ is a generating function for the Bessel functions of integral order, i.e., that

$$\exp\left(\frac{1}{2}\left(s - \frac{1}{s}\right)t\right) = \sum_{n=-\infty}^{\infty} \mathcal{J}_n(t) s^n,$$

and using Laurent operators we see that for

$$A = \frac{1}{2}\begin{bmatrix} \ddots & -1 & & & 0 \\ \ddots & 0 & -1 & & \\ \ddots & 1 & 0 & -1 & \ddots \\ & & 1 & 0 & \ddots \\ & 0 & & 1 & \ddots \end{bmatrix},$$

$$e^{At} = \begin{bmatrix} & \vdots & \vdots & \vdots & \vdots & \\ \cdot\, \mathcal{J}_1(t) & \mathcal{J}_0(t) & \mathcal{J}_{-1}(t) & \mathcal{J}_{-2}(t) & & \cdot\ \cdot \\ \cdot\ \cdot & \mathcal{J}_1(t) & \mathcal{J}_0(t) & \mathcal{J}_{-1}(t) & & \cdot\ \cdot \\ \cdot\ \cdot & \mathcal{J}_2(t) & \mathcal{J}_1(t) & \mathcal{J}_0(t) & \mathcal{J}_{-1}(t)\,\cdot & \\ & \vdots & \vdots & \vdots & \vdots & \end{bmatrix}.$$

Hence the above A along with $b = c = [\cdots 0 \quad 0 \quad 1 \quad 0 \quad 0 \cdots]$ gives a realization for $\mathscr{T}_0(\cdot)$ in $l_2(\mathbb{Z})$. That the spectrum of A is exactly $[i, -i]$ is a well-known fact from [10]. However, this realization is *not* canonical (it is easy to verify that the vector $[\cdots 0 \quad 0 \quad 1 \quad 0 \quad 1 \quad 0 \quad 0 \cdots]$ is orthogonal to $A^i b$ for all i). We are going to use Theorem 8 to reduce the above realization to a canonical one. So let M be the closed linear span of $c, A^*c, A^{*2}c, \cdots$ in $l_2(\mathbb{Z})$. Then M is A^* invariant. But since $A^* = -A$ it is also A invariant, i.e., M reduces A. Since $b = c$, we get by Theorem 8 that $A_1 = A$ restricted to M, b, c is a realization of $\mathscr{T}_0$ which is obviously canonical. Let $\lambda \in \rho(A)$. Then $\lambda I - A$ has a bounded inverse. But since M reduces A, $\lambda I - A_1$ has also a bounded inverse. Hence

$$\rho(A) \subseteq \rho(A_1).$$

Therefore $\rho(A_1)$ is connected. Using the spectral inclusion property, the fact that $\sigma(A) = \sigma(\tilde{\mathscr{T}})$ and the above relation, we have

$$\sigma(\tilde{\mathscr{T}}) \subseteq \sigma(A_1) \subseteq \sigma(A) = \sigma(\tilde{\mathscr{T}}).$$

Thus

$$\sigma(A_1) = \sigma(\tilde{\mathscr{T}})$$

and A_1, b, c is an S-minimal realization.

This example shows that S-minimal realizations can exist for cyclic functions as well. It also shows that there exists no Hilbert space analogue of the finite-dimensional state space isomorphism theorem between two canonical realizations of the same T unless further assumptions are made. (See Helton [14].)

Notice also that nearly the same realization will work for the Bessel function $\mathscr{T}_n$, where

$$\tilde{\mathscr{T}}_n(s) = \frac{1}{\sqrt{s^2+1}}\left(\frac{1}{s+\sqrt{s^2+1}}\right)^n$$

provided we keep A, b as above and take $c = \{\cdots 0 \quad 0 \quad 1 \quad 0 \quad 0 \cdots\}$, with the 1 in the nth place.

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