

THESIS REPORT

Master's Degree

**A Frequency Domain Design for the Control of
a Distributed Parameter System**

by B. Frankpitt

Advisor: J.S. Baras and C.A. Berenstein

M.S. 93-8



*Sponsored by
the National Science Foundation
Engineering Research Center Program,
the University of Maryland,
Harvard University,
and Industry*

**A Frequency Domain Design for the Control of a
Distributed Parameter System**

by
Bernard Frankpitt

Thesis submitted to the Faculty of The Graduate School
of The University of Maryland in partial fulfillment
of the requirements for the degree of
Master of Science
1993

Advisory Committee:

Prof. J. S. Baras	Chairman/Advisor
Prof. C. A. Berenstein	Advisor
Prof. P. S. Krishnaprasad	

ABSTRACT

Title of Thesis: A Frequency Domain Design for the control of a Distributed Parameter System

Name of degree candidate: Bernard Adrian Frankpitt

Degree and Year: Master of Science, 1993

Thesis directed by: Professor John Baras and
Professor Carlos Berenstein

This thesis presents a new approach to frequency domain design of robust controllers for distributed parameter systems. The central idea is to use techniques from complex analysis, that were developed for the solution of the Corona Problem, for the solution to the Bezout equation that arises in the parameterization of stable feedback controllers. An algebraic reformulation of the Bezout equation allows the solution to be computed from the solution of an auxiliary $\bar{\partial}$ equation with a Carleson measure as the inhomogeneous term.

We first show how the Bezout equation arises in the problem of feedback controller design, then we present techniques that are used for its solution. An example is given in which the solution to a Bezout equation derived from an unstable plant with a delay is calculated. Finally this example is extended to show how the techniques developed for the Bezout equation may be used to calculate a sub-optimal solution to the Nehari Problem for a single-input single-output system.

TABLE OF CONTENTS

<u>Section</u>	<u>Page</u>
Table of Contents	ii
List of Figures	iv
Notation	v
1 Introduction	1
2 The Standard Problem	5
2.1 The Standard Problem for an Unstable Plant	5
2.2 Doubly Coprime Factorizations	9
3 Solving the Analytic Bezout Identity	14
3.1 An Algebraic Reformulation of the Bezout Equation	16
3.2 Constructing Bounded Solutions to the Inhomogeneous $\bar{\partial}$ Equation	26
3.3 Examples	41
3.3.1 Example 1	41
3.3.2 Example 2	43

4	The Nehari Problem	49
5	BIBLIOGRAPHY	54

LIST OF FIGURES

<u>Number</u>		<u>Page</u>
2.1	Block diagram for the standard problem	6
3.1	Koszul complex for $m = 2$	20
3.2	m -adic Subdivision of the Half Plane ($m = 3$)	36
3.3	Test pulse applied to deconvolution filters	47
3.4	Pulse responses for filters representing solutions to the Bezout equation	48
4.1	Pulse responses for closed loop transfer function and sensitivity function of closed loop system	53

NOTATION

\tilde{z} The "real conjugate" of z . If \bar{z} denotes the complex conjugate of z , then $\tilde{z} = -\bar{z}$.

\mathcal{H} The half plane $\{z = x + iy \mid x > 0\}$.

H_∞ The algebra of functions of a complex variable that are both analytic and bounded on the right half plane.

$d/d\bar{z}$ Antiholomorphic derivative. If the complex plane is identified with \mathbb{R}^2 by $z = x + iy$, and if $f(z)$ is a function on the complex plane, then $(d/d\bar{z})f(z) = 1/2(df/dx + i df/dy)$.

$\bar{\partial}$ (i) The $\bar{\partial}$ equation is the equation

$$\frac{d}{d\bar{z}}f(z) = \mu$$

which is understood to hold in a distributional sense for a function f and a measure μ on the complex plane.

- (ii) If $f(z)$ is a function on the complex plane (considered as a real manifold of dimension 2), then $\bar{\partial}$ is the exterior derivative

$$\bar{\partial}f(z) = \frac{df}{d\bar{z}} dz.$$

$\rho(z_1, z_2)$ Pseudo-hyperbolic metric in the half plane:

$$\rho(z_1, z_2) = (z_1 - z_2)/(z_1 - \bar{z}_2).$$

$\ell(Q)$ If Q is a square, then $\ell(Q)$ is the length of one side.

1 The identity matrix. For a general module this is a diagonal matrix with the multiplicative identity of the base ring appearing on the diagonal.

CHAPTER
1

Introduction

Robust control of linear systems has been an important research area in control theory for the last 10 years. In 1981 G.Zames [29] introduced a new methodology for designing feedback controllers for linear systems when he recognized that robust control could be achieved by designing a controller that minimized a certain norm of the sensitivity operator. In the case of a linear time invariant plant the sensitivity operator is a convolution operator that maps $L^2[0, \infty)$ into itself; the domain of the operator models a disturbance input signal, and the range the plant output signal. It is a consequence of the Paley Wiener Schwartz theorem [18] that if Laplace transforms of the signal spaces are considered, then the sensitivity operator acts on the transformed signals as a multiplicative operator with symbol in the function algebra H_∞ , and that the operator norm is equal to the H_∞ norm of the symbol. Since the work in this thesis is set entirely in the frequency domain, we call the multiplicative operator on the Laplace transforms of the signals the sensitivity operator. Zames' approach to robust control is to minimize the H_∞ norm of the symbol after it is

weighted by a multiplicative factor. He solved this norm minimization problem by applying a bilinear transform to the feedback operator which converts the sensitivity operator of a stable closed loop system to an affine function of an H_∞ parameter. The optimal feedback operator is obtained by applying the inverse bilinear transform to a value of the parameter that minimizes the norm of the affine expression.

By 1987 H_∞ design — as the methodology is now known — had attracted wide interest and the original scheme of Zames had been extended and refined with the discovery of important links to operator theory, and in particular a link to the Nehari Interpolation Problem. This development is the subject of [14], Bruce Francis' monograph, which provides the point of departure of this thesis. In 1989 Tadmor [26] and Doyle et al. [7] independently presented a new approach to the problem of computing robust controllers that avoids the bilinear transformation and resulting interpolation problem. Instead, the optimal controller is constructed from the solutions of two Riccati equations that are derived from a state space model. The algorithms presented in [7] currently provide the most popular method for designing robust controllers for finite dimensional plants.

Work on robust control for finite dimensional plants has been paralleled by work in the infinite dimensional or distributed parameter case. Frequency domain approaches have been presented by Flamm and Mitter [10] and by Foias and Tannenbaum [11] [12] [13]. Van Kuelen [27] has extended the results in [7].

The principal contribution in this thesis is the presentation of a new fre-

quency domain method for calculating robust, stabilizing controllers for linear distributed parameter systems. The central feature of this method is the use of a constructive solution to the Corona Problem to solve the Bezout Equation over the ring of H_∞ functions. The ingredients are drawn from a number of sources. In [3] Baras and Dewilde presented the concept of a co-prime factorization of an irrational transfer function, and in [1] Baras was the first to show that the existence of a parameterization of stabilizing controllers for distributed systems with strongly coprime factorizations is a direct consequence of the Corona Theorem. In [2] the connection between control theory and the Corona Problem is elaborated, and reference is made to the work of Berenstein et al. that is mentioned below. In [17] Hörmander provided an algebraic technique for reformulating the Corona Problem in terms of the $\bar{\partial}$ equation, and in [20] Jones provided a constructive solution for the $\bar{\partial}$ equation. There is a thorough discussion of the Corona Problem in chapter 8 of Garnett's book [15], and this is the primary source of the material presented in Section 3.2. Another source of motivation has been the work of Berenstein, Taylor and Struppa [5], [4], [25]. Although these authors use the same approach as is presented here to solve Bezout Identities over rings of analytic functions with bounded growth, the singular integral methods that they employ for the solution of the $\bar{\partial}$ equation can not be applied to the H_∞ case.

The presentation of these results, while close to the sources cited, does require some modification: in particular, technical arguments are required to elucidate

the exact nature of the measures that are to be approximated in the computations, and the estimates of the norms of the solutions to the Bezout Equations play a special role in the robust control setting.

The remainder of the work is divided into three chapters: Chapter 2 describes the standard H_∞ control problem from the point of view of Francis [14], and shows how the solution of a Bezout Equation over H_∞ functions allows the stable feedback controllers to be parameterized. Chapter 3 presents the theory behind the solution of the Bezout Equation. And Chapter 4 shows how the theory may be applied to the solution of the Nehari problem for the single input, single output case. Some further work is required before the multi-input multi-output case can be dealt with completely; at present the parameterization of the stabilizing controllers can be calculated by the method presented in Chapter 2, but the solution of the Nehari problem will need a generalization of the theory presented in Chapter 3.

The Standard Problem

The standard problem referred to in the chapter title is a general framework for studying questions of robust control. In particular the sensitivity and mixed sensitivity approaches of Zames [29] [30] to robust stabilization can be recast in this framework. In [14] Bruce Francis treats at length the standard problem for finite dimensional multivariable systems. He uses algebraic, state space techniques to reduce the problem of finding an optimal control in the H_∞ sense for the standard problem to a Nehari problem, which is then solved by a spectral synthesis approach, again with the use of state space techniques. This thesis takes the framework used by Francis for finite dimensional systems and applies it to systems that have irrational transfer functions. To accomplish this, the state space methods employed by Francis are replaced by frequency domain techniques.

2.1 The Standard Problem for an Unstable Plant

Figure 2.1 depicts the system structure for the standard problem. The signals labeled in the diagram are all considered to be vector valued functions in L^2 ,

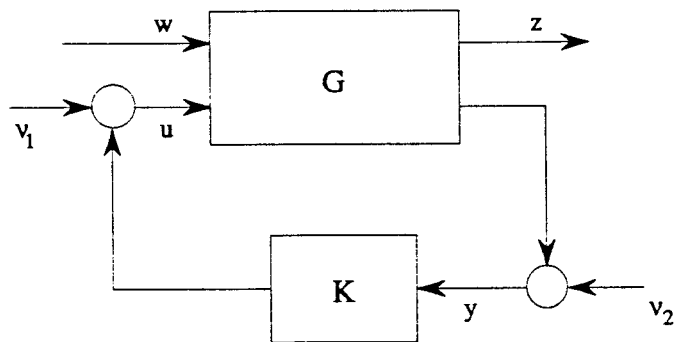


Figure 2.1: Block diagram for the standard problem

labeled in the diagram are all considered to be vector valued functions in L^2 , w models exogenous inputs, z models measured outputs, u the input to the plant from the controller y the output to the controller from the plant. The remaining signals ν_1 and ν_2 are used in the definition of closed loop stability and are not given physical interpretation. The standard problem is solved when a feedback operator K is found which minimizes the operator norm of the closed loop operator that maps w to z , while stabilizing the closed loop system. In this guise the problem is stated as a disturbance rejection problem, but as Francis illustrates in [14] the standard problem is equivalent to a variety of control problems including optimal robustness problems, tracking problems, and model matching problems. In fact the reformulation of the standard problem as a model matching problem is the first step to its solution.

Suppose the plant G is written as a block transfer function matrix

$$\begin{bmatrix} z \\ y \end{bmatrix} = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \begin{bmatrix} w \\ u \end{bmatrix},$$

then the equations for the closed loop system of Figure 2.1 are

$$z = G_{11}w + G_{12}u$$

$$y = G_{21}w + G_{22}u + \nu_2$$

$$u = Ky + \nu_1$$

Closed loop (BIBO) stability is achieved when the nine transfer functions mapping the input signals w , ν_1 , and ν_2 to the signals z , u and y are bounded linear

mappings from $L_2[0, \infty)$ to $L_2[0, \infty)$. Let $\Delta = (\mathbf{1} - KG_{22})$, then the closed loop transfer functions are given by the following matrix.

$$\begin{bmatrix} G_{11} + G_{12}\Delta^{-1}KG_{21} & G_{12}\Delta^{-1} & G_{12}\Delta^{-1}K \\ \Delta^{-1}KG_{21} & \Delta^{-1} & \Delta^{-1}K \\ G_{21} + G_{22}\Delta^{-1}KG_{21} & G_{22}\Delta^{-1} & 1 + G_{22}\Delta^{-1}K \end{bmatrix}$$

In Chapter 4 of [14] Francis proves the results summarized in the following theorem for the case of rational plant and compensator.

Theorem 2.1 [Francis [14]]

Assume that G is stabilizable, then:

(i) *K stabilizes G if and only if K stabilizes G_{22} .*

(ii) *Suppose $G_{22} = NM^{-1} = \widetilde{M}^{-1}\widetilde{N}$ are coprime factorizations of G_{22} , then there exist X, Y , and $\widetilde{X}, \widetilde{Y}$ such that*

$$\begin{bmatrix} \widetilde{X} & -\widetilde{Y} \\ -\widetilde{N} & \widetilde{M} \end{bmatrix} \begin{bmatrix} M & Y \\ N & X \end{bmatrix} = \mathbf{1}. \quad (2.1)$$

and the set of all K stabilizing G_{22} is parameterized by the formulae

$$\begin{aligned} K &= (X - MQ)(X - NQ)^{-1} \\ &= (\widetilde{X} - Q\widetilde{N})^{-1}(\widetilde{Y} - Q\widetilde{M}) \end{aligned}$$

$$Q \in H_\infty.$$

(iii) *With K given by the parameterization in (ii), and with the transfer functions T_1, T_2, T_3 given by*

$$T_1 = G_{11} + G_{12}M\widetilde{Y}G_{21}$$

$$T_2 = G_{12}M$$

$$T_3 = \tilde{M}G_{21},$$

the transfer function from w to z in figure 2.1 equals $T_1 - T_2QT_3$.

The key step in the proof of Theorem 2.1 is the construction of the doubly coprime factorization given in equation (2.1). The finite dimensional state space models and state feedback theory that Francis uses to tackle the rational case do not generalize to irrational transfer functions; for these alternative arguments are needed. Baras [1] was the first to provide a parametrization of all stabilizing controllers for distributed systems with transfer functions that admit co-prime factorizations, although the form of the parameterization that he presents is different from that presented in Theorem 2.1. The next section introduces the concept of a coprime factorization for an H_∞ function, and the techniques that are needed to prove the theorem for irrational transfer functions. The only restriction on the transfer functions is a requirement for the existence of appropriate co-prime factorizations.

2.2 Doubly Coprime Factorizations

Suppose \mathcal{R} is an integral domain, then the following definitions hold:

- 1) Let \mathcal{S} be a set of elements in \mathcal{R} , then d is a greatest common divisor (g.c.d.) of \mathcal{S} if for all $a \in \mathcal{S}$ there exists $b \in \mathcal{R}$ such that $a = bd$, and if d_1

is any other element of \mathcal{R} with this property, then $d = cd_1$ for some $c \in \mathcal{R}$.
g.c.d.s are unique up to multiplication by a unit.

- 2) Two elements of \mathcal{R} are weakly coprime if they have 1 as a g.c.d.
- 3) Two elements $a, b \in \mathcal{R}$ are strongly coprime if there exist $x, y \in \mathcal{R}$ such that

$$ax + by = 1$$

In the integral domain of rational H_∞ functions, strong coprimeness and weak coprimeness are equivalent, however in the whole of H_∞ this need not be the case. For example if λ is an irrational number in the interval $(0, 1)$, then the functions $(1/z)e^{-z} \sinh z$ and $(1/z)e^{-z} \sinh \lambda z$ are weakly coprime but not strongly coprime. An equivalent condition for strong coprimeness in H_∞ is given by the corona theorem which in the context of functions analytic on the disc is stated as:

Theorem 2.2 [Garnett [15]]

If f_1, \dots, f_n are functions in H_∞ that satisfy $\|f_i\|_\infty \leq 1$ and $\max_j |f_j(z)| \geq \delta > 0$ for $|z| < 1$, then there exists a constant C depending only on n and δ , and functions g_1, \dots, g_n such that $f_1g_1 + \dots + f_ng_n = 1$ and $\|g_i\|_\infty \leq C$.

For MIMO systems factorizations are performed over matrices with entries in H_∞ . Again in the matrix case concepts of weak and strong coprimeness are defined [24]. M , a matrix of rank r over an integral domain, is said to be

irreducible if the greatest common divisor of the r dimensional minors of M is 1. Two matrices $N \in \mathcal{R}^{n \times m}$ and $M \in \mathcal{R}^{m \times m}$ are weakly right coprime if the block matrix $\begin{bmatrix} N \\ M \end{bmatrix}$ is irreducible, and two matrices \widetilde{M} and \widetilde{N} are weakly left coprime if the block matrix $[\widetilde{M}, \widetilde{N}]$ is irreducible. Two matrices M and N are strongly right coprime if $\begin{bmatrix} N \\ M \end{bmatrix}$ has a left inverse, and two matrices \widetilde{M} and \widetilde{N} are weakly left coprime if the block matrix $[\widetilde{M}, \widetilde{N}]$ has a right inverse. The following theorem from Rao [23] reduces the problem of determining strong coprimeness in the matrix case to the problem of determining coprimeness in the integral domain.

Theorem 2.3 [Rao]

Let R be an integral domain, and A be an $m \times r$ matrix over R , then A has a right inverse if and only if a linear combination of all $r \times r$ minors with coefficients in R is equal to one.

An analogous theorem holds for left inverses of $r \times m$ matrices.

Rao provides the following method based on the Cauchy-Binet Theorem for constructing the right inverse from the solution of the Bezout Identity in the integral domain. A left inverse may be constructed by the same method with the obvious substitutions.

Let $|A_\beta|$ denote an r -minor of A containing the columns $(\beta_1, \beta_2, \dots, \beta_r)$, and suppose $\sum |A_\beta| c_\beta = 1$ for some c_β in the base ring R . Let $\frac{\partial |A_\beta|}{\partial a_{ki}}$ be the coefficient

of a_{ki} in the minor $|A_\beta|$, and let $b_{ik} = \partial(\sum |A_\beta|c_\beta)/\partial a_{ki}$. Then if B is the matrix with elements b_{ik} , the diagonal elements of the product AB are given by

$$\sum_{i=1}^m a_{ki}b_{ik} = \sum_{\beta} c_{\beta} \left\{ \sum_{i \in \beta} a_{ki} \frac{\partial |A_\beta|}{\partial a_{ki}} \right\} = \sum_{\beta} c_{\beta} |A_\beta| = 1.$$

Consider an off diagonal element $\sum_i a_{ji}b_{ik}$. If D denotes the matrix formed by replacing the k 'th row of A by the j 'th row of A , then

$$\sum_{i=1}^m a_{ji}b_{ik} = \sum_{\beta} c_{\beta} |D_\beta|.$$

But each of the minors D_β must be identically zero, and consequently all the off-diagonal elements of the product are zero and B is the required inverse.

Let $G = \widetilde{M}^{-1}\widetilde{N}$ be a left factorization over matrices with entries in the integral domain H_∞ and let A be the matrix $[\widetilde{N}, \widetilde{M}]$. Then provided that the condition on the minors of A is satisfied, theorem 2.3 provides a matrix B such that

$$AB = [\widetilde{N}, \widetilde{M}] \begin{bmatrix} Y \\ X \end{bmatrix} = \mathbf{1},$$

the identity matrix. Similarly, if $G = NM^{-1}$ is a right factorization and the minors of $[M, N]^t$ satisfy the appropriate condition then an application of the left inverse theorem produces matrices \widehat{X} and \widehat{Y} that solve the equation $\widehat{Y}N + \widehat{X}M =$

1. These two results may be combined to produce the equation

$$\begin{bmatrix} \widetilde{M} & \widetilde{N} \\ \widehat{Y} & -\widehat{X} \end{bmatrix} \begin{bmatrix} X & N \\ Y & -M \end{bmatrix} = \begin{bmatrix} \mathbf{1} & 0 \\ K & \mathbf{1} \end{bmatrix}$$

in which K is some matrix with entries in H_∞ . A simple row operator applied to both sides of the equation gives the doubly coprime factorization

$$\begin{bmatrix} \widetilde{M} & \widetilde{N} \\ \widetilde{Y} & -\widetilde{X} \end{bmatrix} \begin{bmatrix} X & N \\ Y & -M \end{bmatrix} = \begin{bmatrix} \mathbf{1} & 0 \\ 0 & \mathbf{1} \end{bmatrix}$$

in which $\widetilde{Y} = \widehat{Y} - K\widetilde{M}$ and $\widetilde{X} = \widehat{X} + K\widetilde{N}$.

Solving the Analytic Bezout Identity

This chapter describes a method for computing solutions to the Bezout Equation in algebras of holomorphic functions. The method relies on an algebraic reformulation of the problem that reduces the computation to one of finding a solution to the $\bar{\partial}$ equation

$$\frac{dv}{dz} = \frac{dv}{dx} + i \frac{dv}{dy} = f$$

in which f and v lie in appropriate algebras of functions (or more generally distributions) on the complex plane $z = x + iy$. The $\bar{\partial}$ equation and the notation associated with it were developed in the general setting of complex differential geometry. Hörmander's book [16] is the classical reference in the field, and [28] is another good reference. The theory may be simplified considerably in the context of the complex plane; in particular, \mathbb{C} , considered as two dimensional real Euclidean space, may be identified with its tangent space. With this identification a natural almost-complex structure is induced by multiplication by $i = \sqrt{-1}$. This simplification enables the $\bar{\partial}$ equation to be given a naïve interpretation which will be sufficient here.

Consider the space of co-vectors on \mathbb{C} with basis $\{dx, dy\}$. A change of coordinates $z = x + iy$, $\bar{z} = x - iy$ produces a new basis $\{dz, d\bar{z}\}$ ¹. Suppose that $f = f_1(z)dz + f_2(z)d\bar{z}$ is a 1-form on \mathbb{C} , and let the natural projections be denoted by

$$\pi^{1,0}f = f_1dz \quad \pi^{0,1}f = f_2d\bar{z},$$

then if d is the exterior derivative which maps $g(z)$ a function on \mathbb{C} to a 1-form dg , define ∂ and $\bar{\partial}$ to be the operators

$$\partial g = \pi^{1,0}dg, \quad \bar{\partial}g = \pi^{0,1}dg$$

The expressions for these operators in local co-ordinates are

$$\begin{aligned} \partial g &= \frac{dg}{dz}dz = \frac{1}{2} \left(\frac{\partial g}{\partial x} - i \frac{\partial g}{\partial y} \right) dz \\ \bar{\partial}g &= \frac{dg}{d\bar{z}}d\bar{z} = \frac{1}{2} \left(\frac{\partial g}{\partial x} + i \frac{\partial g}{\partial y} \right) d\bar{z}. \end{aligned}$$

It follows from the Cauchy Riemann equations that the kernel of the operator $\bar{\partial}$ is exactly the space of holomorphic functions.

Finally it is worth noting that in the geometric context outlined, the $\bar{\partial}$ problem is a statement about a cohomology group [16], [28]. If \mathcal{A} denotes the space of holomorphic functions on the half plane \mathcal{H} , $\mathcal{E}^{0,0}$ the space of bounded functions on \mathcal{H} and $\mathcal{E}^{0,1}$ the space of antiholomorphic 1-forms on \mathcal{H} , then the inclusion map $i : \mathcal{A} \rightarrow \mathcal{E}^{0,0}$, and the operator $\bar{\partial} : \mathcal{E}^{0,0} \rightarrow \mathcal{E}^{0,1}$ form a sequence

$$0 \rightarrow \mathcal{A} \xrightarrow{i} \mathcal{E}^{0,0} \xrightarrow{\bar{\partial}} \mathcal{E}^{0,1} \rightarrow 0$$

¹What is really meant here is that $\{dz, d\bar{z}\}$ is the dual basis to the basis in the tangent space that diagonalizes the almost-complex structure

and the $\bar{\partial}$ problem is equivalent to the question of whether this sequence is exact at $\mathcal{E}^{0,1}$, or equivalently, whether the cohomology is trivial.

The method of solving Bezout equations over bounded holomorphic functions described in this chapter originates in Carleson's solution to the Corona problem [6]. Hörmander [17] was the first to explicitly cast the Corona problem in an algebraic setting and extend the result to a general class of analytic functions. In [15] Garnett presents a method due to Peter Jones for calculating bounded solutions to the $\bar{\partial}$ equation, and it is on this account that the results in section 3.2 are based.

3.1 An Algebraic Reformulation of the Bezout Equation

Recall [19] that $(A, +, \cdot)$ is an algebra over a ring K if

- (i) $(A, +)$ is a unitary left K -module,
- (ii) $k(a \cdot b) = (ka) \cdot b = a \cdot (kb) \quad \forall k \in K \quad a, b \in A.$

I , a subset of A , is a left ideal of A if

$$a \cdot I = I \quad \forall a \in A,$$

and the left ideal generated by the set $B = \{f_1, \dots, f_N\}$ containing elements of A is the set $\text{span}\{g \cdot f_i \mid g \in A \text{ and } f_i \in B\}$. Let 1 denote the multiplicative identity of A then the left ideal generated by B will coincide with the whole space A if and only if there exist elements $g_1, \dots, g_N \in A$ such that

$$1 = f_1 g_1 + \dots + f_N g_N.$$

In other words, the question of the existence of a solution to the Bezout Identity is equivalent to the question of whether the ideal generated by f_1, \dots, f_N coincides with the whole algebra A .

The algebra that is of particular interest in this thesis is the algebra H_∞ of holomorphic functions on the right half plane with nontangential limits defined almost everywhere on the imaginary axis, and the property that these limits determine an element of L^∞ . The Paley-Wiener-Schwartz theorem [18], [3] allows this algebra to be identified with Transfer Functions of proper causal linear plants. In the calculations that follow, this algebra will be extended to include nonholomorphic functions and distributions on the right hand plane; since the operation of multiplication on two distributions is not always well defined, some care must be exercised when working with these objects.

The algebraic reformulation of the Bezout equation presented in this section was developed by Hörmander in [17] for algebras of holomorphic functions in several complex variables with general growth conditions. Although the key theorem is stated and proved here in general terms, the conditions under which the theorem holds are only established for the case of interest: solving the Bezout equation in the algebra H_∞ .

The algebraic constructions in the next paragraph follow the definitions given by Hörmander in [17]. Garnett also gives a description of the Koszul Complex in an appendix to Chapter 8 of [15]. For a thorough discussion of the use of these constructions in more general settings than presented here, consult Struppa in

[25]. For a discussion of basic concepts that come from differential geometry on complex manifolds, consult [28] or [16].

Suppose A is an algebra of functions (or more generally distributions) defined on the right half complex plane and on which antiholomorphic derivatives $\partial/\partial\bar{z} : A \rightarrow A$ are defined. Define L_r^s as the space of differential forms of type $(0, r)$ ($r = 0, 1$) with vector valued coefficients in $\Lambda^s \mathbb{C}^m$, the vector space of alternating s -forms on \mathbb{C}^m , with components $h_{(j_1, \dots, j_s)}$ in the algebra A . The spaces L_2^s have a single element, the zero form 0. Let $f = (f_1, \dots, f_m)$ be m functions in A satisfying $\partial f_j / \partial \bar{z} = 0$, then a double complex called the Koszul complex is defined on the spaces L_r^s . The first operator $\bar{\partial} : L_r^s \rightarrow L_{r+1}^s$, the antiholomorphic exterior derivative, is defined componentwise by

$$\begin{aligned} h_{(j_1, \dots, j_s)} &\mapsto \frac{\partial h_{(j_1, \dots, j_s)}}{\partial \bar{z}} d\bar{z} && \text{when } r = 0 \\ h_{(j_1, \dots, j_s)} d\bar{z} &\mapsto 0 && \text{when } r = 1. \end{aligned}$$

The second operator $P_f : L_r^{s+1} \rightarrow L_r^s$ operates on a form $h \in L_r^{s+1}$ by acting on the coefficients as follows: suppose that $P_f h$ has coefficients $g_{(j_1, \dots, j_s)}$, then

$$g_{(j_1, \dots, j_s)} = \sum_{k=1}^m h_{(j_1, \dots, j_s, k)} f_k.$$

If $h \in L_r^0$, then $P_f h = 0$, the zero form. From the antisymmetry of P_f it follows that $P_f^2 = 0$, and the analyticity of (f_1, \dots, f_n) ensures that P_f commutes with $\bar{\partial}$. The next Theorem is from Hörmander [17].

Theorem 3.1 [Hörmander]

Suppose that the following conditions are satisfied:

(i) If $g \in L_r^s$ and $P_f g = 0$ then the equation $P_f h = g$ has a solution $h \in L_r^{s+1}$ with $\bar{\partial} h \in L_{r+1}^{s+1}$ when $\bar{\partial} g = 0$.

(ii) $\bar{\partial} g = h$ has a solution $g \in L_r^s$ for every $h \in L_{r+1}^s$ with $\bar{\partial} h = 0$.

Then for every $g \in L_r^s$ with $\bar{\partial} g = P_f g = 0$ one can find $h \in L_r^{(s+1)}$ so that $\bar{\partial} h = 0$ and $P_f h = g$.

Proof:

The premises of the theorem ensure that the rows and columns of the double complex illustrated in the diagram of Figure 3.1 are exact, and the result follows by using these premises to traverse the diagram.

Suppose that $g(z) \in L_0^0$ is a holomorphic function, that is, $\bar{\partial} g = 0$. Then by the first premise there exists $h^1 \in L_0^1$ such that $P_f h^1 = g$ and $\bar{\partial} h^1(z) \in L_1^1$. Commutativity implies that $P_f \bar{\partial} h^1 = \bar{\partial} P_f h^1 = 0$, so again by the first premise there exists $h^2 \in L_1^2$ such that $P_f h^2 = \bar{\partial} h^1$, and by the second premise $h^3 \in L_0^2$ such that $\bar{\partial} h^3 = h^2$. Now let $h = h^1 - P_f h^3$, then

$$\begin{aligned} \bar{\partial} h &= \bar{\partial} h^1 - \bar{\partial} P_f h^3 \\ &= \bar{\partial} h^1 - P_f \bar{\partial} h^3 \\ &= 0 \end{aligned}$$

and $P_f h = P_f(h^1 - P_f h^3) = P_f(h^1) - P_f P_f h^3 = P_f h^1 = g$ as required.

□

$$\begin{array}{ccccccc}
& & 0 & & 0 & & 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
& & \Lambda^2(\mathbb{H}_\infty) & & \Lambda^1(\mathbb{H}_\infty) & & \mathbb{H}_\infty \\
& & \downarrow & & \downarrow & & \downarrow \\
& & L_0^2 & \xrightarrow{P_f} & L_0^1 & \xrightarrow{P_f} & L_0^0 & \xrightarrow{P_f} & 0 \\
& & \downarrow \bar{\partial} & & \downarrow \bar{\partial} & & \downarrow \bar{\partial} & & \\
& & L_1^2 & \xrightarrow{P_f} & L_1^1 & \xrightarrow{P_f} & L_1^0 & \xrightarrow{P_f} & 0 \\
& & \downarrow \bar{\partial} & & \downarrow \bar{\partial} & & \downarrow \bar{\partial} & & \\
& & 0 & & 0 & & 0 & &
\end{array}$$

Figure 3.1: Koszul complex for $m = 2$

Theorem 3.1 can only be useful as a means of solving Bezout Equations for a particular choice of algebra if an explicit means can be found for calculating inverses for the two operators, $\bar{\partial}$ and P_f , appearing in the complex. These inverses must also satisfy the conditions of the theorem. To resolve this question the algebra A and the spaces L_r^s need to be specified more precisely; first some definitions.

Denote the open right half plane by \mathcal{H} , its closure by $\bar{\mathcal{H}}$, and its boundary, the imaginary axis, by $\partial\mathcal{H}$. A distribution u on \mathcal{H} satisfies the equation

$$\frac{\partial u}{\partial \bar{z}} = \mu \quad (3.1)$$

for some measure μ with support on \mathcal{H} if for any continuously differentiable test function ψ with support compactly contained in \mathcal{H} ,

$$\begin{aligned} \left\langle \frac{\partial u}{\partial \bar{z}}, \psi \right\rangle &= - \int_{\mathcal{H}} u(z) \frac{\partial \psi(z)}{\partial \bar{z}} dx dy \\ &= \int_{\mathcal{H}} u(z) d\mu. \end{aligned} \quad (3.2)$$

The measure $dx dy$ in the first integral is the Lebesgue measure on \mathbb{C} . A distribution u that satisfies (3.2) is said to have boundary value ϕ , an L^∞ function on the imaginary axis, if there exists U , an extension of u to $\bar{\mathcal{H}}$ that satisfies:

$$\frac{\partial U}{\partial \bar{z}} = \mu - \phi dz/2i. \quad (3.3)$$

Each side of this formula is to be interpreted as a distribution acting on test functions supported in the closed half plane $\bar{\mathcal{H}}$. The measure $\phi dz/2i$ is a measure on \mathbb{C} with support on the imaginary axis.

The motivation for this definition comes from Stokes' theorem. Suppose that U is a function that equals ϕ on the imaginary axis, and f is some test function that is compactly supported on \mathbb{C} , then $\phi f dz = U f dz$ defines a $(1, 0)$ form on $\partial\mathcal{H}$, and Stokes theorem gives:

$$\begin{aligned} \int_{\partial\mathcal{H}} U f dz &= \int_{\mathcal{H}} d(U f dz) \\ &= \int_{\mathcal{H}} f \bar{\partial} U d\bar{z} \wedge dz + \int_{\mathcal{H}} U \bar{\partial} f d\bar{z} \wedge dz \\ &= 2i \int_{\mathcal{H}} f d\mu + 2i \int_{\mathcal{H}} U \bar{\partial} f d\bar{z} \wedge dz. \end{aligned}$$

Recalling the definition of a derivative for a distribution one can say that U defines a distribution on \mathcal{H} with a derivative that satisfies (3.3).

A measure μ in \mathcal{H} is called a Carleson measure [15] with Carleson constant C if

$$\mu(S) < C l(S)$$

for every square $S \subset \mathcal{H}$ with a side of length $l(S)$ lying on an interval on the imaginary axis.

Appropriate spaces for L^s are now defined as follows: $h \in L^s_0$ if each component h_I is a distribution in \mathcal{H} with boundary value in L^∞ , and each component satisfies

$$\bar{\partial} h_I = \mu \tag{3.4}$$

for some bounded measure μ in \mathcal{H} ; $h \in L^s_1$ if each component $h_I = \mu_I d\bar{z}$ for some Carleson measure μ_I . In [17] Hörmander uses an argument based on the Hahn Banach theorem to prove the existence of solutions in L^s_0 to $dh/d\bar{z} = \mu$ for

a Carleson measure μ in L_1^s ; from this it follows that condition (ii) of Theorem 3.1 is satisfied. The domain of the $\bar{\partial}$ operator does not however coincide with the spaces L_0^s , and this is why the first condition of the theorem requires that $\bar{\partial}h \in L_{r+1}^{s+1}$. These definitions together with (3.4) complete the picture as far as the existence of an inverse for $\bar{\partial}$ is concerned, and it remains only to deal with P_f .

Inversion of the operator P_f is considered in two parts. Suppose that $g \in L_r^1$ with $P_f g = f_1 g_1 + f_2 g_2 = 0$ and let $h \in L_r^2$ be given by

$$h_I = \sum_{j=1}^{s+1} (-1)^{s+1-j} g_{I_j} \frac{\bar{f}_{i_j}}{|f|^2} \quad (3.5)$$

in which I denotes the multi index (i_1, \dots, i_{s+1}) and I_j denotes the multi index $(i_1, \dots, i_{j-1}, i_{j+1}, \dots, i_s)$. Since $s = 1$ in this instance, h has only one component

$$h_{(1,2)} = g_1 \frac{\bar{f}_2}{|f|^2} - g_2 \frac{\bar{f}_1}{|f|^2}$$

with

$$\begin{aligned} (P_f h)_1 &= h_{(1,2)} f_2 = g_1 f_2 \frac{\bar{f}_2}{|f|^2} - g_2 f_2 \frac{\bar{f}_1}{|f|^2} \\ &= g_1 \\ (P_f h)_2 &= h_{(2,1)} f_1 = -g_1 f_1 \frac{\bar{f}_2}{|f|^2} + g_2 f_1 \frac{\bar{f}_1}{|f|^2} \\ &= g_2. \end{aligned}$$

Another straightforward calculation shows that $\bar{\partial} P_f h = 0$ and permits the conclusion that (3.5) provides the appropriate inverse when $s \geq 1$.

When $g \in L^0_r$ the expression (3.5) fails to give an inverse for P_f that satisfies the exterior derivative condition (3.4) on account of the fact that $f \in H_\infty$ does not necessarily imply $\bar{\partial}f \in H_\infty$ (an exception that is exploited later in an example is the case of f being restricted to rational functions). To overcome this hitch, which in fact is the major obstacle in constructing solutions to the Corona Problem, a construction based on the following lemma from [17] is used.

Lemma 3.2 [Carleson - Hörmander] *Let $f_j \in \mathcal{H}$, $j = 1, \dots, n$, and assume that for some $c > 0$*

$$|f_1(z)| + \dots + |f_n(z)| \geq c. \quad (3.6)$$

Then for sufficiently small $\epsilon > 0$ one can find a partition of unity ϕ_j subordinate to the covering of \mathcal{H} by open sets $\mathcal{H}_j = \{z : |f_j(z)| > \epsilon\}$ such that $\partial\phi_j/\partial\bar{z}$, defined in the sense of distribution theory, is a Carleson measure for all j .

This Lemma is a restatement of a result of Carleson's original paper [6] in which he directly constructs the measure. A more recent account of the construction is given in Garnett's book [15]. The difficult part of the lemma is the construction of a partition of the plane into two sets each of which contains the regions of the plane where one of the two functions f_1 or f_2 becomes very small. In general, the boundary between the two sets will be a complicated curve, however in practice, the functions f_1 and f_2 may possess some regularity that allows a boundary curve to be easily chosen. In the example considered at the end of this chapter $f_1(z) = e^{-\tau z}/(1+z)$ and $f_2(z) = (1-z)/(1+z)$ so the restriction on the partition

is that it separate the point $z = 1$ where $f_2 = 0$ from the regions of the plane where $|z|$ is large and $f_1(z)$ tends to zero.

The partition of unity from Lemma 3.2 is used to construct a left inverse for P_f on L_r^0 as follows: let $g \in L_r^0$, then

$$h_I = \sum_{j=1}^{s+1} g_{I_j} \frac{\phi_j}{f_j}. \quad (3.7)$$

This expression is produced from (3.5) by replacing the term $\bar{f}_i/|f|^2$ with ϕ_i/f_i .

With $g \in H_\infty$ and $s = 0$ equation (3.7) becomes

$$h_i = g \frac{\phi_i}{f_i}$$

and $\partial h/\partial \bar{z} = g f_i^{-1} \partial \phi_j/\partial \bar{z}$ which by Lemma 3.2 is a Carleson measure.

The construction of an inverse for P_f together with Lemma 3.2 and the existence theorem for solutions to $\bar{\partial}$ from [17] allow the Corona theorem to be deduced as a corollary to Theorem 3.1.

Theorem 3.3 *If $f_1, \dots, f_n \in H_\infty$ and $|f_1(z)| + \dots + |f_n(z)| \geq c > 0$ then it follows that the ideal generated by f_1, \dots, f_n coincides with the whole of H_∞ .*

In addition, Theorem 3.1 gives a method for explicitly constructing solutions to the Bezout Equation provided that the partition of unity in Lemma 3.2 may be constructed, and that the solution to the inhomogeneous $\bar{\partial}$ equation may be computed when the right hand side is a Carleson measure.

3.2 Constructing Bounded Solutions to the Inhomogeneous $\bar{\partial}$ Equation

In the preceding section the construction of solutions to the Bezout Equation has been reduced to two steps: the construction of the partition of unity ϕ_j , and the construction of bounded solutions to the $\bar{\partial}$ equation. This section describes a technique devised by P. Jones [20] for solving the $\bar{\partial}$ equation; the presentation is based on the account given in Garnett [15]. The problem that needs to be solved is: given μ , a Carleson measure on the right half plane, find a distribution b with bounded boundary values that satisfies

$$\partial b / \partial \bar{z} = \mu.$$

The solution, which is based on a Green's function argument, has three stages: the measure μ is approximated by a sequence of measures μ_j which converge weakly to μ , each μ_j being supported on a finite set of points; the support of each measure μ_j is partitioned in such a way that the pseudo-hyperbolic distance² between any two points in the same partition is bounded from below, and the measure μ_j is subdivided into a corresponding sum $\sum \mu_j^k$ each μ_j^k having support on a distinct set in the partition; finally the $\bar{\partial}$ problem is boundedly solved for each μ_j^k and these solutions are summed to form the approximate solution b_j . The whole procedure is performed in such a way that the sequence of solutions b_j is a uniformly bounded sequence of functions in H_∞ .

²The pseudo-hyperbolic distance between two points in the half-plane is defined as $\rho(z_1, z_2) = (z_1 - z_2) / (z_1 - \bar{z}_2)$

Before the solution is discussed in detail the fundamental solution to the $\bar{\partial}$ operator is introduced, and a result about interpolating Blaschke products is recounted. Let $D \subset \mathbb{C}$ be an open domain with C^1 boundary that contains the origin $z = 0$. The fundamental solution to the $\bar{\partial}$ operator on D is a distribution b that satisfies the identity

$$-\int_D b(z) \frac{\partial \phi(z)}{\partial \bar{z}} dx dy = \phi(0)$$

for any C^∞ function ϕ with support compactly contained in D [18]. In this formula the integral on the left hand side of the identity should be interpreted as the action of the distribution on a test function. The fundamental solution for the $\bar{\partial}$ operator is easily computed. Suppose that ϕ is an arbitrary C^∞ function with support compactly contained in D . Let $U \subset D$ have C^1 boundary and contain the support of ϕ in its interior. Consider the function $\phi(\zeta)/\zeta$, Stokes' theorem gives

$$\begin{aligned} \int_{\partial U} \frac{\phi(\zeta)}{\zeta} d\zeta - \int_{|\zeta|=\epsilon} \frac{\phi(\zeta)}{\zeta} d\zeta &= \int_{|\zeta|>\epsilon} \frac{\partial}{\partial \bar{\zeta}} \left(\frac{\phi(\zeta)}{\zeta} \right) d\bar{\zeta} \wedge d\zeta \\ &= -2i \int_{|\zeta|>\epsilon} \frac{1}{\zeta} \frac{\partial \phi}{\partial \bar{\zeta}} dx dy. \end{aligned}$$

Because $\phi(z) = 0$ on the boundary of U , the first boundary integral is zero, and as $\epsilon \rightarrow 0$ the second integral approaches the limit $i2\pi\phi(0)$. Consequently a fundamental solution for $\bar{\partial}$ is given by the distribution $b(z) = 1/(\pi z)$.

If $z = x + iy$ is a complex number, then the real conjugate of z is defined to be the number $\bar{z} = -x + iy$. The need for this usage results from considering Laplace transforms of system operators; the Laplace transform of a bounded

causal system gives a transfer function which is analytic in the right half plane, so in places where a complex conjugate \bar{z} occurs in the analysis of functions analytic in the upper half plane, it will be natural to substitute the real conjugate \tilde{z} . For instance given a set $\{\zeta_j = \xi_j + i\eta_j, \xi_j > 0\}$ that satisfies the condition

$$\sum \frac{x_n}{1 + |z_n|^2} < \infty$$

a Blaschke product with zeros ζ_j is defined by the expression

$$B(z) = \left(\frac{z-1}{z+1}\right)^m \prod_{\zeta_j \neq 1} \frac{|\zeta_j - 1|}{\zeta_j - 1} \frac{z - \zeta_j}{z - \tilde{\zeta}_j}.$$

The factors $|\zeta_j - 1|/(\zeta_j - 1)$ ensure that the product converges when the sequence $\{|\zeta_j|\}$ is unbounded, and for finite zero sets they may be omitted.

Let $B_0(z)$ be a Blaschke product with a zero set $\{\zeta_j = \xi_j + i\eta_j\}$ that satisfies the condition

$$\prod_{j, j \neq k} \left| \frac{z_k - z_j}{z_k - \tilde{z}_j} \right| \geq \delta > 0, \quad (3.8)$$

then the inverse $1/B_0(z)$ is an analytic function except on the zero set $\{\zeta_j\}$ and is given by the expression

$$1/B_0(z) = 1 + \sum_j \frac{1/B_0'(\zeta_j)}{z - \zeta_j}$$

If $1/B_0(z)$ is considered as a distribution on \mathcal{H} , then it follows from the discussion of the fundamental solution to the $\bar{\partial}$ operator that

$$\begin{aligned} \bar{\partial}(1/B_0(z)) &= \sum_j \frac{\pi}{B_0'(\zeta_j)} \delta_{\zeta_j} \\ &= \sum_j \beta_j \xi_j \delta_{\zeta_j}, \end{aligned} \quad (3.9)$$

where $1 \leq |\beta_j| \leq 1/\delta$.

The following theorem is due to Earl [8][9].

Theorem 3.4 [Earl]

Let $\{z_j\}$ be a sequence in the right half plane such that

$$\prod_{j, j \neq k} \left| \frac{z_k - z_j}{z_k - \bar{z}_j} \right| \geq \delta > 0, \quad k = 1, 2, \dots$$

Then there is a constant K independent of δ such that whenever $\{a_j\} \in l^\infty$, there exists $f(z) \in H^\infty$ such that

$$f(z_j) = a_j, \quad j = 1, 2, \dots,$$

and such that

$$f(z) = K \left(\sup_j |a_j| \right) \delta^{-2} B_{\text{int}}(z),$$

where $B_{\text{int}}(z)$ is a Blaschke product. The zeros $\{\zeta_j\}$ of $B_{\text{int}}(z)$ satisfy

$$\rho(\zeta_j, z_j) = \left| \frac{\zeta_j - z_j}{\zeta_j - \bar{z}_j} \right| \leq \frac{\delta}{3}$$

and

$$\prod_{j, j \neq k} \left| \frac{\zeta_k - \zeta_j}{\zeta_k - \bar{\zeta}_j} \right| \geq \frac{\delta}{3},$$

This theorem is a particular instance of Carleson's interpolation theorem ([15] chapter 7). It has an advantage over Carleson's theorem in that it explicitly gives the form of the interpolating function, in fact the interpolating Blaschke product is constructed in the proof of the theorem. The bound on the norm

of the interpolating function, $K(\sup_j |a_j|)\delta^{-2}$ is not optimal, Carleson's theorem gives the better estimate

$$1/\delta \leq M \leq C(1 - \log \delta)/\delta.$$

in which C is a constant that depends only on $(\sup_j |a_j|)$.

The following technical lemma will be needed in order to apply Earl's Theorem. The content of the lemma and its proof appear in [15] as part of a proof of Carleson's interpolation theorem.

Lemma 3.5 [Garnett] [15]

Suppose that $\{z_j\}$ is a sequence in the right half plane, then the following conditions are equivalent:

(i) *There exists δ such that*

$$\prod_{j, j \neq k} \left| \frac{z_k - z_j}{z_k - \bar{z}_j} \right| \geq \delta, \quad k = 1, 2, \dots$$

(ii) *The points z_j are well-separated in the pseudo-hyperbolic metric,*

$$\rho(z_k, z_j) = \left| \frac{z_k - z_j}{z_k - \bar{z}_j} \right| \geq a > 0, \quad j \neq k,$$

and there is a constant A such that for every square $Q = \{y_0 \leq y \leq y_0 + \ell(Q), 0 < x \leq \ell(Q)\}$,

$$\sum_{z_j \in Q} x_j \leq A \ell(Q)$$

Proof:

The proof is presented only for the direction (ii) \Rightarrow (i) as this is the only direction that is needed later.

First, the following inequality is established:

$$-\log t \leq \frac{-2 \log a}{1 - a^2} \leq (1 + 2 \log \frac{1}{a})(1 - t), \quad \text{for } a^2 < t < 1 \quad (3.10)$$

The left hand inequality follows from the observation that $a^2 \leq t < 1$ implies both $-2 \log a \geq -\log t$ and $(1 - t) \geq (1 - a^2)$ which combine to give

$$-\log t \leq \frac{-2 \log a}{1 - a^2}(1 - t).$$

The right hand inequality follows from the series expansion

$$\log(1 - b) = -b + \frac{b^2}{2} - \frac{b^3}{3} + \dots \geq -b - b^2 - b^3 - \dots = \frac{-b}{1 - b}, \quad 0 < b < 1.$$

Let $b = 1 - a^2$ with $0 < a < 1$, then

$$\begin{aligned} \log a^2 &\geq \frac{1 - a^2}{-a^2} \\ \Rightarrow -\log a^2 + (1 - a^2) \log a^2 &\leq 1 - a^2 \\ \Rightarrow \frac{-2 \log a}{1 - a^2}(1 - t) &\leq (1 + 2 \log \frac{1}{a})(1 - t) \end{aligned}$$

Let $t = |(z_k - z_j)/(z_k - \tilde{z}_j)|^2$, then it follows from statement (ii) of the Lemma that $t > a^2 > 0$ for some $a > 0$ and for all $j \neq k$. Substituting for t in both the left hand and right hand ends of (3.10) gives

$$\begin{aligned} -\log \left| \frac{z_k - z_j}{z_k - \tilde{z}_j} \right|^2 &\leq \left(1 + 2 \log \frac{1}{a}\right) \left(1 - \left| \frac{z_k - z_j}{z_k - \tilde{z}_j} \right|^2\right) \\ &\leq -\left(1 + 2 \log \frac{1}{a}\right) \frac{(z_k - \tilde{z}_j)(\tilde{z}_k - z_j) - (z_k - z_j)(\tilde{z}_k - \tilde{z}_j)}{|z_k - \tilde{z}_j|^2} \\ &\leq \left(1 + 2 \log \frac{1}{a}\right) \frac{4x_j x_k}{|z_k - \tilde{z}_j|^2} \end{aligned}$$

Fix k and sum each side of the inequality over the indices $j \neq k$ to get

$$-\log \prod_{j \neq k} \left| \frac{z_k - z_j}{z_k - \tilde{z}_j} \right|^2 \leq \left(1 + 2 \log \frac{1}{a} \right) \sum_{j \neq k} \frac{4x_j x_k}{|z_k - \tilde{z}_j|^2} \quad (3.11)$$

It remains only to bound the summation in the right hand side of (3.11). With z_k still fixed, consider the following collection of semicircles with center iy_k in the complex plane.

$$S_n = \{z \in \mathcal{H} : |z - iy_k| \leq 2^n x_k\}, \quad n = 0, 1, 2, \dots$$

From the second inequality in statement (ii) of the Lemma it follows that

$$\sum_{z_j + iy_j \in S_n} x_j \leq A 2^{n+1} x_k.$$

Also, if $z_j \in S_0$ then $|z_j - \tilde{z}_k|^2 \geq x_k^2$, and when $z_j \in S_n - S_{n-1}$ with $n \geq 1$, $|z_j - \tilde{z}_k|^2 \geq 2^{2n-2} x_k^2$. Consequently,

$$\begin{aligned} \sum_j \frac{4x_k x_j}{|z_k - \tilde{z}_j|^2} &\leq 4 \sum_{z_j \in S_0} \frac{x_j}{x_k} + 16 \sum_{n=1}^{\infty} \sum_{z_j \in S_n - S_{n-1}} \frac{x_j}{2^{2n} x_k} \\ &\leq 8A + 32A \sum_1^{\infty} 2^{-n} \\ &= 40A \end{aligned}$$

Substituting this result into (3.11) gives

$$\prod_{j, j \neq k} \left| \frac{z_k - z_j}{z_k - \tilde{z}_j} \right| \geq \delta, \quad k = 1, 2, \dots$$

with

$$\delta = \exp \left(-40A \left(1 + 2 \log \frac{1}{a} \right) \right).$$

□

The next two lemmas contain the constructive solution to the $\bar{\partial}$ equation that is presented in Chapter 8 of [15]. The proofs closely follow the work cited, but are given here because they contain the algorithms that are used to compute actual solutions. Earl's Theorem and the discussion preceding it on fundamental solutions provide the basis for calculating $\bar{\partial}$ solutions in the following simple case.

Lemma 3.6 [Garnett [15]]

Let z_j be a finite set of points satisfying (3.8) and let $\mu = \sum \alpha_j x_j \delta_{z_j}$ with $|\alpha_j| \leq 1$. Then the rational function

$$b(z) = K\delta^{-3}(B_2(z)/B_1(z))$$

satisfies $\bar{\partial}b = \mu$ where $B_1(z)$ is a Blaschke product with zeros z_j , $B_2(z)$ is a second Blaschke product, and K is a constant independent of δ .

Proof:

Equation (3.9) states

$$\frac{d}{d\bar{z}} \frac{1}{B_1(z)} = \sum_j \beta_j x_j \delta_{z_j},$$

and Earl's theorem guarantees the existence of a second interpolating Blaschke product $B_2(z)$ such that

$$K\delta^{-3}B_2(z_j) = \alpha_j/\beta_j$$

□

The case of a general Carleson measure μ is tackled by constructing a sequence of approximating measures $\{\mu_n\}$ that converges (weakly) to μ ; each measure in the sequence is supported on a finite set of points and has the form $\mu_n = \sum \alpha_j x_j \delta_{z_j}$. The next lemma presents a method for solving the equations $d/d\bar{z} b_n(z) = \mu_n(z)$ with a uniform bound on the sequence of solutions b_n . The difficulty here is that the bound in Lemma 3.6 depends on the parameter δ which, through Lemma 3.5, is related to the spacing (in the pseudo-hyperbolic metric) of the points in the supporting set $\{z_j\}$. If a general Carleson measure is going to be approximated by a sequence of measures with finite point support, then the spacing of the points in the support of the approximating measures will decrease to zero as the approximations converge. What is needed is a method for decomposing the approximating measures in such a way that the spacing between points of support for each element of the decomposition remains large, yet the sum of the Carleson constants of the elements in the decomposition remains constant.

Lemma 3.7 [Jones-Garnett] [15]

Let $\mu = \sum_{j=1}^M \alpha_j x_j \delta_{z_j}$ be a measure supported on the finite set $\{z_j = x_j + iy_j\}$, with masses $\alpha_j x_j$ at the points z_j , and with Carleson constant $N(\mu) \leq C$. Then there exist an integer N , rational functions $b_p(z)$, and a function

$$b(z) = \frac{1}{N} \sum_{p=1}^{2N} b_p(z)$$

such that each $b_p(z)$ is a function of the type produced in Lemma 3.6, $d/d\bar{z} b(z) =$

$\hat{\mu}$ for a measure $\hat{\mu}$ that is arbitrarily close to μ , and $|b(it)| < KC$ for $t \in \mathbb{R}$ and K a constant independent of μ .

The proof below replaces the dyadic construction of [15] by an m -adic subdivision. Although setting $m = 2$ reduces the complexity of the proof and is sufficient to prove continuity of the inversion, the choice of m will influence the bound achieved on the solution and is important if solutions with small norm are to be calculated.

Proof:

First it is shown that μ may be approximated arbitrarily closely by a new measure $\hat{\mu}$ of the form $C/N \sum x_j \delta_{z_j}$. The support of $\hat{\mu}$ is the same as the support of μ , but each point mass $x_j \delta_{z_j}$ may be repeated a finite, and possibly large number of times in the new sum. If N is chosen to be a sufficiently large positive integer, the coefficients a_j in the finite sum μ may be uniformly approximated to arbitrary accuracy by $a_j \approx n_j/NC$ in which n_j are positive integers and C is the Carleson constant of μ . If each term in the sum $\sum a_j x_j \delta_{z_j}$ is expanded as

$$\alpha_j x_j \delta_{z_j} \approx \frac{C}{N} (x_j \delta_{z_j} + \dots (n_j \text{ times}) \dots + x_j \delta_{z_j})$$

then a renumbering of the terms in the summation gives the approximation

$$\mu \approx \hat{\mu} = \frac{C}{N} \sum_j x_j \delta_{z_j}$$

From here on no distinction will be made between the measure μ and the approximation $\hat{\mu}$

In the second part of the proof a systematic method of decomposing the measure μ is established. The point masses $x_j \delta_{z_j}$ are distributed amongst a finite number of sets in such a way that the distance between any two points in the same set is large.

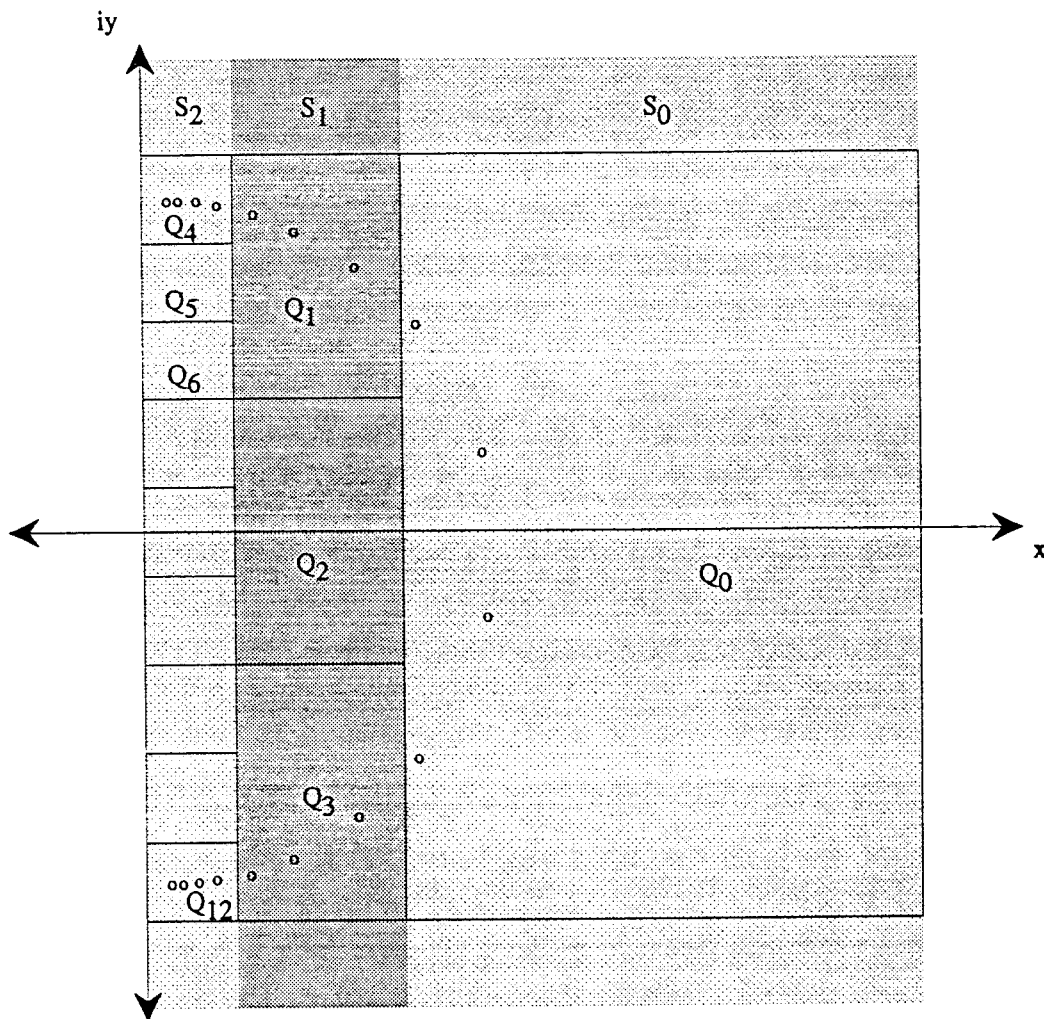


Figure 3.2: m -adic Subdivision of the Half Plane ($m = 3$)

Choose a square Q_0 with $\text{supp } \mu \subset Q_0 \subset \mathcal{H}$ and a side of length $l(Q_0)$

lying on the imaginary axis. This square may be subdivided to form an m -adic sequence of squares of uniform hyperbolic size as follows (Figure 3.2 illustrates the construction for the case $m = 2$). Let Q_1, \dots, Q_m be the m adjacent squares contained in Q_0 each with sides of length $l(Q)/m$, and each having one side on the imaginary axis; continue this subdivision process inductively on each square Q_i until all the squares $Q_{m^n}, \dots, Q_{m^{n+1}-1}$ are outside the support of μ for some n (the process is guaranteed to stop since the support of μ is compactly contained in \mathcal{H}). The right hand section of any m -adic square Q can contain at most mN points z_j because $N(\mu) \leq C$. This allows the points $\{z_j\}$ to be partitioned into $2mN$ sets in such a way that the spacing between any two points in the same set is uniformly bounded from below by $\delta = \frac{m-1}{m+1}$.

For every n , let $S_n = \{z_j : l(Q_0)m^{-n-1} \leq x_j \leq l(Q_0)m^{-n}\}$ and order the elements of each S_n so that $S_n = \{x_{k,n} + iy_{k,n}\}$ with

$$y_{k-1,n} \leq y_{k,n} \leq y_{k+1,n}.$$

Then the set $\{z_j\}$ may be split into mN sequences Y_1, \dots, Y_{mN} such that the points in each S_n are evenly distributed between the Y_r , i.e. if $z_j = x_{k,n} + iy_{k,n} \in S_n$ then $z_j \in Y_r$ if $r = k \bmod mN$. Now suppose that $P \subset \mathcal{H}$ is a fixed square with one side lying on the imaginary axis, let $M_n(P)$ be the number of points in $S_n \cap P$, then each set $Y_r \cap S_n \cap P$ must contain fewer than $1 + M_n(P)/(mN)$ points z_j , and

$$\sum_{Y_r \cap P} x_j \leq \sum_{n: S_n \cap P \neq \emptyset} \left(1 + \frac{M_n(P)}{mN}\right) m^{-n} l(Q_0)$$

$$\begin{aligned}
&\leq m \ell(P) \sum_{n=0}^{\infty} m^{-n} + \frac{1}{mN} \sum_{z_j \in P} mx_j \\
&\leq \frac{m^2}{m-1} \ell(P) + \frac{\mu(P)}{C} \\
&\leq (m+3) \ell(P)
\end{aligned} \tag{3.12}$$

Consider the sets $\{X_p\}$ defined by

$$X_r = Y_r \cap \bigcup_{n \text{ even}} S_n$$

and

$$X_{2r+1} = Y_r \cap \bigcup_{n \text{ odd}} S_n.$$

then the measures $\mu_p = \sum_{z_j \in X_p} x_j \delta_{z_j}$ satisfy $\mu = \sum_p \mu_p$ and provide a decomposition of μ into measures of bounded Carleson constant with well spaced support.

The upper bound on the Carleson constants is given by Equation (3.12) and the lower bound on the separation of two points of support for the measure μ_p

$$\rho(z_i, z_j) \geq \frac{m-1}{m+1} \quad \text{when } z_i, z_j \in X_p. \tag{3.13}$$

This bound is arrived at by the following argument. If $z_i \in S_n$ and $z_j \in S_{n-2}$ then $\rho(z_i, z_j) > (m-1)/(m+1)$ by the definition of the sets S_n . On the other hand, if z_i and z_j are in the same set S_n then it follows, from the fact that the top half of any Q_j contains at most mN points, and the way in which the set Y_r that corresponds to X_p was constructed, that z_i and z_j must be separated by $m-1$ hyperbolic squares each of length $\ell(Q_0)m^{-n}$. Consequently, the distance

between z_i and z_j must be bounded below by

$$\begin{aligned}\rho(z_i, z_j) &\geq \frac{(m-1)m^{-n}}{\sqrt{(m-1)^2m^{-2n} + 4m^{-2n}}} \\ &\geq \frac{m-1}{m+1}.\end{aligned}$$

Lemma 3.6 can now be applied to the measures μ_p to produce functions b_p that satisfy

$$\frac{d}{d\bar{z}}b_p = \mu_p.$$

The constant δ in Lemma 3.6 which determines the bounds on the norms $\|b_p\|$ is estimated by using Lemma 3.5 and the inequalities (3.12) and (3.13). This gives an estimate on the norms $\|b_p\|$ of

$$\|b_p\| \leq K\delta^{-3},$$

in which K is a constant that is independent of the measure μ_p , and δ is given by the formula

$$\delta = \exp\left(120(m+3)\left(1 + 2\log\left(\frac{m+1}{m-1}\right)\right)\right). \quad (3.14)$$

Let

$$b(z) = \frac{C}{N} \sum_{p=1}^{2mN} b_p(z).$$

Then $\bar{\partial}b(z) = C/N \sum z_j \delta_{z_j} = \mu$ and

$$\|b\| \leq 2mKC\delta^{-3}, \quad (3.15)$$

which completes the proof. □

The bound on the norm of the solution b depends on the choice of m both directly through the factor in (3.15), and indirectly through (3.14), the expression for δ . If the procedure given in the proof is being used for computation of solutions to the $\bar{\partial}$ equation, then the question of what value should be chosen for m will arise. When deciding this, particular attention should be paid to the inequality (3.12). The appearance of m on the right hand side of this inequality comes from the constant term 1 in the estimate that each set $Y_r \cap S_n \cap P$ must contain fewer than $1 + M_n(P)/(mN)$ points z_j . This estimate is too large for most of the sets $Y_r \cap S_n \cap P$: take for instance the case illustrated in Figure 3.2 and assume that the Carleson constant for the measure $C = 1$, and that each circle represents a mass of magnitude equal to its distance from the imaginary axis, it follows that $N = 4$ and $m = 3$. If the square P is taken to be equal to Q_0 then although most of the sets $Y_r \cap S_n \cap P$ contain zero points, the estimate assigns them each at least one point. The problem arises from the way the points are assigned to the sets Y_r ; the sets Y_r with small r consistently get more points than those with large r . To mitigate this discrepancy it is sufficient to take each point $z_j = x_{k,n} + iy_{k,n}$ in the set S_n and instead of assigning it to the set Y_r with $r = k \bmod mN$, use the assignment $r = (k + \tau(n)) \bmod mN$ where $\tau(n)$, which acts as an offset, is a function of the set S_n .

The case for a general Carleson measure μ is now straight forward, but since the proof, which involves proving that the approximations converge to a bounded distribution with boundary values, does not add anything to the construction

of the approximations, the reader is referred to Garnett [15] for the details.

Theorem 3.8 [Garnett [15]]

Let $\mu = C \sum \alpha_j x_j \delta_{z_j}$ be a Carleson measure with Carleson constant $N(\mu) \leq C$.

Then there is an H_∞ function $b(z)$ such that

$$\bar{\partial}b(z) = \mu,$$

and the boundary value of b is an L^∞ function with $\|b\|_\infty < C$ for C a positive constant.

3.3 Examples

In this section two examples are presented that indicate how the methods presented in the previous section can be applied to calculate solutions to the Bezout equation.

3.3.1 Example 1

In the first example the Bezout Identity comes from an unstable plant with rational H_∞ factors f_1 and f_2 . Although this simplification allows the problem to be solved with elementary algebraic methods, the example serves well as an illustration of the general method.

$$f_1(z) = \frac{z-1}{z+1}, \quad \text{and} \quad f_2 = \frac{z-2}{z+1},$$

The solution is found by following the construction given in the proof of Theorem 3.1.

$$h_1^1(z) = \frac{1}{2} \frac{z+1}{z-1} \quad \text{and} \quad h_2^1(z) = \frac{1}{2} \frac{z+1}{z-2}$$

solve the equation

$$P_f h^1 = 1$$

in a distributional sense. Taking the exterior derivative of h^1 gives

$$\bar{\partial} h^1 = (\pi\delta_1, 3/2\pi\delta_2),$$

and inverting the operator P_f in

$$\begin{aligned} P_f h^2 &= \bar{\partial} h^1 \\ &= (\pi\delta_1, 3/2\pi\delta_2), \end{aligned}$$

produces the solution $h_{1,2}^2 = -2\pi\delta_1 - 9/2\pi\delta_2$. The Bezout problem is now reduced to solving the equation $\bar{\partial} h^3 = h^2$, or in components

$$\begin{aligned} \frac{d}{d\bar{z}} h_{1,2}^3 &= h_{1,2}^2 \\ &= -2\pi\delta_1 - 9/2\pi\delta_2. \end{aligned} \tag{3.16}$$

Let $B_1(z)$ be the Blaschke Product with zeros at $z = 1$ and $z = 2$, then it follows from equation (3.9) that

$$\frac{d}{d\bar{z}} 1/B_1(z) = -6\pi\delta_1 + 12\pi\delta_2,$$

and equation (3.16) is solved by $h_{1,2}^3 = F_{\text{int}}(z)/B_1(z)$ when $F_1(z)$ is a bounded function that is analytic on the right half plane and interpolates the values

$F_{\text{int}}(1) = 1/3$ and $F_{\text{int}}(1) = -3/8$. Such a function is given by

$$F_{\text{int}}(z) = -\frac{z-2}{z+2} - \frac{9z-1}{8z+1}$$

which produces the solution

$$h_{1,2}^3(z) = -\frac{z+1}{z-1} - \frac{9z+2}{8z-2}$$

The formula for the analytic solution of the Bezout equation h is

$$\begin{aligned} h &= h^1 - P_f h^3 \\ &= (h_1^1 - h_{1,2}^3 f_2, h_2^1 - h_{2,1}^3 f_1) \\ &= \left(\frac{21z+30}{8z+8}, -\frac{13z+19}{8z+8} \right) \end{aligned}$$

The answer is checked by evaluating

$$\begin{aligned} P_f h &= h_1 f_1 + h_2 f_2 \\ &= \frac{21z+30}{8z+8} \frac{z-1}{z+1} - \frac{13z+19}{8z+8} \frac{z-2}{z+1} \\ &= 1, \end{aligned}$$

the expected answer.

3.3.2 Example 2

The second example involves an irrational function. Set

$$f_1(z) = e^{-\tau z} \frac{1}{1+z}, \quad f_2(z) = \frac{z-1}{z+1}$$

The irrationality of the data means that greater care needs to be taken when inverting the operator P_f . The first step is to find an instance of the partition

of unity predicted by Lemma 3.2. The characteristic functions

$$\phi_1 = \chi_{\{|z|>\sigma\}}, \quad \phi_2 = \chi_{\{|z|<\sigma\}}$$

satisfy the requirements when $0 < \operatorname{Re} \sigma < 1$. Formula 3.7 is used to invert the operator P_f in the equation $P_f h^1 = 1$ to give a 1-form valued function h^1 with coefficients

$$h_1^1 = e^{\tau z}(z+1)\phi_1, \quad h_2^1 = \frac{z+1}{z-1}\phi_2$$

The $\bar{\partial}$ derivative of h^1 has components:

$$\begin{aligned} \bar{\partial} h_1^1 &= e^{\tau z}(z+1)\bar{\partial}\phi_1 \\ \bar{\partial} h_2^1 &= \left(\frac{z+1}{z-1}\right)\bar{\partial}\phi_2. \end{aligned}$$

Equation 3.7 is again used to solve $P_f h^2 = \bar{\partial} h^1$ with result

$$h_{1,2}^2 = e^{\tau z} \frac{(z+1)^2}{z-1} \bar{\partial}\phi_2\phi_1 - e^{\tau z} \frac{(z+1)^2}{z-1} \bar{\partial}\phi_1\phi_2$$

Let $\mu = \phi_1\bar{\partial}\phi_2 - \phi_2\bar{\partial}\phi_1$; that this expression defines a measure can be seen as follows. Let ψ be a C^∞ mollifier, and denote by $\tilde{\chi}_{\psi, |z|>\sigma}$ the mollification of the characteristic function, then if $\tilde{\phi}_1$ and $\tilde{\phi}_2$ denote the C^∞ approximations to the functions ϕ_1 and ϕ_2 , and \tilde{D} is the set $\{z : |z| = \sigma\} + \operatorname{supp} \psi$, the distribution

$\tilde{\phi}_1 \bar{\partial} \tilde{\phi}_2 - \tilde{\phi}_2 \bar{\partial} \tilde{\phi}_1$ acts on a test function f as follows:

$$\begin{aligned}
& \int_{\bar{D}} \tilde{\phi}_1 \bar{\partial} \tilde{\phi}_2 f - \tilde{\phi}_2 \bar{\partial} \tilde{\phi}_1 f \, dx \, dy \\
&= i/2 \int_{\bar{D}} \tilde{\phi}_1 \bar{\partial} \tilde{\phi}_2 f - \tilde{\phi}_2 \bar{\partial} \tilde{\phi}_1 f \, dz \wedge d\bar{z} \\
&= -i/2 \int_{\bar{D}} (\tilde{\phi}_2 \bar{\partial} \tilde{\phi}_1 - \tilde{\phi}_1 \bar{\partial} \tilde{\phi}_2) f \, dz \wedge d\bar{z} \\
&= -i/2 \int_{\bar{D}} f \bar{\partial} \tilde{\phi}_1 \, dz \wedge d\bar{z} \\
&= -i/2 \int_{\bar{D}} d(f \tilde{\phi}_1 \, dz) \\
&\quad + i/2 \int_{\bar{D}} \tilde{\phi}_1 \bar{\partial} f \, dz \wedge d\bar{z} \\
&= -i/2 \int_{\partial \bar{D}} f \tilde{\phi}_1 \, dz \\
&\quad + i/2 \int_{\bar{D}} \tilde{\phi}_1 \bar{\partial} f \, dz \wedge d\bar{z}
\end{aligned}$$

The calculation above uses the fact that $\tilde{\phi}_1 + \tilde{\phi}_2 = 1$, and the corollary that $\bar{\partial} \tilde{\phi}_1 = -\bar{\partial} \tilde{\phi}_2$. Consider what happens as the support of the mollifier is allowed to shrink to zero; for any test function f the first term converges to a line integral on ∂D while the second integral, which has a uniformly bounded integrand, converges to zero. It follows that μ is indeed a measure, and is given explicitly by

$$\begin{aligned}
\int_{\operatorname{Re}(z) > 0} f \, d\mu &= -i/2 \int_{|z|=\sigma} f(z) \, dz \\
&= -\sigma/2 \int_{-\pi/2}^{\pi/2} f(\sigma e^{i\theta}) e^{i\theta} \, d\theta
\end{aligned}$$

where in the last integral the substitution $z = \sigma e^{i\theta}$ is made.

Returning to the example, $h_{1,2}^2$ can now be written as

$$h_{1,2}^2 = e^{\tau z} \frac{(z+1)^2}{z-1} \mu(z).$$

The algorithm presented in the previous section is used to solve the equation

$$\bar{\partial}h^3 = h^2, \quad (3.17)$$

and the final solution is $h = h^1 - P_f h^3$. A computer program has been written to calculate the solution to Equation (3.17) for the example given. The software generates a finitely supported approximation to the measure μ , decomposes the measure into a sum of measures each with points of support well-spaced with respect to the pseudo-hyperbolic distance, and solves the Blaschke Product interpolation problem using an iterative scheme. The result of the algorithm is a rational H_∞ function that approximates the solution to the $\bar{\partial}$ equation. by increasing the number of points in the support of the approximating measure, the L^∞ norm of the error in the solution may be reduced to lie within an arbitrary distance from zero.

Figures 3.3 and 3.4 illustrate the solutions obtained for Example 2 when $\tau = 0.4$ and $\sigma = 4$. The graphs in Figure 3.4 show the responses of the deconvolution filters to the pulse input illustrated in Figure 3.3. That the solutions solve the Bezout Equation is clear from the algorithm, and the causal nature of the responses in Figure 3.4 indicates that the solutions closely approximate holomorphic functions. The sharp corners that occur in the pulse responses are a characteristic of infinite dimensional filters that may incorporate pure delays

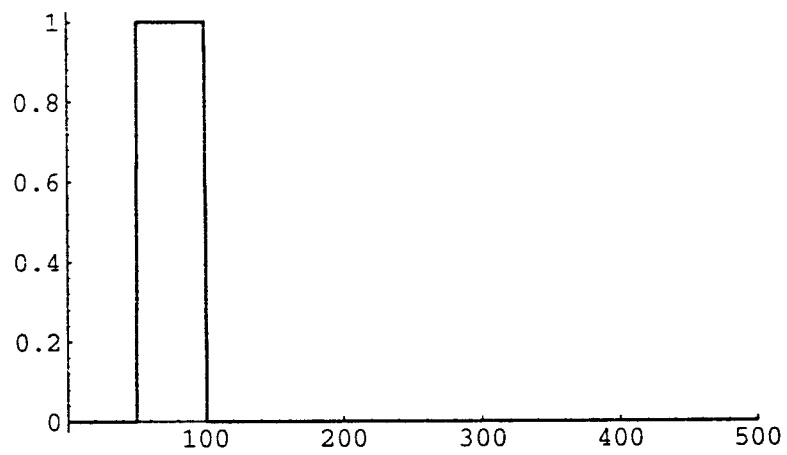


Figure 3.3: Test pulse applied to deconvolution filters

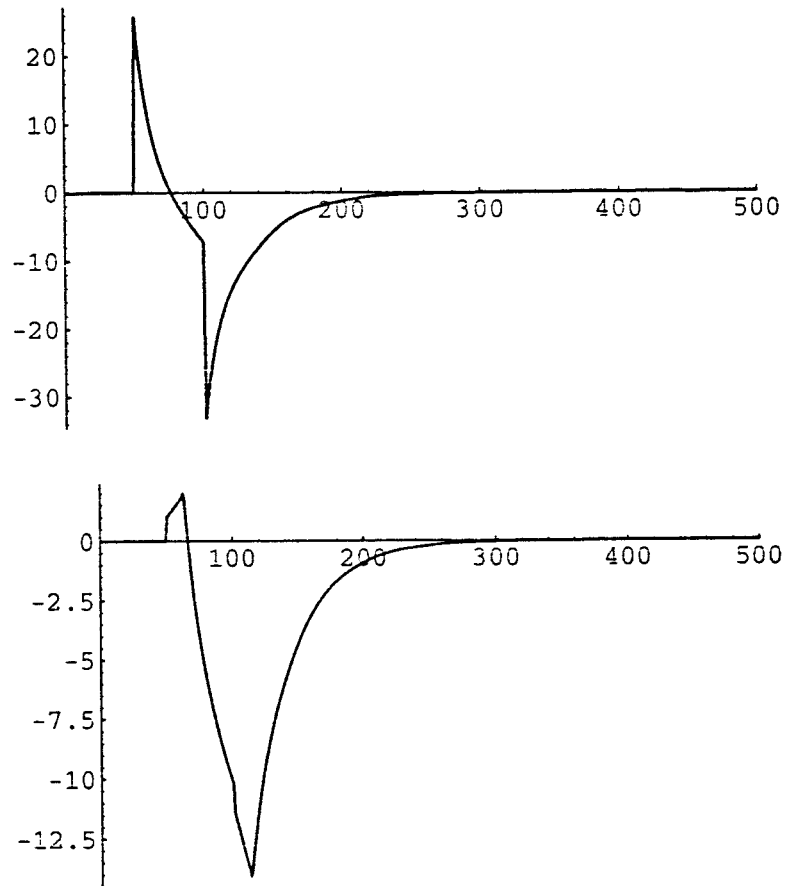


Figure 3.4: Pulse responses for filters representing solutions to the Bezout equation

The Nehari Problem

The final chapter demonstrates how the solution to the Bezout Equation presented in the previous chapter may be used to calculate sub-optimal controllers at least for the case of single input single output systems. While the method may not produce the “best” sub-optimal solutions, it does have the advantage that it requires relatively little information about the open loop transfer function. In particular, an inner outer factorization is not required.

Recall the definition of T_1 , T_2 , and T_3 in Theorem 2.1

$$T_1 = G_{11} + G_{12}M\tilde{Y}G_{21}$$

$$T_2 = G_{12}M$$

$$T_3 = \tilde{M}G_{21},$$

let

$$\eta = \inf_{Q \in H_\infty} \|T_1 - T_2QT_3\|,$$

then Theorem 2.1 formulates the sub-optimal controller design problem as a

search for a matrix Q with H_∞ entries that satisfies

$$\|T_1 - T_2QT_3\| \leq \eta + \epsilon \quad (4.1)$$

for some acceptably small ϵ . Suppose Q is one such matrix, then there exists $P \in H_\infty$, with $\|P\| \leq \eta + \epsilon$ and

$$T_1 - T_2QT_3 = P \in H_\infty,$$

or, with a small rearrangement,

$$T_1 = P1 + T_2QT_3 \quad (4.2)$$

in which 1 is the function $1(z) = 1$. The suboptimal controller design problem can now be restated as: given T_1, T_2 and $T_3 \in H_\infty$ find $P, Q \in H_\infty$ that satisfy equation (4.2) and such that $\|P\| \leq \eta + \epsilon$.

In the SISO case the operators in (4.2) are members of the commutative algebra of H_∞ , and the equation reduces to the Bezout Equation that was solved in the previous chapter. Let \mathcal{Z} denote the zero set of T_2T_3 , then a lower bound on η is given by

$$\eta \geq \eta_0 = \sup_{z \in \mathcal{Z}} T_1(z).$$

This observation provides the basis for the following approach to the Nehari Problem. Choose a neighborhood Ω of \mathcal{Z} such that the measure $\mu_\Omega = T_1(z)^2 / (T_2(z)T_3(z)) \bar{\partial} \chi_{\partial\Omega}$ is a Carleson measure that is supported on the boundary of Ω and has Carleson constant $N(\mu_\Omega)$. Following the notation of Theorem 3.1, define f_1 as the constant function 1, let $f_2 = T_2(z)T_3(z)$, $g(z) = T_1(z)$, and

associate with Ω the characteristic functions $\phi_1 = \chi_\Omega$ and $\phi_2 = \chi_{\mathbb{C} \setminus \Omega}$, then, the equation $P_f h^1 = g$ is solved by

$$\begin{aligned} h_1^1 &= T_1(z)\phi_1(z) \\ h_2^1 &= \frac{T_1(z)}{T_2(z)T_3(z)}\phi_2(z). \end{aligned}$$

The function h_1^1 is uniformly bounded, and the minimum upper bound of its magnitude is given by

$$\eta_1 = \sup_{z \in \mathcal{Z}} |T_1(z)|.$$

Equation (4.2) can be rewritten as $P_f h = g$, and if $h_{1,2}^3$ is the solution to the equation $\bar{\partial} h^3 = \mu_\Omega$, then from the proof of Theorem 3.1 an H_∞ solution is given by

$$h = (P, Q) = h^1 - P_f h^3.$$

If $b(z) = h_{1,2}^3(z)$ then the solution may be rewritten in terms of its components P and Q as follows

$$\begin{aligned} P(z) &= T_1(z)\phi_1(z) - T_2(z)T_3(z)b(z) \\ Q(z) &= \frac{T_1(z)}{T_2(z)T_3(z)}\phi_2(z) + b(z) \end{aligned}$$

This gives the following estimate for the norm in the Nehari problem

$$\|P(z)\| \leq \eta_1 + \|T_2(z)T_3(z)b(z)\|$$

and leads to the conclusion that the quality of the suboptimal solution calculated will depend on the the choice of Ω and the ability to find solutions to the equation

$\bar{\partial}b = \mu_\Omega$ with small H_∞ norm and rapid decay away from the support of μ_Ω . The method presented in Chapter 3 for the solution of the $\bar{\partial}$ problem is not optimal in this sense; in particular, as was mentioned, the bound given for the interpolation operator of Theorem 3.4 is not optimal, and better interpolation operators have been presented in [21] and [22]. An obvious avenue for future work would be to investigate whether these operators give a clear computational advantage.

The technique presented above was used to compute a controller for the example given in Section 3.3.2. The first graph of Figure 4.1 shows the response of the closed loop transfer function to the test pulse of Figure 3.3 applied at the input. The second graph shows the response of the sensitivity function to the same input.

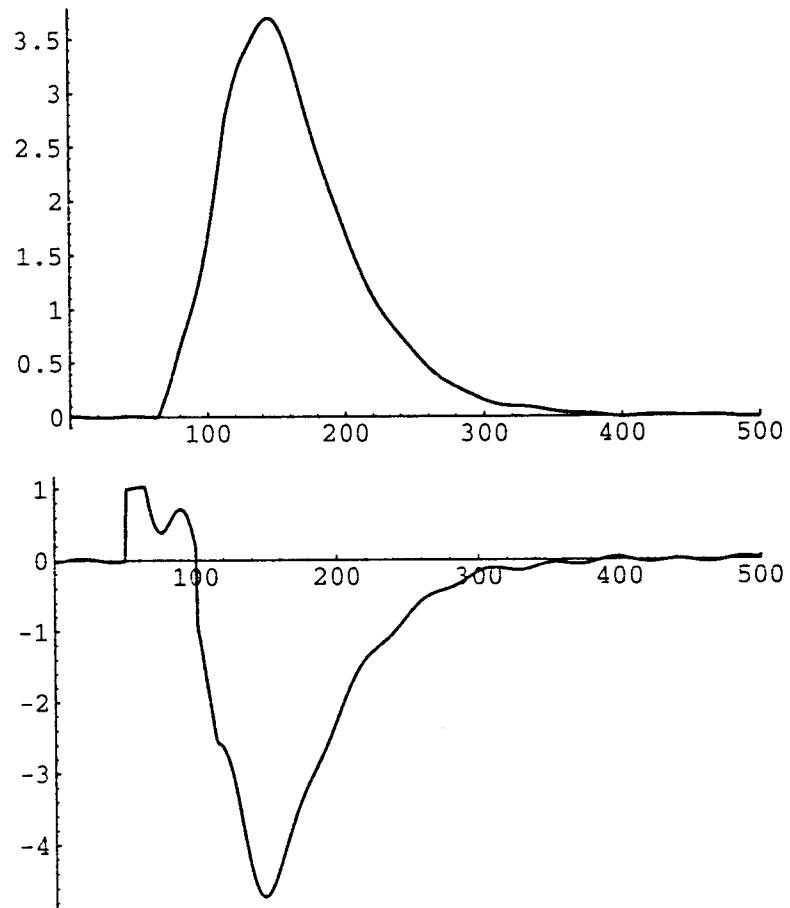


Figure 4.1: Pulse responses for closed loop transfer function and sensitivity function of closed loop system

BIBLIOGRAPHY

- [1] John S. Baras. Frequency domain design of linear distributed systems. In *Proceedings of the 19th IEEE Conference On Decision and Control*, pages 728 – 732, 1980. This paper was also presented in a preliminary form at The 4'th International Symposium on the Mathematical Theory of Networks and Systems, Delft, The Netherlands, July 3–6, 1979.

- [2] John S. Baras. Complex variable methods in the modelling and control of distributed parameter systems. Unpublished notes for a paper presented at the International Symposium on the Mathematical Theory of Networks and Systems, Amsterdam, 1989.

- [3] John S. Baras and Patrick Dewilde. Invariant subspace methods in linear multivariable-distributed systems and lumped-distributed network synthesis. *Proceedings of the IEEE*, 64:160 – 178, 1976.

- [4] Carlos A. Berenstein and Daniele C. Struppa. l -inverses for polynomial matrices of non constant rank. *Systems & Control Letters*, 6:309–314, Jan 1986.

- [5] Carlos A. Berenstein and B.A. Taylor. Interpolation problems in \mathbb{C}^n with applications to harmonic analysis. *Journal D'Analyse Mathématique*, 38:188 – 254, 1980.
- [6] Lennart Carleson. Interpolation by bounded analytic functions and the corona problem. *Ann. of Math.*, 76:547 – 559, 1962.
- [7] John C. Doyle, Keith Glover, Pramod P. Khargonekar, and Francis Bruce A. State-space solutions to standard H_2 and H_∞ control problems. *IEEE Transactions on Automatic Control*, 34:831 – 846, 1989.
- [8] J.P. Earl. On the interpolation of bounded sequences by bounded functions. *J. London Math. Soc. (2)*, 2:544 – 548, 1970.
- [9] J.P. Earl. A note on bounded interpolation in the unit disk. *J. London Math. Soc. (2)*, 13:419 – 423, 1976.
- [10] D.S. Flamm and S.K. Mitter. H_∞ -sensitivity for delay systems, part 1. *Systems and Control Letters*, 9:17 – 24, 1987.
- [11] Ciprian Foias and Allen Tannenbaum. On the nehari problem for a certain class of L_∞ -functions appearing in control theory. *Journal of Functional Analysis*, 74:146 – 159, 1987.
- [12] Ciprian Foias and Allen Tannenbaum. On the four block problem, i. *Operator Theory: Advances and Applications*, 32:93 – 112, 1988.

- [13] Ciprian Foias and Allen Tannenbaum. On the four block problem, ii: the singular system. *Integral Equations and Operator Theory*, 11:726 – 767, 1988.
- [14] Bruce A. Francis. *A course in H^∞ control theory*. Springer Verlag, first edition, 1987.
- [15] John B. Garnett. *Bounded Analytic Functions*. Academic Press, first edition, 1981.
- [16] Lars Hörmander. *An Introduction to Complex Analysis in Several Variables*. Van Nostrand, first edition, 1966.
- [17] Lars Hörmander. Generators for some rings of analytic functions. *Bull. Amer. Math. Soc.*, 73:943 – 949, 1967.
- [18] Lars Hörmander. *The Analysis of Linear Partial Differential Operators I*. Springer Verlag, second edition, 1990.
- [19] Thomas W. Hungerford. *Algebra*. Springer Verlag, second edition, 1974.
- [20] Peter Jones. Carleson measures and the Fefferman - Stein decomposition of $BMO(\mathbb{R})$. *Ann. of Math.*, 111:197 – 208, 1980.
- [21] Peter W. Jones. L^∞ estimates for the $\bar{\partial}$ problem in a half-plane. *Acta Mathematica*, 150:137 – 152, 1983.

- [22] N.K. Nikol'skii. *Treatise on the Shift Operator*. Springer-Verlag, first edition, 1986.
- [23] K.P.S. Bashkara Rao. On generalized inverses of matrices over integral domains. *Linear Algebra and its Applications*, 49:179 – 189, 1983.
- [24] Malcolm C. Smith. On stabilization and the existence of coprime factorizations. *IEEE Transactions on Automatic Control*, 34:1005 – 1007, 1989.
- [25] Daniele Carlo Struppa. The fundamental principle of systems of convolution equations. *Memoirs of the A.M.S.*, 273:1 – 167, 1983.
- [26] Gilead Tadmor. Worst case design in the time domain: The maximum principle and the standard H_∞ problem. *Mathematics of Control, Signals, and Systems*, 3:301 – 324, 1990.
- [27] B Van Kuelen. L^∞ -compensation with mixed sensitivity as a broadband matching problem. *Systems & Control Letters*, 4:125 – 129, 1984.
- [28] R. O. Wells. *Differential Analysis on Complex Manifolds*. Springer-Verlag, second edition, 1980.
- [29] G. Zames. Feedback and optimal sensitivity: Model reference transformations, multiplicative seminorms, and approximate inverses. *IEEE Transactions on Automatic Control*, 26:301 – 320, 1981.

- [30] G. Zames and Bruce A. Francis. Feedback, minimax sensitivity, and optimal robustness. *IEEE Transactions on Automatic Control*, 28:585 – 601, 1983.