BAYESIAN

SEQUENTIAL HYPOTHESIS TESTING

by

David C. Mac Enany

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ABSTRACT

Title of Thesis: Bayesian Sequential Hypothesis Testing

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Thesis Directed by : Dr. John S. Baras

Professor and Head of the Systems Research Center

Electrical Engineering Department

In this thesis, optimality results are presented for Bayesian problems of sequential hypothesis testing. Conditions are given which are sufficient to demonstrate the existence and optimality of threshold policies and others are given which help characterize these policies. The general results are applied to solve three specific problems where the observations respectively arise from a time-homogeneous diffusion, a progressive process observed through white Guassian noise, and a time-homogeneous Poisson process. It is shown that threshold policies are optimal in all three cases. An exact formula for the Bayesian cost in the Poisson case will be presented for the first time.

Acknowledgements

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Introduction

The principal goal of this thesis is to demonstrate the optimality of certain statistical procedures, called threshold policies, as applied to problems of Bayesian sequential hypothesis testing. The importance of Bayesian sequential hypothesis testing is due not only to its applicability to many practical signal detection problems, but also because it provides a theoretical foundation with which to prove optimality results for other sequential statistical procedures.

The mathematical theory available to formulate and obtain optimality results in sequential analysis roughly divides into two main categories, the first of which can be called the 'dynamic programming' approach, and the second which might be called the 'excessive function' approach. Naturally, both approaches share many similarities but are sufficiently different to permit this broad classification. The methodology adopted in this thesis is best described as falling into the second category, but represents a significant modification to the usual development [DYNKIN], [SHIRYAYEV 77], [THOMPSON].

Loosely speaking, the excessive function theory approach extends well-known results for sequences of independent and identically distributed random variables to continuous-time (see also [KEL'BERT]). In this thesis, a more direct development is taken, one which yields a rich interplay between analytical and probabilistic ideas. While it is true that that the excessive function theory is more general and hence more powerful, it is also true that the approach contained herein requires considerably less

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specialized mathematical machinery. Thus one can argue that it renders important optimality results more accessible to a wider audience. Moreover, although it is less generally applicable, it is nonetheless powerful enough that with it one can obtain the solutions to three important applications, one of which, to this author's knowledge, has never before appeared.

The organization of the thesis by chapter is from the general to the specific, and is given as follows.

In Chapter I is given a general, detailed treatment of certain types of semimartingales, special cases of which will be needed later in the applications chapter. Also given, is a proof of the existence and uniqueness of solutions to a functional differential equation. It was a tough decision to include this theorem in the first chapter, but it was felt that this type of differential equation is so intrinsic to the problems involving Poisson-type processes, that relegating the result to an unread appendix would be cheating the reader out of valuable insight into the nature of these types of problems, and their associated difficulties. An interesting historical note is that one of the earliest (post-Euler) references to equations of this type is due to Poisson himself [POISSON].

In Chapter II, the general Bayesian sequentail hypothesis testing problem is discussed and formulated. A principal goal of the chapter is to show how one can greatly reduce the complexity of the search for the optimal test.

In Chapter III, the principal optimality result is given, the essence of which is contained in Lemma 3.1.1. Also included are theorems whose importance is paramount in the discovery of the unique test which is optimal in a particular application.

In Chapter IV, the results of the previous chapters are applied to solve two Bayesian sequential detection problems, one involving a fully-observed time-homogeneous diffusion, and the other involving a partially-observed progressively measurable semimartingale process. Also, the Bayesian sequential hypothesis testing

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problem based on observations of a Poisson process is solved. Explicit formulae for the risk and thresholds are presented.

After reading the thesis, one should be able to appreciate the difficulty in handling non-diffusion type processes in sequential analysis, especially in view of the formulae obtained for the Poisson case. It seems safe to say that future work in sequential analysis will deal with ever more exotic processes as demanded by applications, and that therefore the research in this area will of necessity focus more on approximate and asymptotic methods. It appears that a good starting point for such research is with the relatively recent functional central limit theory [HELLAND].

Chapter I
Prerequisite Considerations

1.0   Introduction

This chapter will establish some important results which will be used later on. Each theorem will be presented with slightly more generality than will be necessary in the hope that the essence of each is more clearly understood.

Theorem 1.1.1 demonstrates that special types of semimartingales always escape intervals in finite time. This result will be used to show that particular hypothesis testing strategies—threshold policies based on these semimartingales—in effect bifurcate the sample space Ω, in the sense that they terminate in finite time almost surely under either hypothesis.

Theorem 1.1.2 establishes a weak form of the differential rule for functions of locally finite variation semimartingales with piecewise monotone sample paths. The usual generalized Itô differential rule is stated for twice continuously differentiable functions of general semimartingales. This is sometimes specialized to semimartingales driven by discontinuous martingale processes where the smoothness requirement is weakened to 'once continuously differentiable'. It will become necessary in later chapters to consider stochastic differentials of functions even less smooth, specifically functions which are piecewise right continuous and have piecewise right continuous derivatives. In fact, functions of this type are intrinsic to the theory of discontinuous

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semimartingales. The reason is that the formal application of the usual differential rule to discontinuous semimartingales very often leads to functional differential equations [e.g. HIBEY, SEGALL] of a type whose solutions are generically not continuously differentiable, and which do not, therefore, satisfy the assumptions which led to them.

Theorem 1.2.1 establishes the existence and uniqueness of the solution to a certain functional advance differential equation. Its solution yields an important quantity arising in the sequential Bayesian hypothesis testing problem involving time-homogeneous Poisson processes.

1.1 Semimartingale Prerequisites

Consider a probability triple $(\Omega, \mathcal{F}, P)$ equipped with a right-continuous filtration $\mathcal{F}_t, t \geq 0$. In this thesis, the following definition of semimartingale will be used.

Definition 1.1.1 *A* $(P, \mathcal{F}_t)$-**semimartingale** *is a random process X which has the decomposition,*

$$X_t = X_0 + M_t + A_t \quad t \geq 0 \quad P\text{-a.s.} \tag{1.1.1}$$

*where X_0 is a $(P, \mathcal{F}_t)$-r.v., $|X_0| < \infty$ P-a.s., M is corlol $(P, \mathcal{F}_t)$-**local martingale**, with $M_0 = 0$, and A is a **right-continuous**, $\mathcal{F}_t$-adapted process, initially zero and of* **locally integrable variation** *[ELLIOTT]. The value set for X will be denoted as $E \subseteq \mathbb{R}$.*

Without loss of generality (wlog) then, take the **predictable** version of A and note that the random variable A_∞ exists [ELLIOTT]. In most of this thesis, the full generality of the definition will not be needed. However, as stated previously, the results given in this chapter are somewhat more general than will be necessary in later chapters but that this is done for clarity and because the general results were no harder to obtain.

Since a principal theme of the thesis is to consider threshold policies and hence first exit times of processes from intervals, one is naturally interested in the properties that should be satisfied by a process upon which one intends to base a threshold policy. One obvious requirement is that such a process should eventually exit the threshold interval. The next theorem gives three conditions which are sufficient to guarantee such behavior for a semimartingale. It is an adaptation and generalization of Lemma 17.7 in [LIPTSER & SHIRYAYEV 78].

Theorem 1.1.1 *Let X be a $(P, \mathcal{F}_t)$-semimartingale as in definition 1.1.1, and suppose $P\{|A_\infty| = \infty\} = 1$, while $P\{|A_t| = \infty\} = 0 \,\forall 0 \leq t < \infty$. Furthermore, suppose*

$|E[A_{\tilde{\tau}}]| \geq E[\|A_{\tilde{\tau}}\|]$ *for any* $(P, \mathcal{F}_t)$*-stopping time* $\tilde{\tau}$*. Then,* X *will escape any open interval containing* X_0 *in finite time,* P*-a.s.*

Proof: Choose $X_0 = 0$ P-a.s. wlog and choose a, b to satisfy $-\infty < a < 0 < b < \infty$, a, b otherwise arbitrary. Define $\tau = \inf\{t \geq 0 : X_t \notin (a, b)\}$ and $\sigma_n = \inf\{t \geq 0 : \int_0^t |dA_s| \geq n\}, n \geq 0$. Note that τ and σ_n are both $(P, \mathcal{F}_t)$-stopping times and thus so is $\tau \wedge \sigma_n$. Moreover, since A is of locally integrable variation, $\sigma_n < \infty$ P-a.s. $\forall n \geq 0$, and $\sigma_n \leq \sigma_{n+1} \uparrow \infty$ P-a.s.

Next, assume that for all $n \geq 0$ it is true that,

$$a \leq X_{\tau \wedge \sigma_n} \leq b. \tag{1.1.2}$$

Otherwise, there exists an $n \geq 0$ satisfying $\tau \wedge \sigma_n = \tau < \sigma_n < \infty$ P-a.s., and such that $X_{\tau \wedge \sigma_n} = X_\tau \notin (a, b)$, from which the theorem follows. So assuming 1.1.2 holds for all $n \geq 0$, the remainder of the proof will proceed by *reductio ad absurdum.* From 1.1.2 then, using the fact that M is a $(P, \mathcal{F}_t)$-local martingale one obtains,

$$a - b < a \leq E[A_{\tau \wedge \sigma_n}] \leq b < b - a, \tag{1.1.3}$$

since clearly $a - b < a$, and $b < b - a$. Rewriting 1.1.3 yields,

$$|E[A_{\tau \wedge \sigma_n}]| < b - a < \infty, \tag{1.1.4}$$

and since $\sigma_n \uparrow \infty$ P-a.s., then $A_{\tau \wedge \sigma_n} \to A_\tau$ P-a.s, and invoking the bounded convergence theorem yields $|E[A_\tau]| < b - a$. From this it follows that $E[\|A_\tau\|] < b - a$ by hypothesis, and hence,

$$E[\|A_\tau\|] < \infty. \tag{1.1.5}$$

But then,

$$\infty > E[\|A_\tau\|] = E[1_{\{\tau=\infty\}}\|A_\tau\|] + E[1_{\{\tau<\infty\}}\|A_\tau\|]$$
$$\geq E[1_{\{\tau=\infty\}}\|A_\tau\|] = E[1_{\{\tau=\infty\}}\|A_\infty\|], \tag{1.1.6}$$

and since $|A_\infty| = \infty$ P-a.s., it must be true that $\tau < \infty$ P-a.s. That is to say, X will exit (a, b) almost surely in finite time. Since a and b were arbitrary modulo X_0, the result is shown. ∎

Note that the result can be extended to a process whose **compensator** almost surely takes on one of two not necessarily infinite values at infinity, as long as one can exhibit a suitably regular, bijective function which maps those values into plus and minus infinity. The theorem then holds for the semimartingale resulting from the function acting on the original process, and hence for the original process by inversion.

Perhaps it should also be stressed that the theorem states only a sufficiency result. For instance, it is clear that no martingale can satisfy its hypotheses, because martingales lack (nontrivial) compensators. On the other hand, using the same notion as in the preceding paragraph, a bijective function of a martingale might yield a process with a nontrivial compensator which indeed satisfies the theorem. In fact, in Chapter IV there will be a need to guarantee that the exit times of a certain martingale (an *a posteriori* probability) are almost surely finite. The approach taken to accomplish this has just been outlined and utilizes Theorem 1.1.1.

When a semimartingale as in definition 1.1.1 escapes an open interval, questions naturally arise as to its whereabouts at the time of escape. The next definition provides a handle with which to phrase such questions.

Definition 1.1.2 *Let X be a real $(P, \mathcal{F}_t)$-semimartingale, taking values in $E \subseteq \mathbb{R}$, and let $X_0 \in I_0, P$-a.s., where $I_0 \subset E$, is some nonempty open interval. Define the sets,*

$$\bar{\Omega}_0 = \{\omega \in \Omega \ : \ X_0(\omega) \in I_0\},$$

$$\bar{\Omega}_\infty = \{\omega \in \Omega \ : \ X_\infty(\omega) \in I_0\},$$

and suppose τ is given as,

$$\tau = \inf\{t \geq 0 \ : \ X_t \notin I_0\}.$$

Then the X-boundary of I_0 is defined and denoted as,

$$\partial_x I_0 = \{x \in E \ : \ x = X_{\tau(\omega)}(\omega) \text{ for some } \omega \in \bar{\Omega}_0 \setminus \bar{\Omega}_\infty\}.$$

In addition, the X-closure of I_0 is defined and denoted as,

$$[I_0]_x = I_0 \cup \partial_x I_0.$$

It will also be convenient to define the upper and lower X-boundary of I_0 as $\partial_x^+ I_0$ and $\partial_x^- I_0$, respectively, i.e., $\partial_x I_0 = \partial_x^- I_0 \cup \partial_x^+ I_0$, and $x' \in \partial_x^+ I_0$ implies $x' > x$ for all $x \in \partial_x^- I_0$, and vice versa.

Note that $\partial_x I_0$ is P-a.s. nonempty if X satisfies the hypotheses of Theorem 1.1.1, and often $\partial I_0 \subseteq \partial_x I_0$ for any such X, where ∂I_0 is the usual boundary of I_0 consisting of its endpoints. In fact, if X is a continuous semimartingale satisfying Theorem 1.1.1, then $\partial I_0 = \partial_x I_0$ P-a.s. Also note that neither $\partial_x I_0$ nor $[I_0]_x$ are necessarily closed sets. For instance, if X is a real right continuous semimartingale driven by a discontinuous martingale whose jumps are positive, then $\partial_x^+ I_0$ is generically a semi-open interval, closed on the left and open on the right.

As it was just indicated, the notion of the boundary of an interval with respect to a jump-type process is not trivial as it is for continuous process. The simplest case is where the jumps of the process are always positive (negative) while the sample paths are strictly nonincreasing (nondecreasing) in between the jumps. Fortunately, this is the situation in the binary hypothesis testing of counting processes. It is also fortunate because a weaker form of the generalized Itô can be given for a process which has such sample paths, and because one soon discovers that the differential rule, as it is usually stated, is inadequate to handle the types of functions which arise in the hypothesis testing.of such processes. Specifically, the Itô rule for jump processes is usually given for functions which are once continuously differentiable, whereas functions which are less smooth need to be considered. The next definition gives a general description of the type of functions which arise.

Definition 1.1.3 *Let $F : E \to \mathbb{R}$, with $E \subseteq \mathbb{R}$, and suppose that for all but a countable set of real numbers, $D \subseteq \mathbb{R}$, F is n-times continuously differentiable, $n \geq 0$.*

Further, at each point $x_0 \in D$ assume that $\lim_{x \downarrow x_0} F^{(k)}(x)$ exists for $k = 0, 1, ..., n$, and define this limit to be the kth derivative of F at $x_0 \in D$. Also suppose that $\lim_{x \uparrow x_0} F^{(k)}(x)$ exists and is finite for all $x_0 \in D$, $k = 0, 1, ..., n$. Then, F is said to be a $C^{n+}(E)$ function or one writes $F \in C^{n+}(E)$.

Loosely then, a $C^{n+}(E)$ function is a function which is n-times, piecewise right-continuously differentiable, with left-hand limits, and only a countable set of points that prevents it from being n-times continuously differentiable. This set of points is called the set of **breakpoints** of the function.

The notion of monotonic sample paths introduced above is useful when considering C^{1+} functions of jump semimartingales. For instance, if one applies a right-discontinuous function, such as the greatest integer function, to a right-continuous jump semimartingale with piecewise nondecreasing sample paths, then the resulting process will be right-continuous. However, applying the same function to right-continuous semimartingale with piecewise nonincreasing sample paths yields a left-continuous process.

The next result is an extension of the usual differential rule to include C^{1+} functions of **locally finite variation** semimartingales which have piecewise nondecreasing sample paths.

Theorem 1.1.2 *Let X be a $(P, \mathcal{F}_t)$-semimartingale of locally finite variation with piecewise nondecreasing sample paths. Suppose a function F is given such that $F \in C^{1+}(E)$. Then $F(X)$ is a semimartingale, and with equality denoting P-indistinguishability* [BREMAUD],

$$F(X_t) = F(X_0) + \int_0^t F'(X_{s-})\, dX_s + \sum_{0 < s \leq t} \{\Delta F(X_s) - F'(X_{s-})\Delta X_s\}, \quad t \geq 0 \quad (1.1.7)$$

where ΔX_t denotes $X_t - X_{t-}$ for all $t > 0$, any process X.

Proof: First note that the right continuity and left-hand limits of F are necessary in order to yield a corlol process. To see this, consider an arbitrary $\omega \in \Omega$. Let

$t \geq 0$, then, because X is piecewise monotone increasing, there exist $\{t_n\}_{n=0}^{\infty}$, $t_n \downarrow t$ such that $X_{t_n}(\omega) \downarrow X_t(\omega)$. But then $F(X_{t_n}(\omega)) \to F(X_t(\omega))$ because F is right-continuous, and hence $F(X)$ is continuous on the right. For emphasis note that in general,

$$\lim_{x \uparrow c} F(x) = F(c-) \neq F(c) = \lim_{x \downarrow c} F(x) = F(c+) \qquad (1.1.8)$$

It is clear that $F(X_t)$ has limits on the left. Similarly, the right continuity of F' and implied left-hand limits yield $F'(X_{s-})$ locally bounded, and so the integral in 1.1.7 makes sense.

Let $\mathcal{D}$ be the set of disjoint, consecutive open intervals constructed from the set of breakpoints of F, say $\{d_k\}$, so that $\mathcal{D} = \{..., (d_{-1}, d_0), (d_0, d_1), (d_1, d_2), ...\}$. Write $D_i \in \mathcal{D}$ for $D_i = (d_i, d_{i+1})$. Without loss of generality, suppose $X_0 \in D_{i_0}$ for some $D_{i_0} \in \mathcal{D}$. Define τ_0 as the first time after $t = 0$ that X escapes D_{i_0} i.e., $\tau_0 = \inf\{t \geq 0 : X_t \notin (d_{i_0}, d_{i_0+1})\}$. Now, as long as $t \in [0, \tau_0)$, the usual differential rule [SNYDER] for jump processes holds for $F(X_t)$ since F is assumed strictly continuously differentiable on D_{i_0}. Hence for $t \in [0, \tau_0)$,

$$F(X_t) = F(X_0) + \int_0^t F'(X_{s-}) \, dX_s + \sum_{0 < s \leq t} \{\Delta F(X_s) - F'(X_{s-}) \Delta X_s\} \qquad (1.1.9)$$

Since $F(X_t)$ is corlol, then,

$$F(X_{\tau_0-}) = F(X_0) + \int_0^{\tau_0 -} F'(X_{s-}) \, dX_s + \sum_{0 < s \leq \tau_0 -} \{\Delta F(X_s) - F'(X_{s-}) \Delta X_s\} \qquad (1.1.10)$$

Decomposing dX_s as $dX_s = dX_s^c + \Delta X_s$ [WONG] one obtains,

$$\int_0^{\tau_0 -} F'(X_{s-}) \, dX_s = \int_0^{\tau_0} F'(X_{s-}) \, dX_s^c + \sum_{0 < s \leq \tau_0 -} F'(X_{s-}) \Delta X_s \qquad (1.1.11)$$

Combining 1.1.10 and 1.1.11 yields,

$$F(X_{\tau_0-}) = F(X_0) + \int_0^{\tau_0} F'(X_{s-}) \, dX_s^c + \sum_{0 < s \leq \tau_0 -} \Delta F(X_s) \qquad (1.1.12)$$

Now the possibilities as to whether X jumps a $t = \tau_0$ and/or F has discontinuities at $X_{\tau_0 -}$ and/or X_{τ_0} reduce to the cases $\Delta F(X_{\tau_0}) = 0$ or $\Delta F(X_{\tau_0})$ is nonzero but bounded (implicitly F is locally bounded). Either way, 1.1.12 yields,

$$F(X_{\tau_0 -}) + \Delta F(X_{\tau_0}) = F(X_0) + \int_0^{\tau_0} F'(X_{s-})\, dX_s^c + \sum_{0 < s \le \tau_0 -} \Delta F(X_s) + \Delta F(X_{\tau_0})$$

$$(1.1.13)$$

and so,

$$F(X_{\tau_0}) = F(X_0) + \int_0^{\tau_0} F'(X_s)\, dX_s^c + \sum_{0 < s \le \tau_0} \Delta F(X_s) \qquad (1.1.14)$$

Since $X_{\tau_0} \in D_{i_1}$ for some $D_{i_1} \in \mathcal{D}$ one can define $\tau_1 = \inf\{t \ge \tau_0 : X_t \notin D_{i_1}\}$. Then, if $t \in [\tau_0, \tau_1)$, $X_t \in D_{i_1}$, and F is continuously differentiable there so, arguing as before it follows that,

$$F(X_{\tau_1 -}) = F(X_{\tau_0}) + \int_{\tau_1}^{\tau_1 -} F'(X_{s-})\, dX_s^c + \sum_{\tau_1 < s \le \tau_2 -} \{\Delta F(X_s) - F'(X_{s-})\Delta X_s\}$$

$$(1.1.15)$$

and from this,

$$F(X_{\tau_1}) = F(X_{\tau_0}) + \int_{\tau_0}^{\tau_1} F'(X_{s-})\, dX_s^c + \sum_{\tau_0 < s \le \tau_1} \Delta F(X_s) \qquad (1.1.16)$$

By induction, for any $n \in \mathbb{N}$,

$$F(X_{\tau_n}) = F(X_{\tau_{n-1}}) + \int_{\tau_{n-1}}^{\tau_n} F'(X_{s-})\, dX_s^c + \sum_{\tau_{n-1} < s \le \tau_n} \Delta F(X_s) \qquad (1.1.17)$$

where $\tau_n = \inf\{t \ge \tau_{n-1} : X_t \notin D_{i_n}\}$, and where $D_{i_n} \ni X_{\tau_{n-1}}$. Performing the recursion indicated in 1.1.17 one obtains,

$$F(X_{\tau_n}) = F(X_0) + \int_0^{\tau_n} F'(X_{s-})\, dX_s^c + \sum_{0 < s \le \tau_n} \Delta F(X_s) \qquad (1.1.18)$$

Note that $\tau_n \le \tau_{n+1}$, and that since X is of locally finite variation, X can only cross $\partial \mathcal{D}$ a finite number of times in finite time, P-a.s.. So if τ is any stopping time which is P-a.s.-finite, there exists $n_0 \in \mathbb{N}$ such that $\tau \wedge \tau_{n_0} = \tau$ P-a.s., and 1.1.18 yields,

$$F(X_\tau) = F(X_0) + \int_0^\tau F'(X_{s-})\, dX_s^c + \sum_{0 < s \le \tau} \Delta F(X_s) \qquad (1.1.19)$$

or,

$$F(X_\tau) = F(X_0) + \int_0^\tau F'(X_{s-})\, dX_s + \sum_{0 < s \leq \tau} \{\Delta F(X_s) - F'(X_{s-})\Delta X_s\} \quad (1.1.20)$$

Since τ is any P-a.s.-finite stopping time, 1.1.7 follows from 1.1.20 *a fortiori*.   ∎

Hence, the usual differential rule in the non-diffusion case holds under slightly weaker conditions on the function F than are customarily imposed, if one is willing to assume slightly stronger conditions on the sample paths of X. There are two key ingredients to the proof.ii The first is that F must respect the sample path properties of X such that $F(X)$ is corlol. For example, if X has nondecreasing sample paths, the F must be at least piecewise right-continuous. The second ingredient is that X must have locally finite variation. This yields a boundary behavior which is sufficiently simple to permit the argument to proceed as in the proof. Without this assumption, for instance in the diffusion case, this type of argument is confounded at the discontinuities, where such a process can cross the boundary infinitely often in an arbitrarily small time period.

This concludes the general semimartingale results which will be needed in the applications chapter. In the next section, a result is presented which will be necessary in the hypopthesis testing problem dealing with Poisson processes.

1.2 A Functional Differential Equation

The next theorem plays a key role in Bayesian hypothesis testing problem in the case of point process observations.

Theorem 1.2.1 *Let a, b, c, and u be given such that $0 < a < b < 1$, and both c and u are positive. Suppose a function $E(x)$ is given which is continuous for all $x \in (0,1)$. Consider the following functional-advance differential equation,*

$$U R(x) = -c \qquad x \in (0,b), \tag{1.2.1}$$

where the linear operator U is defined via,

$$U R(x) = -x(1-x)R'(x) + (u+x)\Big[R\big(\frac{u+1}{u+x}x\big) - R(x)\Big],$$

where the boundary conditions are given by,

$$\begin{aligned} R(a) &= E(a) \\ R(x) &= E(x) \qquad \forall x \in [b, \frac{u+1}{u+b}b) \end{aligned} \tag{1.2.2}$$

and with the functional requirement,

$$R(x) \quad \text{is continuous at each} \quad x \in (0,b). \tag{1.2.3}$$

Then a unique solution exists for all $x \in (0,b)$.

Proof: Before getting into the details of the proof a short informal discussion will be given, since problems of this type are somewhat peculiar (see [EL'SGOL'TS & NORKIN]). Consider the fact that as long as $x \in [b - \frac{b(1-b)}{u+b}, b)$ that,

$$R\big(\frac{u+1}{u+x}x\big) = E\big(\frac{u+1}{u+x}x\big), \tag{1.2.4}$$

and thus one may write,

$$R'(x) = F(x, R(x)) \qquad x \in J_1[b], \tag{1.2.5}$$

where $J_1[b)$ is the stated semi-open interval and where,

$$F(x,R) = G(x) - \frac{u+x}{x(1-x)}R, \qquad (1.2.6)$$

with G(x) a known continuous function given by,

$$G(x) = \frac{c + (u+x)E\left(\frac{u+1}{u+x}x\right)}{x(1-x)}. \qquad (1.2.7)$$

Hence, if one can prove the existence and uniqueness of a continuous solution to the ODE 1.2.5, with arbitrary initial condition, then the process may be repeated inductively to the left, each induction step involving a proof of the existence and uniqueness of an ODE with a different forcing function obtained from the previous induction step. The induction process is continued until the ODE under consideration lives on the interval containing $x = a$. On this interval, call it J_a, the ODE solution of interest is the one passing through $(a, E(a))$, which may or may not be an initial condition for that interval. In either case, assuming a unique trajectory exists passing through $(a, E(a))$, then this solution is extended to the right hand endpoint of J_a, yielding a value for R at this point. This value of R is then taken to be the initial condition for the solution obtained on the previous induction step. Note that matching the value at this endpoint is justified since the proof of the existence and uniqueness on the previous step was specified with an arbitrary initial condition. The process of matching endpoints is then continued to the right until the starting interval $J_1[b)$ is reached, at which time the matching procedure is terminated since one is not free to choose the value of $E(x)$ at $x = b$. However, after the induction step on J_a, and after the endpoint matching procedure, one will have proven the existence of a unique continuous solution to the following problem,

$$U\tilde{R}(x) = -c \qquad x \in [A, b)$$

$$\tilde{R}(x) = E(a) \qquad\qquad\qquad (1.2.8)$$

$$\tilde{R}(x) = E(x) \qquad \forall x \in J_0[b),$$

where A is the left hand endpoint of J_a, and $J_0[b]$ is the semi-open interval in 1.2.2. The only difference between 1.2.8 and 1.2.1, is that the solution to 1.2.1 lives on all of $(0, b)$, whereas the solution to 1.2.8 does not. However, consider the problem,

$$U\hat{R}(x) = -c \qquad x \in (0, A)$$
$$\hat{R}(x) = \tilde{R}(A^+) \qquad\qquad (1.2.9)$$
$$\hat{R}(x) = \tilde{R}(x) \qquad \forall x \in J_a.$$

If the existence and uniqueness of a continuous solution to 1.2.9 can be shown, then by combining this and the result for 1.2.8, it follows that,

$$R(x) = \hat{R}(x)1\{x < A\} + \tilde{R}(x)1\{x \geq A\}, \qquad (1.2.10)$$

will solve the original problem.

The problem 1.2.8 will be handled first, dropping the tildes for notational convenience. To start, it is necessary to define the discretization intervals and towards this end, consider the advance Σ defined as,

$$\Sigma^0 x = x, \ \ \Sigma^1 x = \frac{u+1}{u+x}x, \ \ \Sigma^n x = \Sigma^{n-1}(\Sigma x). \qquad (1.2.11)$$

If one asks that Σ^{-1} satisfy $\Sigma\Sigma^{-1} = \Sigma^{-1}\Sigma = \Sigma^0$, then this yields,

$$\Sigma^{-1} x = \frac{ux}{u+1-x}, \qquad (1.2.12)$$

and Σ^{-1} is called the one step retardation. For any $x \in (0, 1)$, it follows easily that,

$$0 < \Sigma^{-n} x < x < \Sigma^n x < 1 \ \ \forall n \geq 1, \qquad (1.2.13)$$

and hence the significance of the terminology 'advance' and 'retardation'. With these notions, one can define,

$$J_n[b] = [\Sigma^{-n}b, \Sigma^{-(n-1)}b) \ \ n \geq 0, \qquad (1.2.14)$$

as the nth discretization interval; this agrees with the previous definition given for $J_0[b]$ given above. Note that there will be occasion to use the notation $J_n(x)$, with the obvious interpretation. Observe that $\Sigma^{-n}x$ ($\Sigma^n x$) is monotone decreasing (increasing) and bounded below (above). Hence, there exists $n = n_a$ such that $\Sigma^{-n_a} b \le a$, and such that this is the smallest such number satisfying the inequality. In the previous notation then, $J_a = J_{n_a}[b]$, and $A = \Sigma^{-n_a} b$. Thus, the existence and uniqueness of a soution to 1.2.18 will follow if one can prove this for solutions of

$$R'_n(x) = F_n(x, R_n(x)) \qquad x \in J_n[b]$$
$$R_n(\Sigma^{-n} b) = R_n,$$

$\qquad\qquad (1.2.15)$

for each $1 \le n < n_a$, and in addition do the same for a solution of,

$$R'_{n_a}(x) = F_{n_a}(x, R_{n_a}(x)) \quad x \in J_{n_a}(b)$$
$$R_{n_a} = E(a),$$

$\qquad\qquad (1.2.16)$

where in 1.2.15, R_n is an arbitrary constant and,

$$F_n(x, R) = G_n(x) - \frac{u + x}{x(1 - x)} R, \qquad\qquad (17)$$

where,

$$G_n(x) = \frac{c + (u = x) R_{n-1}\left(\frac{u+1}{u+x} x\right)}{x(1 - x)}, \qquad\qquad (1.2.18)$$

with,

$$R_0(x) = E(x) \qquad x \in J_0[b]. \qquad\qquad (1.2.19)$$

Let n=1, for the first induction step. Then it is a simple consequence of elementary ODE theory that a unique solution to 1.2.15 exists passing through any point interior to the set $(J_1[b], \mathbb{R})$. To prove a unique solution exists starting from the boundary point $(\Sigma^{-1} b, \mathbb{R})$, one can choose $\epsilon_1 > 0$ such that that $\Sigma^{-1} b - \epsilon_1 > 0$, and then define $\bar{J}_1(b) = (\Sigma^{-1} b - \epsilon_1, b)$. Using these open intervals, consider the auxiliary ODE,

$$\bar{R}'(x) = F_1(x, \bar{R}(x)) \qquad x \in \bar{J}_1(b)$$
$$\bar{R}(\Sigma^{-1} b) = R_1.$$

$\qquad\qquad (1.2.20)$

Now apply ODE theory to 1.2.20. Specifically, $F_1(x, R)$ is continuous for all (x, R) in the open set $(\bar{J}_1(b), \mathbb{R})$. In view of 1.2.19, F_1 has a continuous first partial derivative in R for all $(x, R) \in (\bar{J}_1(b), \mathbb{R})$. So, applying the standard results (see HALE, Theorem 3.1), $\bar{R}$ satisfying 1.2.20 exists and is unique. Therefore, if one defines the restriction,

$$R_1(x) = \bar{R}(x) \qquad x \in J_1[b], \tag{1.2.21}$$

then $R_n(x)$ uniquely exists satisfying 1.1.15 for $n = 1$. Notice from 1.2.15 that $R_1'(x)$ is continuously differentiable for all $x \in J_n(b)$. Therefore, $R_1(x)$ and thence also $G_2(x)$ is continuous there. The proof for any $n < n_a$ is the same as for $n = 1$ with the following observations. First, realize that one can always find an ϵ_n, $n \geq 1$, satisfying $\Sigma^{-n} b - \epsilon_n > 0$, in view of 1.2.13. Thus for any $n \geq 1$, $\bar{J}_n(b)$ is well-defined. Second, from 1.2.17, it is clear that $F_n(x, R)$ has a continuous first partial in R for all $(x, R) \in ((0, 1), \mathbb{R})$, and hence this is true for all $(x, R) \in (\bar{J}_n(b), \mathbb{R})$. Lastly, it should be clear that $F_n(x, R)$ is continuous for all $(x, R) \in (\bar{J}_n(b), \mathbb{R})$. It is clearly true for n=1, has been shown to be true for $n = 2$, and is true by induction in general. In summary, a unique solution to 1.2.15 with arbitrary initial condition has been shown to exist for each n, $1 \leq n \leq n_a$. The proof of the existence of a unique solution to 1.2.16 is the analogous. If $a = \Sigma^{-n_a} b$, then the proof is exactly the same. If $a > \Sigma^{-n_a} b$, then there is no need to consider the auxiliary problem since then 'a' is interior to $J_{n_a}[b]$, and the proof follows directly.

To begin the endpoint matching process, note that since F_n is so well behaved for all $(x, R) \in ((0, 1), \mathbb{R})$, then the solution on $J_{n_a}[b]$ can be extended naturally to include the right-hand endpoint (see HALE, Lemma 2.1). Then define $R_{n_a-1}(\Sigma^{-(n_a-1)} b) = R_{n_a}(\Sigma^{-(n_a-1)} b)$. Continue this process of extension and matching up to and including $J_2[b]$. The unique solution to 1.2.8, for a, b as given, has now been demonstrated to exist.

To complete the proof, it is necessary to extend the solution continuously to the

left, i.e., solve the problem given in 1.2.19. First observe that the solution to 1.2.8 can be extended naturally to the left-hand endpoint of $J_{n_a}[b]$ (again, see HALE, Lemma 2.1). Thus solving 1.2.9 is equivalent to proving the existence and uniqueness of a solution to (hats removed for notational politeness),

$$R_n'(x) = F_n(x, R_n(x)) \qquad x \in J_n[b]$$
$$R_n((\Sigma^{-(n-1)}b)^-) = R_n,$$

(1.2.22)

for all $n > n_a$, with $R_{n_a+1} = R_{n_a}(\Sigma^{-n_a}b)$. The proof in this case is also by induction on the analogous 'overbarred' problem (see 1.2.20), where now $\bar{J}_n(b) = (\Sigma^{-n}b, \Sigma^{-(n-1)}b+\epsilon_n)$, and again, $\epsilon_n > 0$ can always be chosen so that $\Sigma^{-(n-1)}b+\epsilon_n < 1$. This time, at each step of the induction, the solution is extended naturally to the left, providing an initial condition on the next induction step. Finally, since for any $x \in (0, b)$, there exists a smallest n_x such that $\Sigma^{-n_x} < x$, the solution to 1.2.9 is secured. Recalling 1.2.10, the theorem has been shown.   ∎

Corollary *If in addition, $E(x)$ is n-times continuously differentiable on $J_0[b]$, $n \geq 1$, then the unique solution to the problem is also n-times continuously differentiable except possibly for $x \in \{\Sigma^{-k}b : k = 0, ...n\}$. Also, if $R(b^-) = R(b)$ for b given, then $R(x)$ is n-times continuously differentiable for all $n \geq 1$, and $x < b$.*

Proof: This fact follows directly from 1.2.1, by solving for the derivative and using induction.   ∎

This concludes the current section and chapter. In the next chapter, the problem of Bayesian sequential hypothesis testing is presented.

Chapter II
Bayesian Sequential Hypothesis Tests

2.0 Introduction

In this chapter, the Bayesian sequential hyptothesis testing problem is presented. The first section establishes the probabilistic framework which will be used throughout the thesis. In addition, the class of possible soutions to the sequential hypothesis testing problem is defined. Within this class of so called *admissible policies*, is the class of *threshold policies* which is also defined. The second section gives the particulars of the general Bayesian problem, however, only the binary case will be presented.

In section 2.3, the first steps are taken in reducing the complexity of the minimization problem posed by the Bayes risk. Theorem 2.3.1 shows that for any policy $\tilde{u} = (\tilde{\tau}, \tilde{\delta}) \in \mathcal{U}$, one can replace $\tilde{\delta}$ with a certain $\mathcal{F}_{\tilde{\tau}}$-measurable random variable and by so doing obtain a new policy whose risk is not larger. The immediate consequence is that the infimum over all admissible policies is no smaller than the infimum over their first component. Next, it is shown that threshold policies are unchanged by the replacement procedure of Theorem 2.3.1 — an obvious property to check before attempting to prove that such policies are optimal. The section ends by showing that without loss of generality, one can assume that the cost due to wrong decisions enjoys a certain symmetry property.

2.1   Preliminaries

In this section, the problem of testing two statistical hypotheses will be presented using the Bayesian risk approach. The following probablistic framework forms the starting point of the investigation.

On the measurable space $(\Omega, \mathcal{F})$, there are given two probability measures P_i, i=0,1. In addition, there is given a filtration $\{\mathcal{O}_t : t \in T\}$, where $T \subseteq [0, \infty)$ is the time parameter set, either continuous or discrete. If $T = [0, \infty)$, then $\mathcal{O}_t$ is taken to be right continuous. The $\mathcal{O}_t$ filtration should be thought of as being the history of some observable process. There is also given Θ, the hypothesis parameter set. For instance, in the simplest case of binary hypotheses, $\Theta = \{\theta_0, \theta_1\}$.

The general hypothesis testing problem can be loosely described as discovering a plan, or policy, which dictates when to stop and what to decide, based upon the available information, and which also satisfies a given performance criterion. Any policy then, ought to be a pair consisting of a stopping time and a deciding rule, and an optimum policy is one which achieves a specified performance. In view of this, it is clear that in seeking optimum policies, the search is naturally limited to a class of candidate policies whose membership in the class, or *admissibility*, is defined by practical measurability requirements. That is to say, each component of an eligible policy pair should at least, *and at most*, be measurable with respect to some σ-algebra which contains all the information that might possibly be observed. A natural definition of admissible policy then, is given in terms of the observation filtration, $\mathcal{O}_t$.

Definition 2.1.1 *An admissible policy, u, is a pair, $u = (\tau, \delta)$, where τ is a $(P, \mathcal{O}_t)$-stopping time taking values in T, and δ is a $(P, \mathcal{O}_\tau)$-random variable taking values in Θ. The class of admissible policies will be denoted as $\mathcal{U}$.*

In the binary hypothesis testing problem, the choice of $u \in \mathcal{U}$ prescribes a policy which specifies, via τ, when the observation is to be terminated, and specifies via δ,

which hypothesis is to be accepted. That is to say if $\delta = i$, then the hypothesis $\theta = \theta_i$ will be accepted, $i = 0, 1$.

An important subclass of admissible policies are the threshold policies. This is due both to their simple specification and remarkable optimality properties.

Definition 2.1.2 *Let X be an O_t-adapted process. A* **threshold policy** $\bar{u} \in \mathcal{U}$ *is a pair, $\bar{u} = (\bar{\tau}, \bar{\delta})$, where,*

$$\bar{\tau} = \inf\{t \geq 0 : X_t \notin (a, b)\}$$

and,

$$\bar{\delta} = \begin{cases} 1 & X_{\bar{\tau}} \geq b \\ 0 & X_{\bar{\tau}} \leq a \end{cases}$$

where $a \leq b$ are the **thresholds**, *and (a, b) is called the* **continuation interval**. *The class of theshold policies will be denoted as $\bar{\mathcal{U}}$; note that, $\bar{\mathcal{U}} \subseteq \mathcal{U}$.*

Observe that for a given filtration, there are as many possible threshold policies as there are ways to choose a triple (X, a, b) which satisfies the definition. For a given O_t-adapted process X however, the hope is that one can find *a priori* a particular threshold pair such that the threshold policy so defined is optimal in some sense. For our purposes, X will be taken to be either a certain *a posteriori* probability or likelihood ratio process, the important similarity being that they both satisfy Theorem 1.1.1.

Consider an arbitrary admissible policy $u \in \mathcal{U}$. For each $\omega \in \Omega$, the policy $u = (\tau, \delta)$ can be described very generally as incurring two kinds of losses, one due to the cost of waiting to decide, and the other due to making a wrong decision. A reasonably general characterization of the cost of the observation time, or *running cost*, is given by $\int_0^\tau c_s \, ds$, where $\{c_t, t \geq 0\}$ is some O_t-adapted process which serves as a suitable cost measure. For instance, $c_t = c \geq 0$, a positive constant, is often chosen with the intention of using an apparently simple running cost to capture the behavior: 'the longer it takes to decide, the more it costs'. Of course, this is not

the only cost process which captures this behavior, nor is it necessarily simple, for the simplicity of a running cost is often best judged not by its specification, but by how easily it yields to mathematical analysis. Indeed, a fundamental tradeoff in the modeling of a sequential testing problem is choosing a running cost which can be worked with mathematically, *and still captures a desired behavior.*

Next, consider that each admissible policy can make two types of incorrect decisions,

$$
\begin{cases}
\delta = 0 & \text{while} \quad \theta = \theta_1 \\
\delta = 1 & \text{while} \quad \theta = \theta_0.
\end{cases}
$$

Naturally, it will be desirable, to minimize in some way the probabilities of these errors.

This concludes the section. In the next, the discussion will focus on the details of the Bayesian formulation.

2.2 Bayesian Formulation

Let θ be a $\{\theta_0, \theta_1\}$-valued, $\mathcal{F}$-measurable random variable, $\theta : \Omega \to \Theta$. Fix an arbitrary $\pi \in [0, 1]$ and define a probability measure P_π on $(\Omega, \mathcal{F})$ via,

$$P_\pi\{\omega : \theta(\omega) = \theta_1\} = \pi \quad , \quad P_\pi\{\omega : \theta(\omega) = \theta_0\} = 1 - \pi, \qquad (2.2.1)$$

such that for each set $A \in \mathcal{F}$ there holds,

$$P_\pi\{A\} = \pi P_1\{A\} + (1 - \pi)P_0\{A\}. \qquad (2.2.2)$$

In words, it is assumed that θ is a binary random variable taking on the values θ_0 or θ_1 with the a priori probabilities $1 - \pi$ and π, respectively. The *a posteriori* probability of the $\theta = \theta_1$ hypothesis is then defined as $\pi_t = P_\pi\{\theta = \theta_1 | \mathcal{O}_t\}$, $t \in T$, and since $\mathcal{O}_0 = \{\emptyset, \Omega\}$, it follows that $P_\pi\{\pi_0 = \pi\} = 1$.

Consider an arbitrary admissible policy $u \in \mathcal{U}$ as defined above. For each $\omega \in \Omega$, the policy $u = (\tau, \delta)$ can be described as incurring two kinds of losses, one due to the cost of waiting to decide, the other due to making wrong decisions. On average therefore, a natural definition for the average running cost is given by $E_\pi[\int_0^\tau c_s ds]$, where $\{c_t, t \geq 0\}$, is some $\mathcal{O}_t$-adapted process which yields a desired cost behavior. Later $\{c_t, t \geq 0\}$ will be specialized to $\{c(\pi_t), t \geq 0\}$, where $c(\pi), \pi \in [0, 1]$ serves as a reasonable cost rate function.

To describe the cost of making wrong decisions, define

$$w(\theta, \delta) = \begin{cases} c^0 & \text{if} \quad \theta = \theta_1, \delta = 0 \qquad c^0 > 0 \\ c^1 & \text{if} \quad \theta = \theta_0, \delta = 1 \qquad c^1 > 0 \\ 0 & \text{if} \quad \delta = \theta, \end{cases} \qquad (2.2.3)$$

yielding the average cost of incorrect decisions as $E_\pi[w(\theta, \delta)]$. Observe that it is without loss of generality that there is no cost levied for correct decisions in 2.2.3. Now, one can easily show that

$$E_\pi[w(\theta, \delta)] = c^0 P_1\{\delta = 0\} + c^1 P_0\{\delta = 1\}.$$

Putting the pieces together yields,

$$\rho(\pi, u) = E_\pi \Big[\int_0^\tau c_s ds + w(\theta, \delta) \Big] \tag{2.2.4}$$

as the total P_π-average Bayesian cost, or risk, of the policy $u = (\tau, \delta)$.

The following definition serves to characterize the optimality of admissible policies for the Bayesian formulation.

Definition 2.2.1 *Let $I \subseteq [0, 1]$. The policy $u^* \in \mathcal{U}$ is said to be I-Bayesian if*

$$\rho(\pi, u^*) = \inf_{u \in \mathcal{U}} \rho(\pi, u) \qquad \forall \pi \in I \tag{2.2.5}$$

If $u^* \in \mathcal{U}$ is $[0, 1]$-Bayesian, it is said to be Bayesian.

Consider that any reasonable, admissible policy will be at least $\{0, 1\}$-Bayesian, and so for convenience of notation, I will be understood to denote $I \cup \{0, 1\}$ unless noted otherwise.

Given this definition, one of the principal goals of this thesis can be succinctly stated as demonstrating that there exists a threshold policy, say with thresholds a, b, which is at least (a, b)-Bayesian, and preferably Bayesian, for a given observation filtration. It is this goal which motivates the use of the suggestive 'I' notation in the definition.

This concludes the section. The next sections begins by showing how the infimum over all admissible policies appearing in 2.2.5 can be replaced by a much simpler minimization.

2.3 Optimal Stopping

Having considered general aspects of Bayesian hypothesis testing in the previous sections, the focus here is to reduce the complexity of the search for the optimal policy. Towards this end it is first shown that finding a Bayesian policy can be reduced to a problem of optimal stopping, i.e., one can restrict the search for the optimal policy, an infimum operation over both policy components, to a search for an optimal stopping policy, an infimum operation over the stopping time component alone. This is accomplished by showing that any policy pair can be replaced by another pair whose risk is no larger. The new pair is obtained from the original by replacing its decision component with a new decision variable which has an explicit representation in terms of the *a posteriori* probability; the stopping time component is kept the same. The immediate question arises as to the effect of this replacement procedure on threshold policies, and it is shown that such policies remain unchanged—an obvious property to check before one tries to prove that threshold policies are Bayesian! Next it is shown that without loss of generality one can consider a certain symmetric and normalized version of the Bayesian formulation.

The next theorem is a generalization of Lemma 4.1.1 in [SHIRYAYEV 77].

Theorem 2.3.1 *Let $\tilde{u} = (\tilde{\tau}, \tilde{\delta}) \in \mathcal{U}$ be arbitrary, and define $u = (\tau, \delta) \in \mathcal{U}$ via, $\tau = \tilde{\tau}$ and,*

$$\delta = \begin{cases} 1 & c^0 \pi_\tau \geq c^1(1 - \pi_\tau) \\ 0 & c^0 \pi_\tau < c^1(1 - \pi_\tau). \end{cases} \tag{2.3.1}$$

Then, for all $\pi \in [0, 1]$,

$$\rho(\pi, \tilde{u}) \geq \rho(\pi, u) = E_\pi[\int_0^\tau c_s ds + e(\pi_\tau)] \tag{2.3.2}$$

where,

$$e(\pi) = \min\{c^0 \pi, c^1(1 - \pi)\}. \tag{2.3.3}$$

and $\{c_t : t \geq 0\}$ is an $\mathcal{O}_t$-adapted, nonnegative process.

Proof: If $\pi = 0$ or 1, then the assertions are easy to check, so suppose $\pi \in (0,1)$.

From definition 2.2.1, it is clear that since $\{\tilde{\delta} = i\} \in \mathcal{O}_\tau$ for $i = 0, 1$, there holds,

$$E_\pi[1\{\tilde{\delta} = 0\}1\{\theta = 1\}] = E_\pi[1\{\tilde{\delta} = 0\}E_\pi[1\{\theta = 1\}|\mathcal{O}_\tau]]$$
$$= E_\pi[1\{\tilde{\delta} = 0\}\pi_\tau]. \tag{2.3.4}$$

and similarly,

$$E_\pi[1\{\tilde{\delta} = 1\}1\{\theta = 0\}] = E_\pi[1\{\tilde{\delta} = 1\}(1 - \pi_\tau)]. \tag{2.3.5}$$

So, combining 2.2.3, 2.2.4, 2.3.4, and 2.3.5 yields,

$$\begin{aligned}
\rho(\pi, \tilde{u}) &= E_\pi\left[\int_0^{\tilde{\tau}} c_s ds + w(\theta, \tilde{\delta})\right] \\
&= E_\pi\left[\int_0^{\tilde{\tau}} c_s ds + c^0 1\{\tilde{\delta} = 0\}1\{\theta = 1\} + c^1 1\{\tilde{\delta} = 1\}1\{\theta = 0\}\right] \\
&= E_\pi\left[\int_0^{\tilde{\tau}} c_s ds + c^0 (1 - \tilde{\delta})\pi_{\tilde{\tau}} + c^1 \tilde{\delta}(1 - \pi_{\tilde{\tau}})\right] \\
&= E_\pi\left[\int_0^{\tau} c_s ds + c^0 (1 - \tilde{\delta})\pi_\tau + c^1 \tilde{\delta}(1 - \pi_\tau)\right] \\
&\geq E_\pi\left[\int_0^{\tau} c_s ds + e(\pi_\tau)\right] = \rho(\pi, u),
\end{aligned} \tag{2.3.6}$$

and the theorem is shown.   ∎

This theorem is a welcome simplification of the problem for it says that in the search for Bayesian decision policies, one need only consider policies $u = (\tau, \delta)$ with δ given by 2.3.1 and with the risk given by,

$$\rho(\pi) = \inf_{u \in \mathcal{U}} \rho(\pi, u) = \inf_{\tau \in \mathcal{T}} \rho(\pi, \tau),$$

where,

$$\rho(\pi, \tau) = E_\pi\left[\int_0^{\tau} c_s ds + e(\pi_\tau)\right], \tag{2.3.7}$$

and where $\mathcal{T}$ denotes the class of $(P_\pi, \mathcal{O}_t)$-stopping times. In fact, without loss of generality it is sufficient to take $\mathcal{T}$ to be the subclass of P_π-a.s. finite stopping times if the following technical condition holds,

$$\text{(T):} \qquad E_\pi\left[\int_0^\infty c_s ds\right] = \infty,$$

Intuitively, the condition (T) is entirely natural since usually one does not impose a running cost which fails to force decisions in finite time. As a result, *it will be assumed throughout the remainder of this thesis that condition (T) holds.* As a sidenote, be aware that the cost due to making wrong decisions is often called the 'terminal cost' in the Bayesian formulation. This usage is a direct consequence of the theorem, and equation 2.3.1 in particular.

Observe that the theorem is very general. No assumptions were made regarding the continuity of either the time parameter or state-space; one has only to interpret the integrals in the usual way. The extension to multiple hypotheses is tedious but straightforward. In addition, notice from definition 2.1.2 that a threshold policy $(\bar{\tau}, \bar{\delta}) \in \bar{\mathcal{U}}$ is entirely specified by its first component (via the thresholds). That is to say, once $\bar{\tau}$ is known, $\bar{\delta}$ is completely determined, a characteristic not shared by admissible policies in general. The theorem above however shows that in minimizing the risk, the infimum over all policies is no smaller than the infimum over all policies which share this characteristic.

In light of the above theorem, a natural question to ask is what policy results when a threshold policy $(\bar{\tau}, \bar{\delta})$ is exchanged for the policy $(\bar{\tau}, \delta)$, with δ as in 2.3.1. If $\bar{u}$ is a threshold policy based upon the *a posteriori* probability, the next theorem shows that as long as the candidate threshold policy never prescribes stopping where the terminal cost is at its maximum, then $(\bar{\tau}, \bar{\delta})$ and $(\bar{\tau}, \delta)$ are identical. Note that the maximum value of the terminal cost is $\frac{c^1}{c^0 + c^1}$.

Theorem 2.3.2 *Let $I_0 = (a, b)$, with $0 < a \leq \frac{c^1}{c^0 + c^1} \leq b < 1$, $a \neq b$, and consider the threshold policy $(\bar{\tau}, \bar{\delta})$ based on I_0 and $\{\pi_t : t \geq 0\}$ (see def. 2.1.2). Then,*

$$\bar{\delta}(\omega) = \delta(\omega) \qquad \forall \omega \in \Omega,$$

with δ as in Theorem 2.3.1.

Proof: Suppose $\omega \in \Omega$ is such that, $\pi_{\bar{\tau}(\omega)}(\omega) \geq b$. Then from 2.3.1,

$$\frac{\pi_{\bar{\tau}}}{1 - \pi_{\bar{\tau}}} \geq \frac{b}{1 - b}.$$

On the other hand, $b > c^1/(c^0 + c^1)$, and so,

$$\frac{b}{1 - b} > \frac{c^1}{c^0}.$$

Thus $\pi_{\bar{\tau}} \geq b$ implies $c^0 \pi_{\bar{\tau}} \geq c^1(1 - \pi_{\bar{\tau}})$. Now let $\omega \in \Omega$ be such that,

$$\frac{\pi_{\bar{\tau}(\omega)}(\omega)}{1 - \pi_{\bar{\tau}(\omega)}(\omega)} \geq \frac{c^1}{c^0},$$

so that upon rearranging one obtains,

$$\pi_{\bar{\tau}(\omega)}(\omega) \geq \frac{c^1}{c^0 + c^1} \geq a,$$

where the second inequality follows by hypothesis. Hence for such an $\omega \in \Omega$, it follows that, $\pi_{\bar{\tau}} \geq b$ in view of the definition of $\bar{\tau}$. From this one concludes that $c^0 \pi_{\bar{\tau}} \geq c^1(1 - \pi_{\bar{\tau}})$ implies $\pi_{\bar{\tau}} \geq b$ and therefore,

$$\pi_{\bar{\tau}(\omega)}(\omega) \geq b \iff c^0 \pi_{\bar{\tau}(\omega)}(\omega) \geq c^1(1 - \pi_{\bar{\tau}(\omega)}(\omega)) \qquad \forall \omega \in \Omega.$$

Similarly, one can show that,

$$\pi_{\bar{\tau}(\omega)}(\omega) \leq a \iff c^0 \pi_{\bar{\tau}(\omega)}(\omega) < c^1(1 - \pi_{\bar{\tau}(\omega)}(\omega)) \qquad \forall \omega \in \Omega.$$

Comparing definition 2.1.2 and equation 2.3.1 completes the argument.   ∎

Another consequence of Theorem 2.3.1 is that if one considers $(P_\pi, \mathcal{O}_t)$-adapted cost rates of the form $c_t = c_t(\pi_t), t \geq 0$, where $c_t(\pi)$, $\pi \in [0,1], t \geq 0$ is some suitable, nonnegative cost rate function, it follows that allowing different multipliers on the terminal cost is a false generality. This is easily seen using the new form of the risk given in Theorem 2.3.1. In fact, let

$$\tilde{c}^0 = c^1(1 - \pi) + c^0\pi = \tilde{c}^1 \qquad (2.3.8)$$

and take $\tilde{\pi} = c^0\pi/\tilde{c}^0$ for any $\pi \in [0,1]$. Set $\tilde{c}_t(\pi) = c_t(\tilde{c}^0\pi/c^0)$, and also

$$\tilde{e}(\pi) = \min\{\tilde{c}^0\pi, \tilde{c}^0(1-\pi)\}.$$

Then notice,

$$E_\pi[\int_0^\tau c_s(\pi_s)ds + e(\pi_\tau)] = E_\pi[\int_0^\tau \tilde{c}_s(\tilde{\pi}_s)ds + \tilde{e}(\tilde{\pi}_\tau)]$$

$$= E_{\tilde{\pi}}[\int_0^\tau \tilde{c}_s(\tilde{\pi}_s)ds + \tilde{e}(\tilde{\pi}_\tau)],$$

where $P_{\tilde{\pi}}$ is defined via 2.2.1 and 2.2.2 with $\pi \in [0,1]$ replaced by $\tilde{\pi} \in [0, \frac{c^0}{\tilde{c}^0}]$. Thus, the original Bayesian formulation based on the *a posteriori* probability process can be replaced by an equivalent formulation based on $\{\tilde{\pi}_t : t \geq 0\}$. Of course, the process $\{\tilde{\pi}_t : t \geq 0\}$ may no longer have the interpretation as an *a posteriori* probability. Analogously, the family $\{P_{\tilde{\pi}} : 0 \leq \tilde{\pi} \leq \frac{c^0}{\tilde{c}^0}\}$ is no longer strictly a family of probability measures, but of course as is usually the case it is only necessary that it be a family of finite measures to obtain the essence of the arguments contained herein. In view of this equivalence, it is sufficient mathematically to consider the symmetric case $c^0 = c^1$, in questions of optimality. In this case the risk is given by

$$\rho(\pi, \tau) = E_\pi[\int_0^\tau c_s(\pi_s)ds + e(\pi_\tau)],$$

where,

$$e(\pi) = \min\{c^0\pi, c^0(1-\pi)\} = c^0\min\{\pi, 1-\pi\}.$$

So, letting $\bar{c}_t = c_t/c^0$ and $\bar{e} = e/c^0$, it is clear that minimizing $\rho(\pi, \tau)$ is equivalent to minimizing

$$\rho(\pi, \tau) = E_\pi[\int_0^\tau \bar{c}_s(\pi_s)ds + \bar{e}(\pi_\tau)].$$

Hence, it is sufficient to consider the case of symmetric, normalized terminal cost. For the remainder of this thesis, therefore, the risk may be assumed to be,

$$\rho(\pi, \tau) = E_\pi[\int_0^\tau c(\pi_s)ds + e(\pi_\tau)],$$

where,

$$e(\pi) = \begin{cases} \pi & \pi \le 1/2 \\ 1 - \pi & \pi \ge 1/2. \end{cases} \qquad (2.3.9)$$

Specific quantities in the general case can be obtained for the symmetric case and transformed back by the relation 2.3.8.

Having obtained the special form of the terminal cost in 2.3.9 with no loss of generality, one might wonder what this symmetry implies about the risk. To motivate the discussion, consider the set defined via,

$$C_0 = \{\pi \,:\, \rho(\pi) < e(\pi)\}.$$

The set C_0 is called the **natural continuation** set for the obvious reason that one intuitively expects to continue collecting data as long as the risk is less'than the terminal cost. In addition to C_0, one can define the **natural stopping set** as,

$$S_0 = \{\pi \,:\, \rho(\pi) = e(\pi)\}.$$

It is easy to see that the two sets are disjoint and that their union is the whole interval, $[0, 1]$. In general however, because the risk is not known a priori, it is difficult to squeeze more useful information about the risk out of C_0 or S_0. Indeed, this is the essence of the problem since clearly, the goal of proving threshold policies are optimal is precisely equivalent to showing that C_0 is open and connected. That is to say, one believes intuitively that the stopping *interval* for a threshold policy should be identical to C_0, if the policy is Bayesian. One might also expect that the risk arising from a symmetric terminal cost ought to be symmetric also. However, this is not the case! What is true is that a symmetric terminal cost implies that S_0 and C_0 are symmetric sets (about $\pi = \frac{1}{2}$) and that the risk is symmetric on C_0, in certain problems where the observations arise from diffusion-type semimartingales (see section 4.1). It is also true that the risk is not symmetric on C_0 in the Poisson

process case (see section 4.2). In addition, it appears that the sets S_0 and C_0 are not symmetric in this case either, although the author was not able to show this.

This concludes the section and chapter. As a final note, be aware that the main reason for demonstrating the symmetry property above is so that qualitative understanding of the problem can be gained. Another reason is to simplfy some arguments later chapters. However, in the chapter on applications, the results will be stated for a non-symmetric terminal cost so as to facilitate quantitative analysis.

Chapter III
Sufficient and Necessary Conditions

3.0 Introduction

Having considered the general aspects of Bayesian hypothesis testing, the next goal is to show under what conditions threshold policies based on the *a posteriori* probability are Bayesian optimal. The purpose of this chapter is to identify such conditions, and indicate how to use them in a particular application to demonstrate optimality as well as compute the thresholds. The chapter is organized as follows.

Section 3.1 gives a set of three conditions which are sufficient to guarantee that threshold policies are Bayesian. Two lemmas are given which help characterize the risk and the continuation interval. The main theorem, which is based on these lemmas, then demonstrates the sufficiency of the conditions mentioned. The section concludes with a uniqueness result.

In section 3.2 the conditions laid out in section 3.1 are imposed and consequent properties are derived which guide the search for the risk and continuation interval in applications.

3.1 Sufficient Conditions

The purpose of this section is to spell out a set of three conditions which are sufficient to prove that there exists a unique threshold policy which is Bayesian. In addition, the conditions will show how to compute both the Bayesian risk and the

optimal thresholds.

From the last chapter, it is clear that the process of fundamental interest is the *a posteriori* probability process $\{\pi_t : t \geq 0\}$. In the remainder of this thesis, it will be assumed that $\{\pi_t : t \geq 0\}$ is in fact a time-homogeneous Markov process. In the applications which follow, it will be shown how results for less restricted $\{\pi_t : t \geq 0\}$ processes can in fact be obtained from the results contained in this chapter.

Since the goal of this thesis is to demonstrate the optimality of threshold policies, it is natural that the topic of first exit times of a process from an open interval should arise. In turn then, it might be expected that the notion of characteristic operator should appear. This is indeed the case, but it will be used in a sense less general than usual. For instance, using the previous probabilistic set-up, define

$$Ur(\pi) = \lim_{J \downarrow \{\pi\}} \frac{E_\pi[r(\pi_{\tau_J}) - r(\pi_0)]}{E_\pi[\tau_J]}, \qquad \pi \in (0,1), \qquad (3.1.1)$$

where $r \in \mathcal{D}(U)$, and $\mathcal{D}(U)$ is the domain of definition of U, i.e., the set of all functions such that the limit exists. In addition, let $\tau_J = \{t \geq 0 : \pi_t \notin J\}$, with J an interval containing π, and let the limiting process consists of a series of intervals shrinking down to the point $\{\pi\}$ (see DYNKIN). Now given that $\{\pi_t : t \geq 0\}$ is as above, and supposing that $r : (0,1) \to \mathbb{R}$ satisfies the hypotheses of a generalized Itô rule for a particular $\{\pi_t : t \geq 0\}$, one can write

$$r(\pi_\tau) - r(\pi_0) = \int_0^\tau U^{\pi_s} r(\pi_s) ds + M_\tau, \qquad (3.1.2)$$

for all $(P_\pi, \mathcal{F}_t)$-stopping times satisfying $E_\pi[\tau] < \infty$, where M is some P_π-local martingale. That is, because $r(\pi_\tau)$ has a stochastic integral representation, the limit in 3.1.1 exists. Clearly, 3.1.2 is a convenient and compact representation. If a function $r \in \mathcal{D}(U)$ admits a stochastic integral representation, it will be written as $r \in \hat{\mathcal{D}}(U)$.

Let U denote the **characteristic operator** for the *a posteriori* probability process $\{\pi_t : t \geq 0\}$ and $\mathcal{D}(U)$ denote its domain of definition. Let $I_0 \subset E$ be some

open interval and $r : E \to \mathbb{R}$ be some function in $\hat{\mathcal{D}}(U)$. Consider the following set of conditions:

$$\text{(C1):} \qquad Ur(\pi) = -c(\pi) \qquad \forall \pi \in E$$

$$\text{(C2):} \qquad r(\pi) = e(\pi) \qquad \forall \pi \in \partial_\pi I_0$$

$$\text{(C3):} \qquad r(\pi) < e(\pi) \qquad \forall \pi \notin \partial_\pi I_0$$

The three conditions as a whole will be referred to as the conditions, (C1-3), and separately as (C1), (C2), etc.

The remainder of this section is devoted to proving that the conditons (C1-3) are sufficient to guarantee that there exists a threshold policy which is Bayesian. The first lemma goes a long way towards showing this.

Lemma 3.1.1 *Suppose there exists a pair* (r, I_0) *satisfying the conditions (C1-3). Then,*

$$\rho(\pi) = r(\pi) \qquad \forall \pi \in [I_0]_\pi, \tag{3.1.3}$$

where $\rho(\cdot)$ *is the Bayes risk.*

Proof: First note that (C2) and (C3) together imply,

$$e(\pi) \geq r(\pi) \qquad \forall \pi \in E. \tag{3.1.4}$$

Next, in view of (C1), one obtains,

$$\begin{aligned}
\rho(\pi) &= \inf_{\tau \in \mathcal{T}} \rho(\pi, \tau) \\
&= \inf_{\tau \in \mathcal{T}} E_\pi \Big[\int_0^\tau c(\pi_s) ds + e(\pi_\tau) \Big] \tag{3.1.5} \\
&= r(\pi) + \inf_{\tau \in \mathcal{T}} E_\pi [e(\pi_\tau) - r(\pi_\tau)] \qquad \forall \pi \in E.
\end{aligned}$$

Hence, combining 3.1.4 and 3.1.5 yields,

$$\rho(\pi) \geq r(\pi) \qquad \forall \pi \in E. \tag{3.1.6}$$

On the other hand, if one defines the $(P_\pi, \mathcal{O}_t)$-stopping time $\bar{\tau}$ as,

$$\bar{\tau} = \inf\{t \geq 0 : \pi_t \notin I_0\}, \qquad (3.1.7)$$

then, from (C1) and (C2) it follows that,

$$\rho(\pi, \bar{\tau}) = r(\pi) \qquad \forall \pi \in [I_0]_\pi. \qquad (3.1.8)$$

Thus, 3.1.6 and 3.1.7 together yield,

$$\rho(\pi) = r(\pi) \qquad \forall \pi \in [I_0]_\pi, \qquad (3.1.9)$$

and the lemma is shown.   ■

Because of 3.1.9 and (C3), it is convenient and suggestive to call $r(\cdot)$ the subrisk. Likewise, it is appropriate to call I_0 the continuation interval.

The next lemma indicates what the risk is on the remainder of E.

Lemma 3.1.2 *Suppose there exists a pair (r, I_0) such that the conditions (C1-3) hold. Then,*

$$\rho(\pi) = e(\pi) \qquad \forall \pi \notin I_0 \qquad (3.1.10)$$

Proof: Suppose that the *a posteriori* probability is P_π-a.s. initially outside of $[I_0]_\pi$. Consider first the cases $\pi = 0$ and $\pi = 1$. First note that,

$$\rho(\pi) \geq 0 \qquad \pi = 0, 1, \qquad (3.1.11)$$

but also $\rho(\pi, 0) = 0$ and therefore $\rho(\pi) = 0$. But then,

$$\rho(\pi) = e(\pi) \qquad \pi = 0, 1, \qquad (3.1.12)$$

since $e(\pi) = 0$, for $\pi = 0, 1$.

Next, consider the stopping time specifying the first time that $\{\pi_t : t \geq 0\}$ reenters $[I_0]_\pi$,

$$\tilde{\tau} = \inf\{t \geq 0 : \pi_t \in [I_0]_\pi\}, \qquad (3.1.13)$$

where $\tilde{\tau}(\omega) = \infty$ if it never reenters for $\omega \in \Omega$.

Since $e(\pi)$ is linear for all $\pi \notin I_0$, it follows that,

$$\int_0^{\tau \wedge \tilde{\tau}} U^{\pi \cdot} e(\pi_s) ds = 0 \qquad \forall \tau \in \mathcal{T}, \qquad (3.1.14)$$

since it is an obvious fact that affine functions are annihilated by the characteristic operator of a martingale. Using 3.1.14, one obtains for all $\pi \notin I_0$ and $\tau \in \mathcal{T}$,

$$
\begin{aligned}
\rho(\pi, \tau) &= E_\pi[\int_0^\tau c(\pi_s) ds + e(\pi_\tau)] \\
&= e(\pi) + E_\pi[\int_0^\tau c(\pi_s) ds + \int_{\tau \wedge \tilde{\tau}}^\tau U^{\pi \cdot} e(\pi_s) ds] \\
&= e(\pi) + E_\pi[\int_0^\tau c(\pi_s) ds + 1\{\tau > \tilde{\tau}\} \int_{\tilde{\tau}}^\tau U^{\pi \cdot} e(\pi_s) ds] \qquad (3.1.15) \\
&= e(\pi) + E_\pi[\int_0^\tau c(\pi_s) ds + 1\{\tau > \tilde{\tau}\} e(\pi_\tau)] - E_\pi[1\{\tau > \tilde{\tau}\} e(\pi_{\tilde{\tau}})] \\
&\geq e(\pi) + E_\pi[1\{\tau > \tilde{\tau}\}(\int_0^\tau c(\pi_s) ds + e(\pi_\tau))] - E_\pi[1\{\tau > \tilde{\tau}\} e(\pi_{\tilde{\tau}})],
\end{aligned}
$$

where the last line holds since the running cost is nonnegative. Concentrating on the first expectation on the right-hand side of the preceding inequality, the lower bounding procedure continues,

$$
\begin{aligned}
E_\pi[1\{\tau > \tilde{\tau}\}(\int_0^\tau c(\pi_s) ds + e(\pi_\tau))] \\
&= E_\pi[1\{\tau > \tilde{\tau}\} E_\pi[\int_0^\tau c(\pi_s) ds + e(\pi_\tau)|\tau > \tilde{\tau}, \pi_{\tilde{\tau}(\omega)}(\omega) \in \partial_\pi I_0]] \\
&= E_\pi[1\{\tau > \tilde{\tau}\} \rho(\pi_{\tilde{\tau}}, \tau)] \\
&\geq E_\pi[1\{\tau > \tilde{\tau}\} \rho(\pi_{\tilde{\tau}})] \qquad \forall \pi \notin I_0, \tau \in \mathcal{T}. \\
&\hspace{10cm} (3.1.16)
\end{aligned}
$$

Combining 3.1.15 and 3.1.16 yields,

$$\rho(\pi, \tau) \geq e(\pi) + E_\pi[1\{\tau > \tilde{\tau}\}(\rho(\pi_{\tilde{\tau}}) - e(\pi_{\tilde{\tau}}))] \quad \forall \pi \notin I_0, \tau \in \mathcal{T}. \qquad (3.1.17)$$

The idea now of course is to kill the expectation in 3.1.17 by claiming that $\pi_{\tilde{\tau}}$ is in the π-boundary of I_0, which is where it is known that the risk equals the terminal cost, from Lemma 3.1.1. A subtlety has arisen here however, in the case where $\{\pi_t : t \geq 0\}$ is a real right-continuous $(P_\pi, \mathcal{O}_t)$-martingale with jumps. Letting,

$$S^+ = \{\pi \notin [I_0]_\pi\} \cap \{\pi > \frac{1}{2}\}, \tag{3.1.18}$$

and,

$$S^- = \{\pi \notin [I_0]_\pi\} \cap \{\pi < \frac{1}{2}\}, \tag{3.1.19}$$

then either $\partial_\pi^- S^+ \subset \partial_\pi^+ I_0$, or $\partial_\pi^+ S^- \subset \partial_\pi^- I_0$, and respectively $\pi_{\tilde{\tau}} \in \partial_\pi^+ I_0$ for $\pi_0 \in S^-$, or $\pi_{\tilde{\tau}} \in \partial_\pi^- I_0$ for $\pi_0 \in S^+$, P_π-a.s. If $\{\pi_t : t \geq 0\}$ has a continuous negative drift, and the jump term is positive and assigned to the $\theta = 1$ hypothesis, then $\partial_\pi^- S^+ \subset \partial_\pi^+ I_0$ is guaranteed. This is the case of interest here, so the argument will proceed with this assumption.

Now, from 3.1.13 and assuming $\partial_\pi^- S^+ \subset \partial_\pi^+ I_0$, P_π-a.s., it follows that

$$\pi_{\tilde{\tau}} \in \partial_\pi I_0 \qquad P_\pi\text{-a.s.,} \tag{3.1.20}$$

and therefore,

$$E_\pi[1\{\tau > \tilde{\tau}\}(\rho(\pi_{\tilde{\tau}}) - e(\pi_{\tilde{\tau}}))] = 0 \quad \forall \pi \notin I_0, \tau \in \mathcal{T}, \tag{3.1.21}$$

since condition (C2) and 3.1.9 require,

$$\rho(\pi) = r(\pi) = e(\pi) \qquad \forall \pi \in \partial_\pi I_0. \tag{3.1.22}$$

Thus upon substituting 3.1.21 into 3.1.17 there follows,

$$\rho(\pi) = \inf_{\tau \in \mathcal{T}} \rho(\pi, \tau) \geq e(\pi) \qquad \forall \pi \in S^+. \tag{3.1.23}$$

On the other hand, in view of the fact that $e(\pi)$ is concave, a consequence of Jensen's Inequality is,

$$
\begin{aligned}
\rho(\pi) &= \inf_{\tau \in \mathcal{T}} E_\pi [\int_0^\tau c(\pi_s)ds + e(\pi_\tau)] \\
&\leq \inf_{\tau \in \mathcal{T}} [E_\pi \int_0^\tau c(\pi_s)ds + e(E_\pi[\pi_\tau])] \\
&= \inf_{\tau \in \mathcal{T}} E_\pi \int_0^\tau c(\pi_s)ds + e(\pi) \\
&= e(\pi) \qquad\qquad \forall \pi \in [0,1].
\end{aligned}
\tag{3.1.24}
$$

Hence, 3.1.23 and 3.1.24 yield,

$$
\rho(\pi) = e(\pi) \qquad \forall \pi \in S^+.
\tag{3.1.25}
$$

By a similar argument one can show that,

$$
\rho(\pi) = e(\pi) \qquad \forall \pi \in S^-.
\tag{3.1.26}
$$

Combining 3.1.25 and 3.1.26 gives,

$$
\rho(\pi) = e(\pi) \qquad \forall \pi \notin I_0,
\tag{3.1.27}
$$

which is the desired result.   ∎

The preceding lemmas are the main ingredients to the principal optimality result proved next. A short recapitulation is in order. Lemma 3.1.1 showed that the three conditions imply that the risk and the subrisk are identical on the π-closure of the continuation interval. Hence, by itself the lemma suggests that threshold policies are at least $[I_0]_\pi$-Bayesian. Putting the two lemmas together yields the following.

Theorem 3.1.3 *Suppose there exists a pair (r, I_0) satisfying the conditions (C1-3). Then threshold policies are Bayesian.*

Proof: Suppose (C1),(C2), and (C3) hold, and define $\tau^* = \bar{\tau}$ as in 3.1.7. Then 3.1.8 and 3.1.9 give,

$$
\rho(\pi, \tau^*) = \rho(\pi) \cdot \qquad \forall \pi \in [I_0]_\pi,
\tag{3.1.28}
$$

i.e., τ^* solves the optimal stopping problem for all $\pi \in [I_0]_\pi$. Since τ^* is of threshold type, it follows that the policy $u^* = (\tau^*, \delta^*)$ exists and is $[I_0]_\pi$-Bayesian, where δ^* is given by,

$$\delta^* = \begin{cases} 1 & \text{if } \pi_{\tau^*} \geq b \\ 0 & \text{if } \pi_{\tau^*} \leq a, \end{cases}$$

for $I_0 = (a, b)$.

It follows from Lemma 3.1.2 that,

$$\rho(\pi) = e(\pi) \qquad \forall \pi \notin I_0. \tag{3.1.29}$$

On the other hand, $\tau^* = 0 \, P_\pi$-a.s. $\forall \pi \notin I_0$, and then,

$$\begin{aligned} \rho(\pi, \tau^*) &= E_\pi \big[\int_0^{\tau^*} c(\pi_s) ds + e(\pi_{\tau^*}) \big] \\ &= E_\pi [0 + e(\pi)] \\ &= e(\pi). \end{aligned} \tag{3.1.30}$$

Hence τ^* solves the optimal stopping problem for all $\pi \notin I_0$. Combining this fact with the above argument for $\pi \in [I_0]_\pi$ shows that there exists a policy, in particular $u^* = (\tau^*, \delta^*)$, τ^* as in 3.1.7 and δ^* as above, such that u^* is Bayesian. $\blacksquare$

The final theorem is a uniqueness result showing that a pair (r, I_0) satisfying conditions (C1-3) is unique on $[I_0]_\pi$ whenever r is concave on I_0. This is called **essential uniqueness** of the pair.

Theorem 3.1.4 *Let (r, I_0) satisfy the conditions (C1-3). If in addition r is concave on I_0, the pair is essentially unique.*

Proof: Let (s, J_0) be another pair satisfying the conditions, with s concave on J_0. The proof will proceed by contradiction; suppose $I_0 \neq J_0$. From Theorem 2.3.2, one may assume that $I_0 \cap J_0 \neq \emptyset$. So either I_0 or J_0 is strictly contained in the other, or they overlap. In the first situation, suppose wlog $I_0 \subset J_0$, strictly. In this case, $\partial I_0 \subset J_0$ and so letting $\pi \in \partial I_0$, it follows from Lemma 3.1.1 that,

$$\rho(\pi) = r(\pi) = e(\pi) \qquad \forall \pi \in \partial I_0. \tag{3.1.31}$$

On the other hand, it is also a consequence of Lemma 3.1.1 that,

$$\rho(\pi) = s(\pi) < e(\pi) \qquad \forall \pi \in J_0 \supset \partial I_0. \tag{3.1.32}$$

As a result, neither strictly contains the other, and so they must overlap. Define the nonempty interval $I^* = I_0 \cap J_0$. Choose $\pi \in \partial I^*$ and notice either $\pi \in \partial I_0 \cap J_0$ or $\pi \in \partial J_0 \cap I_0$. However, if either is true, then arguing as in 3.1.31 and 3.1.32 will lead to a contradiction, thus indicating that the original supposition is false and thence $I_0 = J_0$.

It is now a simple application of Lemma 3.1.1 to the pairs (r, I_0), (s, I_0) which demonstrates,

$$r(\pi) = s(\pi) \qquad \forall \pi \in [I_0]_\pi. \tag{3.1.33}$$

Thus the pair (r, I_0) is essentially unique in the sense that $I_0 = J_0$ and 3.1.33 holds. In particular,

$$r(\pi) = \rho(\pi) = s(\pi) \qquad \forall \pi \in [I_0]_\pi, \tag{3.1.34}$$

suggesting the terminology.   ∎

The nonuniqueness of $r(\pi)$ off of $[I_0]_\pi$ is irrelevant to any questions concerning the risk. This is fortunate because r is not in general unique there.

In the next section, the conditions (C1-3) are used to discover important properties that the pair must satisfy. Such necessary conditions guide the search for pair in the applications chapter.

3.2   Necessary Conditions

In the last section, the goal was to demonstrate the sufficiency of the conditions (C1-3) in proving that threshold policies are Bayesian. The emphasis in this section is in using these conditions to discover properties which the subrisk and the continuation interval must satisy. Such necessary conditions have their primary importance in applications, by providing characterizations that guide the search for the subrisk and continuation interval pair which obeys (C1-3).

The next theorem is very important in applications in that it provides a way to demonstrate the concavity of the subrisk based on its smoothness properties, quite a bargain since a direct proof of concavity can be much more difficult than a verification of smoothness. Note then the intimate connection between this theorem and the condition (C3) in an applications setting.

Theorem 3.2.1 *Let (r, I_0) exist satisfying (C1), and suppose $r(\pi)$ is continuous on $[I_0]_\pi$. Then $r(\pi)$ is concave on $[I_0]_\pi$ if and only if $r \in \mathcal{C}^{2+}([I_0]_\pi)$.*

Proof: If $r(\pi)$ is concave on $[I_0]_\pi$, then $r \in \mathcal{C}^{2+}([I_0]_\pi)$ *a fortiori*. Suppose then that $r \in \mathcal{C}^{2+}([I_0]_\pi)$. The proof will proceed by contradiction. Suppose $r(\pi)$ is not concave on $[I_0]_\pi$. Then it is convex on some subinterval $J \subset [I_0]_\pi$ in view of the fact that $r \in \mathcal{C}^0 \cap \mathcal{C}^{2+}([I_0]_\pi)$. Define,

$$\tau_J = \inf\{t \geq 0 : \pi_t \notin J\}, \qquad (3.2.1)$$

and suppose $\pi_0 \in J$, P_π-a.s. Now, consider any stopping time $\sigma \in \mathcal{T}$, where $\sigma < \tau$, P_π-a.s.. Since $r(\pi)$ is convex on J, one obtains,

$$E_\pi[r(\pi_\sigma)] \geq r(E_\pi[\pi_\sigma]) = r(\pi) \qquad \pi \in J, \qquad (3.2.2)$$

where the inequality is an application of Jensen's Inequality, and the equality follows since $\{\pi_t : t \geq 0\}$ is a $(P_\pi, \mathcal{O}_t)$-martingale. On the other hand,

$$U r(\pi) < 0 \quad \Longrightarrow \quad E_\pi[r(\pi_\tau)] < r(\pi) \quad \forall \tau \in \mathcal{T}, \pi \in E, \qquad (3.2.3)$$

i.e., condition (C1) guarantees that $\{r(\pi_t) : t \geq 0\}$ is bounded above by its initial value. As a result, 3.2.1 is a clear contradiction, and therefore $r(\pi)$ is concave on $[I_0]_\pi$. ∎

Corollary *If (r, I_0) exists satisfying (C1) and $r \in \mathcal{C}^0 \cap \mathcal{C}^{2+}(E)$ for any set $E \subset (0,1)$, then $r(\pi)$ is concave on E.*

Proof: Replace $[I_0]_\pi$ in the theorem by E. ∎

The next theorem is used in applications to obtain condition (C3) from (C1), (C2), and Theorem 3.2.1.

Theorem 3.2.2 *Let there exist a pair (r, I_0) satisfying the conditions (C1) and (C2). Suppose in addition that $r(\pi)$ is concave on $[I_0]_\pi$. Then,*

$$r(\pi) < e(\pi) \qquad \forall \pi \in I_0,$$

if,

$$r'(a) \leq e'(a)$$
$$r'(b^-) \geq e'(b), \qquad\qquad (3.2.4)$$

where, $I_0 = (a, b)$, $0 < a < \frac{1}{2} < b < 1$.

Proof: First note that since $r(\pi)$ satisfies (C1), then it is not affine. Next note that $r(\pi)$ is continuous at $\pi = a$ and $\pi = b$. It is continuous at $\pi = b$ because $\pi = b$ is not an endpoint of $[I_0]_\pi$, and because concave functions are continuous except possibly at their endpoints. It is continuous at $\pi = a$ because $r \in \hat{\mathcal{D}}(U)$ implies that $r(\pi)$ is at least right-continuous, per convention.

Then it is clear from the fact that $r(\pi)$ is concave that,

$$r(\pi) < l_a(\pi) \qquad \forall \pi \in I_0, \qquad\qquad (3.2.5)$$

where,

$$l_a(\pi) = r(a) + r'(a)(\pi - a). \qquad\qquad (3.2.6)$$

Likewise, it follows that,

$$r(\pi) < l_b(\pi) \qquad \forall \pi \in I_0, \tag{3.2.7}$$

where,

$$l_b(\pi) = r(b) + r'(b^-)(\pi - b). \tag{3.2.8}$$

Since condition (C2) imposes: $r(a) = e(a)$, $r(b) = e(b)$, then one observes that,

$$e(\pi) = l_a(\pi)1\{\pi < \frac{1}{2}\} + l_b(\pi)\{\pi \geq \frac{1}{2}\} \qquad \forall \pi \in [0,1], \tag{3.2.9}$$

taking 3.2.4 into account. The combination of 3.2.5, 3.2.7, and 3.2.9 gives,

$$r(\pi) < e(\pi) \qquad \forall \pi \in I_0, \tag{3.2.10}$$

the desired result.   ∎

This concludes the chapter. In the next, the results of this chapter and those of the first will be used to solve three problems arising in applications. The first two deal with the Bayesian problem when the observations come from a diffusion, and the third when one is observing a time-homogeneous Poisson process.

Chapter IV
Applications in Sequential Testing

4.0   Introduction

In this chapter three problems in Bayesian sequential hypothesis testing will be formulated and solved. Given the framework established in the second chapter, formulation here means specifying the two hypothesized probability measures, and the observation σ-algebra. Two of the problems arise when one observes a diffusion and seeks to minimize a given Bayes risk. In the first, it is assumed that one of two constant drifts is responsible for the observations. Actually, the detection version of the hypothesis testing problem is solved because it is mathematically equivalent while notationally simpler. A constant, positive cost-rate is chosen for this problem which is proportional to the the square of the nonzero drift. It is shown that a threshold policy based on thea *posteriori* probability is optimal, and the risk and thresholds are given.

The second problem considered is the case where a diffusion with a drift which is assumed to be merely progressively measurable is the basis for the detection problem. It is shown that, if one chooses a constant, positive cost-rate which is proportional to the square of the stochastic drift, then again, a threshold policy is optimal. In fact, it is shown that the risk and thresholds are identical to those in the homogeneous case.

The last problem considers the case where under each hypothesis, it is assumed

that the observations are due to a constant rate Poisson process. Here the hypothesis testing problem is presented because the detection problem—a constant rate versus a rate of one—is no real notational bargain. A threshold policy is shown to be Bayesian, and explicit formulas for the risk and thresholds are given.

4.1   Diffusion Observations

This section will draw on the formulation and notation laid down in sections 2.1 and 2.2 on the Bayesian Formulation to the sequential hypothesis testing problem. Following the briefest review, the problem will be specialized to the case where the filtration history arises from observations of a continous semimartingale driven by a standard Wiener process.

Let $\pi \in [0, 1]$ be arbitrary throughout the remainder of the section, unless otherwise noted. This will save the continual trouble of having to restate the obvious, and this will not be done, except for emphasis in some of the numbered equations.

Recall that on the measurable space, $(\Omega, \mathcal{F})$ there is given a family of probability measures $\{P_\pi : 0 \leq \pi \leq 1\}$, satisfying 2.2.2, and a $\{\theta_0, \theta_1\}$-valued random variable, θ, with the *a priori* probability distribution given in 2.2.1. Based on the observation filtration, $\mathcal{O}_t$, one wants to choose a value for θ which minimizes the risk, defined in 2.2.4. Values for θ are chosen by a two-step procedure. First, the decision to terminate the observation procedure is made according to a $(P_\pi, \mathcal{O}_t)$-stopping time, say τ, and then the value for θ is chosen according to a $(P_\pi, \mathcal{O}_\tau)$-binary random variable, say δ (see definition 2.1.1). A particular application begins by specifying the nature of $\{\mathcal{O}_t : t \geq 0\}$, the σ-algebra generated by the observations.

It is assumed that the random variable θ is unobservable, but that one can observe a continuous random process $\{y_t : t \geq 0\}$ whose statistics under each of the hypotheses—$\theta = \theta_0$, $\theta = \theta_1$—are governed by the probability measures P_0 and P_1, respectively. To wit, for each $\omega \in \Omega$, the observation process has the stochastic differentials,

$$dy_t(\omega) = \begin{cases} h_t^0(\omega)dt + dw_t(\omega), & \text{if } \theta(\omega) = \theta_0; \\ h_t^1(\omega)dt + dw_t(\omega), & \text{if } \theta(\omega) = \theta_1, \end{cases}$$

where, $\{w_t : t \geq 0\}$ is a $(P_\pi, \mathcal{F}_t)$-**standard Wiener process** which is P_π-independent of θ, and where $\{h_t^i : t \geq 0\}$ is an $\mathcal{F}_t$-progressive process satisfying, $E_i[\int_0^t |h_s^i|ds] < \infty$,

$i = 0, 1$. Now, from section 2.5, the above hypothesis testing problem was shown, without loss of generality, to be equivalent to a detection problem, and so, letting $\theta_0 = 0$, and $\theta_1 = 1$, the problem can be recast in the detection format, in which case $\{y_t : t \geq 0\}$ has the $(P_\pi, \mathcal{F}_t)$-stochastic differential

$$dy_t = \theta h_t dt + dw_t \qquad t \geq 0, \tag{4.1.1}$$

with the obvious requirements on $\{h_t : t \geq 0\}$, and $\{w_t : t \geq 0\}$.

From 4.1.1, one takes the observation filtration, $\mathcal{O}_t$, to be the σ-algebra generated by the $\{y_t : t \geq 0\}$ process. Having done this, recall from section 2.2 that the *a posteriori* probability is $\pi_t = P_\pi\{\theta = 1 | \mathcal{O}_t\}$, $t \geq 0$. This permits the total average risk to be given as

$$\rho(\pi, \tau) = E_\pi \left[\int_0^\tau c(\pi_s) ds + e(\pi_\tau) \right], \tag{4.1.2}$$

where $c_t(\pi)$ is a given cost-rate function, and $e(\pi)$ is the terminal cost. Note that 4.1.2 is free of δ, as achieved in section 2.3.

Having defined the observation filtration, the immediate goal is to compute a stochastic differential for the $\{\pi_t : t \geq 0\}$ process. From 4.1.1 it is clear that $P_0 \ll P_1$, so take,

$$\Lambda_t(\omega) = \frac{dP_1}{dP_0}(\mathcal{O}_t)(\omega), \tag{4.1.3}$$

to be the likelihood ratio for the problem, i.e., it is the Radon-Nikodym derivative of the $\mathcal{O}_t$-restriction of the measure P_1 with respect to the $\mathcal{O}_t$-restriction of the measure P_0. If $\pi = 1$, then $P_\pi\{\pi_t = 1\} = 1$ with P_π probability one, for all $t \geq 0$. Suppose $\pi < 1$; in Appendix I, it is shown in this case that (see A.1.11),

$$\pi_t = \frac{\frac{\pi}{1-\pi}\Lambda_t}{1 + \frac{\pi}{1-\pi}\Lambda_t} = \phi(\Lambda_t), \tag{4.1.4}$$

and so, applying the Itô differentiation rule one obtains,

$$\pi_t - \pi_0 = \int_0^t \frac{\pi_s(1 - \pi_s)}{\Lambda_s} d\Lambda_s - \int_0^t \frac{\pi_s^2(1 - \pi_s)}{\Lambda_s^2} d[\Lambda, \Lambda]_s. \tag{4.1.5}$$

The likelihood ratio for this problem is well known and is given by,

$$\Lambda_t = \exp\{\int_0^t \hat{h}_s dy_s - \frac{1}{2}\int_0^t \hat{h}_s^2 ds\}, \tag{4.1.6}$$

where,

$$\hat{h}_t = E_1[h_t|\mathcal{O}_t]. \tag{4.1.7}$$

From 4.1.3 and 4.1.6, one can obtain,

$$\Lambda_t = 1 + \int_0^t \hat{h}_s \Lambda_s dy_s. \tag{4.1.8}$$

Now, if one observes that (see A.1.7),

$$E_\pi[\theta h_t|\mathcal{O}_t] = \pi_t \hat{h}_t, \tag{4.1.9}$$

then a straightforward calculation shows that there exists a $(P_\pi, \mathcal{O}_t)$-standard Wiener martingale, say $\{\bar{w}_t : t \geq 0\}$, such that $\{y_t : t \geq 0\}$ has stochastic differential,

$$dy_t = \pi_t \hat{h}_t dt + d\bar{w}_t. \tag{4.1.10}$$

Combining 4.1.5, 4.1.8, and 4.1.10 yields,

$$\pi_t - \pi_0 = \int_0^t \pi_s(1-\pi_s)\hat{h}_s dy_s - \int_0^t \pi_s^2(1-\pi_s)\hat{h}_s^2 ds$$
$$= \int_0^t \pi_s(1-\pi_s)\hat{h}_s d\bar{w}_s \qquad \forall t \geq 0, \tag{4.1.11}$$

and so,

$$d\pi_t = \pi_s(1-\pi_s)\hat{h}_s d\bar{w}_s. \tag{4.1.12}$$

It is clear from 4.1.12 that since $\{\bar{w}_t : t \geq 0\}$ is a continuous martingale that $\partial_\pi I_0 = \partial I_0 = \{a, b\}$, if $I_0 = (a, b)$ is some interval such that $a < \pi_0 < b$, P_π-a.s., and $0 < a < b < 1$. It may not be clear that $\{\pi_t : t \geq 0\}$ can be guaranteed to satisfy Theorem 1.1.1, i.e., can be guaranteed to escape any such interval. To see that this is so, first note that $\pi_t = \Phi(L_t)$, where $\Phi : \mathbb{R} \to [0,1]$ is the bijective function given

implicitly by 4.1.4 and the relation, $L_t = \log \Lambda_t$, i.e., $\Phi(L) = \phi(\log(L))$; L is called the log-likelihood ratio. Now, equations 4.1.6 and 4.1.10 show,

$$dL_t = (\pi_t - \frac{1}{2})\widehat{h}_t^2 dt + \widehat{h}_t \bar{w}_t \qquad t \geq 0, \tag{4.1.13}$$

which yields the $(P_\pi, \mathcal{O}_t)$-compensator for L as,

$$A_t^L = (\pi_t - \frac{1}{2})\widehat{h}_t^2 \qquad t \geq 0. \tag{4.1.14}$$

Since $P_i \ll P_\pi$, $i = 0, 1$, it follows that,

$$P_\pi\{|A_\infty^L| = \infty\} = 1 \quad \Longleftrightarrow \quad P_i\{|A_\infty^L| = \infty\} = 1 \qquad i = 0, 1, \tag{4.1.15}$$

and therefore, as prerequisites to Theorem 1.1.1, it is assumed that,

$$\text{(A1):} \qquad P_\pi\{|A_\infty^L| = \infty\} = 1 \quad \forall \pi \in [0, 1];$$

$$\text{(A2):} \qquad P_\pi\{|A_t^L| = \infty\} = 0 \quad \forall \pi \in [0, 1], \, t < \infty.$$

These assumptions are necessary to ensure that $\{\pi_t : t \geq 0\}$ will escape I_0 in finite time P_π-a.s., and thence guarantee that the threshold policy with continuation interval I_0 will eventually terminate. In view of 4.1.15, the equivalent assumptions, recast in terms of the measures P_i, $i = 0, 1$ are,

$$\text{(A1)}' : \qquad P_i\{\int_0^\infty \widehat{h}_s^2 ds = \infty\} = 1 \quad i = 0, 1;$$

$$\text{(A2)}' : \qquad P_i\{\int_0^t \widehat{h}_s^2 ds = \infty\} = 0 \quad i = 0, 1, \, t < \infty.$$

The third condition of Theorem 1.1.1 can be shown directly as,

$$
\begin{aligned}
|E_\pi[A_t^L]| &= |(1 - \pi)E_0[A_t^L] + \pi E_1[A_t^L]| \\
&= |(\pi - 1)E_0[\int_0^t \frac{1}{2}\widehat{h}_s^2 ds] + \pi E_1[\int_0^t \frac{1}{2}\widehat{h}_s^2 ds]| \\
&= |(\pi - 1)E_0|A_t^L| + \pi E_1|A_t^L|| \\
&= |E_\pi|A_t^L| - \pi E_1|A_t^L| + \pi E_1|A_t^L|| \\
&= E_\pi|A_t^L|
\end{aligned}
\tag{4.1.18}
$$

Hence, by Theorem 1.1.1, L is guaranteed to escape any open interval in finite time P_π-a.s., and since $\pi_t = \Phi(L_t)$ where Φ is bijective, then the same holds true for the *a posteriori* probability.

Next, it is desired to compute the characteristic operator for $\{\pi_t : t \geq 0\}$ on $\widehat{D}(U)$. So, letting $r \in \widehat{D}(U)$, an application of the Itô rule gives,

$$r(\pi_t) - r(\pi_0) = \int_0^t \pi_s^2(1 - \pi_s)^2 \widehat{h}_s^2 r''(\pi_s) ds + \int_0^t \pi_s(1 - \pi_s) \widehat{h}_s r'(\pi_s) d\bar{w}_s, \quad (4.1.19)$$

using 4.1.12. Two cases will be considered.

For the first case, suppose $h_t = h$, $t \geq 0$, where h is some deterministic, nonzero constant. Thus, (A1)$'$ and (A2)$'$ hold. In addition, suppose that the cost-rate function is given as $c(\pi) = ch^2$, with c a positive deterministic constant; this choice of cost-rate entails no loss of generality since $c > 0$ is otherwise arbitrary. In addition, it follows that condition (T) holds, thus $E_\pi[\int_0^\infty c_s ds] = \infty$. This is a classic Bayesian set-up: a constant running cost risk with a homogeneous diffusion. From 4.1.19, the characteristic operator is found to be,

$$Ur(\pi) = h^2 \pi^2 (1 - \pi)^2 r''(\pi) \qquad \forall \pi \in (0, 1). \quad (4.1.20)$$

The application of the results in sections 3.2 and 3.3 begins by setting up the problem specified by conditions (C1) and (C2), which is:

$$\begin{cases} Ur(\pi) = -ch^2 & \forall \pi \in (0, 1); \\ r(\pi) = e(\pi) & \pi \in \{a, b\}. \end{cases}$$

Substituting 4.1.20 into the above and dividing by h^2 yields,

$$\begin{cases} \pi^2(1 - \pi)^2 r''(\pi) = -c & \forall \pi \in (0, 1); \\ r(\pi) = e(\pi) & \pi \in \{a, b\}, \end{cases} \quad (4.1.21)$$

which is a family of ODE's indexed by the boundary points $\{a, b\}$. Motivated by Theorem 3.2.1, it is required that a solution to 4.1.21 at least satisfies,

$$r(\pi) \text{ is continuous } \forall \pi \in [a, b]. \quad (4.1.22)$$

Clearly, the solution to 4.1.21-22 exists, is unique, and is easily obtained in closed form as,

$$r(\pi) = \frac{b - \pi}{b - a} e(a) + \frac{\pi - a}{b - a} [e(b) - d(b) + d(a)] + d(\pi) - d(a), \qquad (4.1.23)$$

where,

$$d(\pi) = c(1 - 2\pi) \log \frac{\pi}{1 - \pi}.$$

Now it is clear that $r(\pi)$ is concave for all $\pi \in (0, 1)$ directly from 4.1.21. However, it is also true that 4.1.21 shows that $r \in C^{2+}(E)$, with $E = [a, b]$, and so invoking Theorem 3.2.1, it follows that $r(\pi)$ is concave for all $\pi \in [a, b]$. Obviously, this weaker concavity argument is trivial and unnecessary, but it is given for pedagogical reasons—to show at which step Theorem 3.2.1 is applied. It will turn out that the concavity of the subrisk is not a 'free gift' in the discontinuous martingale case considered in the next section.

Since $r(\pi)$ is concave for all $\pi \in (0, 1)$, then condition (C3) is satisfied if one imposes,

$$r'(a) = e'(a), \qquad r'(b) = e'(b). \qquad (4.1.24)$$

The solution $\{a^*, b^*\}$ to 4.1.24 is easily shown to exist (see below). Its uniqueness is also easily shown, but follows *a fortiori* in view of Theorem 3.2.2 simply from its existence. Again, in the jump process case, this is a welcome device to proving uniqueness because the existence proof will be difficult enough.

In summary, the three sufficient conditions, (C1-3), have been shown to be in force. Thus, the following theorem has been demonstrated to be true.

Theorem 4.2.1 *In the problem of sequential detection based on observations of the homogeneous diffusion process,*

$$y_t = \theta h t + w_t \qquad h \neq 0; \quad t \geq 0,$$

with running cost,

$$E_\pi[\int_0^\tau c_s ds] = E_\pi[ch^2\tau],$$

there exists a Bayesian rule, $u^* = (\tau^*, \delta^*)$, which is a threshold policy given by,

$$\tau^* = \inf\{t \geq 0 : \pi_t \notin (a^*, b^*)\}$$

$$\delta^* = \begin{cases} 1, & \text{if } \pi_{\tau^*} \geq b^*; \\ 0, & \text{if } \pi_{\tau^*} \leq a^*. \end{cases} \tag{4.1.25}$$

The Bayes risk is given by,

$$\rho(\pi) = \begin{cases} e(\pi) & \pi \notin (a^*, b^*); \\ \frac{b^*-\pi}{b^*-a^*}e(a^*) + \frac{\pi-a^*}{b^*-a^*}[e(b^*) - d(b^*) + d(a^*)] + d(\pi) - d(a^*) & \text{otherwise,} \end{cases}$$

where,

$$d(\pi) = c(1 - 2\pi) \log \frac{\pi}{1 - \pi}. \tag{4.1.26}$$

The constants a^* and b^* are uniquely determined from the system of transcendental equations,

$$e'(a^*) - d'(a^*) = f(a^*, b^*);$$
$$e'(b^*) - d'(b^*) = f(a^*, b^*), \tag{4.1.27}$$

with,

$$f(a, b) = \frac{e(b) - e(a) - (d(b) - d(a))}{b - a}. \tag{4.1.28}$$

Proof: Given largely in the preceding discussion. Note that the conditions (T), (A1), and (A2) are satisfied trivially because $h \neq 0$ and because $c(\pi) = ch^2$, with $c > 0$. Also note that $\tau^* < \infty$, $P_\pi - a.s.$, in view of Theorem 1.1.1. The system to solve for the optimal thresholds a^*, b^*, is obtained from 4.1.23 and 4.1.24. In order to show that a^*, b^* exist satisfying 4.1.27, first observe that according to the results of section 2.3 one may assume $e(\pi)$ to be symmetric and normalized without loss of generality (see 2.3.9). Having made this assumption, it follows from 4.1.27 that

$$e'(a^*) = 1 = -e'(b^*). \tag{4.1.29}$$

Letting $\varphi(\pi) = d'(\pi)$, it is clear that $\varphi : (0,1) \to (-\infty, \infty)$ is monotone decreasing and antisymmetric about $\pi = \frac{1}{2}$, i.e., $\varphi(\pi) = -\varphi(1 - \pi)$. Note also from 4.1.28 that,

$$
\begin{aligned}
f(\pi, 1 - \pi) &= f(\pi, 1 - \pi) \\
&= \frac{1 - (1 - \pi) - \pi - (d(1 - \pi) - d(\pi))}{(1 - \pi) - \pi} \\
&= 0.
\end{aligned}
\tag{4.1.30}
$$

Thus it is easy to verify that (a^*, b^*) solves 4.1.27 if one chooses $b^* = 1 - a^*$ and a^* solves,

$$
\varphi(a^*) = 1.
\tag{4.1.31}
$$

In view of the comments about φ above, it follows that a solution to 4.1.31 exists and is unique. In view of Theorem 3.1.4 the solution (a^*, b^*) so constructed is also unique. This completes the proof. ∎

A particular choice of cost-rate constant for an instance of the above problem is purely a design decision, modulo the positivity requirement. There is no choice which is uniformly good for all problems. However, the given choice, $c(\pi) = ch^2$, is intuitively satisfying since it not only enjoys positivity, but moreover exhibits the appealing property of penalizing the detector more, or less, depending upon whether the magnitude of the drift is greater, or smaller, than one, and does so according to the square of the magnitude. One might say that this choice of cost-rate is reasonable in that it reflects a designer's modest desire to expect better performance from the detector (faster decisions on average) in 'favorable' problems (large drifts to detect), and to allow worse performance (slower decisions on average) in 'hard' problems (drifts close to zero). Apparently, such a designer believes 'not all filtrations are created equal', to put it colloquially.

A by-product of this choice of cost is that the drift is factored out of the risk, and therefore the risk is independent of any particular drift. This is a significant advantage because it implies immediately that the results of Theorem 4.1.1 can be extended to

a more general case. To see this, first consider that the proper generalization of the
cost choice discussed above is to take the running cost as,

$$E_\pi[\int_0^\tau c_s ds] = E_\pi[\int_0^\tau c\widehat{h}_s^2 ds], \tag{4.1.32}$$

in the case where the drift of $\{y_t : t \geq 0\}$ is in general only known to be $(P\pi, \mathcal{F}_t)$-
progressive. For this choice of cost, it is clear that $c_t = c\widehat{h}_t^2$, $c > 0$, $t \geq 0$, is a
$(P_\pi, \mathcal{O}_t)$-progressive process. Having made this choice of running cost, one can prove
the following.

Theorem 4.1.2 *Assume (A1) and (A2) hold. In the problem of sequential detection
based on observations of the process (see 4.1.1),*

$$y_t = \theta \int_0^t h_s ds + w_t \qquad t \geq 0, \tag{4.1.33}$$

with running cost,

$$E_\pi[\int_0^\tau c_s ds] = E_\pi[\int_0^\tau c\widehat{h}_s^2 ds] \qquad c > 0, t \geq 0, \tag{4.1.34}$$

*the threshold policy, $u^* = (\tau^*, \delta^*)$, given in the homogeneous case, is Bayesian. More-
over, the risk and thresholds are exactly the same.*

Proof: First note that (T) follows from (A1).

The computation of the risk is given by,

$$\begin{aligned}
\rho(\pi) &= \inf_{\tau \in \mathcal{T}} E_\pi[\int_0^\tau c_s ds + e(\pi_\tau)] \\
&= \inf_{\tau \in \mathcal{T}} E_\pi[\int_0^\tau c\widehat{h}_s^2 ds + e(\pi_\tau)] \\
&= \inf_{\tau \in \mathcal{T}} E_\pi[-\int_0^\tau \pi_s^2(1-\pi_s)^2\widehat{h}_s^2 r''(\pi_s) ds + e(\pi_\tau)],
\end{aligned} \tag{4.1.35}$$

where the last line follows from 4.1.21. Now, from 4.1.19 and 4.1.23,

$$\rho(\pi) = r(\pi) + \inf_{\tau \in \mathcal{T}} E_\pi[e(\pi_\tau) - r(\pi_\tau)], \tag{4.1.36}$$

but this is the same risk as in the homogeneous case, and so the result is shown.   ▪

A special case of the above result is given in [LaVIGNA], using an argument based upon a theorem due to Shiryayev [LIPTSER & SHIRYAYEV 77] for the probelm of Wald sequential detection [WALD]. The proof given here has one advantage in that it makes it clear how to generalize the problem to include other cost-rates.

This concludes the section. In the next, the Bayesian problem is considered for the homogeneous Poisson case, which is very similar to the homogeneous diffusion case.

4.2   Point Process Observations

This section will consider the sequential hypothesis testing problem from the Bayesian viewpoint when the observations arise from a counting process. The notation and basic set-up are given in sections 2.1 and 2.2.

Recall that on a measurable space $(\Omega, \mathcal{F})$, there is given a family of probability measures $\{P_\pi : 0 \leq \pi \leq 1\}$, satisfying 2.2.2, and a binary valued random variable, θ, with the *a priori* probability distribution given in 2.2.1. Based on the observation filtration, $\mathcal{O}_t$, one wants to choose a value for θ which minimizes the risk, defined in 2.2.4. The manner in which values for θ are chosen has two parts. First, the decision to terminate the observation procedure is made according to a $(P_\pi, \mathcal{O}_t)$-stopping time, say τ, and second, a value for θ is chosen according to a $(P_\pi, \mathcal{O}_\tau)$-binary random variable, say δ (see definition 2.1.1). A particular application begins by specifying the nature of $\{\mathcal{O}_t : t \geq 0\}$, the σ-algebra generated by the observations.

It is assumed that the random variable θ is unobservable, but that one can observe a counting process $\{n_t : t \geq 0\}$ whose statistics under each of the hypotheses—$\theta = 0$, $\theta = 1$—are governed by the probability measures P_0 and P_1, respectively. Specifically, for each $\omega \in \Omega$, the observation process is a counting process with semimartingale representation,

$$
n_t(\omega) = \begin{cases} \int_0^t \lambda_s^0(\omega)ds + m_t(\omega), & \text{if } \theta(\omega) = 0; \\[2mm] \int_0^t \lambda_s^1(\omega)ds + m_t(\omega), & \text{if } \theta(\omega) = 1, \end{cases}
$$

where, $\{m_t : t \geq 0\}$ is a $(P_\pi, \mathcal{F}_t)$-martingale, and where $\{\lambda_t^i : t \geq 0\}$ is an $\mathcal{F}_t$-predictable process satisfying $E_i \int_0^t |\lambda_s^i|ds < \infty$, $i = 0, 1$. Thus, it is assumed that the observation process is a $(P_\pi, \mathcal{F}_t)$-semimartingale with stochastic differential,

$$
dn_t = \left(\theta\lambda_t^1 + (1 - \theta)\lambda_t^0\right)dt + dm_t \qquad t \geq 0, \tag{4.2.1}
$$

while the observation filtration is the σ-algebra generated by the this process.

The next step is to compute a (P_π, O_t)-semimartingale representation for the *a posteriori* probability, $\pi_t = P_\pi\{\theta = 1|O_t\}$, $t \geq 0$, as a precursor to minimizing the total average risk given in 2.3.7. To begin, note that the likelihood ratio for the problem is well known and is given by,

$$\Lambda_t = \exp\Big[\int_0^t \log \frac{\hat{\lambda}_s^1}{\hat{\lambda}_s^0} dn_s - \int_0^t (\hat{\lambda}_s^1 - \hat{\lambda}_s^0)ds\Big] \quad t \geq 0, \tag{4.2.2}$$

where,

$$\hat{\lambda}_t^i = E_i[\lambda_t^i|O_{t-}] \qquad t \geq 0. \tag{4.2.3}$$

From 4.2.2 it follows that,

$$d\Lambda_t = \Lambda_{t-}\frac{\hat{\lambda}_t^1 - \hat{\lambda}_t^0}{\hat{\lambda}_t^0}\big(dn_t - \hat{\lambda}_t^0 dt\big). \tag{4.2.4}$$

Now, using 4.1.4, one obtains,

$$\pi_t - \pi_0 = \int_0^t \frac{\pi_s(1 - \pi_s)}{\Lambda_s}d\Lambda_s^c + \sum_{0 < s \leq t} \Delta\phi(\Lambda_s), \tag{4.2.5}$$

and from 4.2.4

$$d\Lambda_t^c = -\Lambda_s(\hat{\lambda}_s^1 - \hat{\lambda}_s^0)ds. \tag{4.2.6}$$

The computation for $\Delta\phi(\Lambda_t)$ is given as,

$$\begin{aligned}
\Delta\phi(\Lambda_t) &= \frac{\frac{\pi}{1-\pi}\Delta\Lambda_t}{\big(1 + \frac{\pi}{1-\pi}\Lambda_t\big)\big(1 + \frac{\pi}{1-\pi}\Lambda_{t-}\big)} \\
&= \pi_{t-}\big(1 - \pi_t\big)\frac{\Delta\Lambda_t}{\Lambda_{t-}} \\
&= \pi_{t-}\big(1 - \pi_t\big)\frac{\hat{\lambda}_t^1 - \hat{\lambda}_t^0}{\hat{\lambda}_t^0} \\
&= \frac{\pi_{t-}\big(1 - \pi_t\big)}{u_t},
\end{aligned} \tag{4.2.7}$$

where the last line also serves to define $\{u_t : t \geq 0\}$. In addition, note that,

$$\begin{aligned}
1 - \pi_t &= \frac{1}{1 + \frac{\pi}{1-\pi}\Lambda_t} \\
&= \frac{1}{1 + \frac{\pi}{1-\pi}\Lambda_{t-} + \frac{\pi}{1-\pi}\frac{\Lambda_{t-}}{u_t}} \\
&= \frac{(1 - \pi_{t-})u_t}{u_t + \pi_{t-}}.
\end{aligned} \tag{4.2.8}$$

Combining 4.2.5, 4.2.6, 4.2.7, and 4.2.8 yields,

$$\pi_t - \pi_0 = -\int_0^t (\hat{\lambda}_s^1 - \hat{\lambda}_s^0)\pi_s(1 - \pi_s)ds + \sum_{0 < s \leq t} \frac{\pi_{s-}(1 - \pi_{s-})}{u_s + \pi_{s-}}$$

(4.2.9)

$$= -\int_0^t (\hat{\lambda}_s^1 - \hat{\lambda}_s^0)\pi_s(1 - \pi_s)ds + \int_0^t \frac{\pi_{s-}(1 - \pi_{s-})}{u_s + \pi_{s-}}dn_s,$$

where the second line follows since $\{m_t : t \geq 0\}$, and therefore $\{n_t : t \geq 0\}$, has unity jumps. Now, if one observes that (see A.1.7),

$$E_\pi[\theta\lambda_t^1 + (1 - \theta)\lambda_t^0 | \mathcal{O}_{t-}] = \hat{\lambda}_t^1 \pi_{t-} + \hat{\lambda}_t^0(1 - \pi_{t-}),$$

(4.2.10)

then a straightforward calculation shows that there exists a $(P_\pi, \mathcal{O}_t)$-martingale, say $\{\bar{m}_t : t \geq 0\}$, such that $\{n_t : t \geq 0\}$ has the $(P_\pi, \mathcal{O}_t)$-stochastic differential,

$$dn_t = (\hat{\lambda}_t^1 \pi_{t-} + \hat{\lambda}_t^0(1 - \pi_{t-}))dt + d\bar{m}_t,$$

(4.2.11)

Substituting 4.2.11 into 4.2.9 gives,

$$\pi_t - \pi_o = \int_0^t \frac{\pi_{s-}(1 - \pi_{s-})}{u_s + \pi_{s-}}d\bar{m}_s.$$

(4.2.12)

Having computed 4.2.12, the next step is to investigate is under what conditions a process as in 4.2.12 can be guaranteed to escape an interval and hence serve as the basis of a threshold policy.

Consider an interval $I_0 = (a, b)$ such that $0 < a < b < 1$, and $a < \pi_0 < b$, P_π-a.s. From 4.2.2, 4.2.11, and the relation $L_t = \log \Lambda_t$, there holds,

$$dL_t = \left[(\hat{\lambda}_t^1 \pi_{t-} + \hat{\lambda}_t^0(1 - \pi_{t-})) \log \frac{\hat{\lambda}_t^1}{\hat{\lambda}_t^0} - (\hat{\lambda}_t^1 - \hat{\lambda}_t^0)\right]dt + \log \frac{\hat{\lambda}_t^1}{\hat{\lambda}_t^0}d\bar{m}_t,$$

(4.2.13)

yielding the $(P_\pi, \mathcal{O}_t)$-compensator for L as,

$$A_t^L = (\hat{\lambda}_t^1 \pi_{t-} + \hat{\lambda}_t^0(1 - \pi_{t-})) \log \frac{\hat{\lambda}_t^1}{\hat{\lambda}_t^0} - (\hat{\lambda}_t^1 - \hat{\lambda}_t^0).$$

(4.2.14)

As in section 4.1, the assumptions (A1) and (A2) must hold in order to apply Theorem 1.1.1 and thus ensure that L will escape I_0 in finite time P_π-a.s. In view of 4.1.5, the assumptions (A1) and (A2) can be recast in terms of the measures P_i, $i = 0, 1$, equivalently as,

$$(A1)'' : \qquad P_i\{\int_0^\infty \varphi(\hat{\lambda}_s^{1-i}, \hat{\lambda}_s^i) ds = \infty\} = 1 \quad i = 0,1;$$

$$(A2)'' : \qquad P_i\{\int_0^t \varphi(\hat{\lambda}_s^{1-i}, \hat{\lambda}_s^i) ds = \infty\} = 0 \quad i = 0,1, \ t < \infty.$$

where,

$$\varphi(\lambda^1, \lambda^0) = \lambda^1 \log \frac{\lambda^1}{\lambda^0} - (\lambda^1 - \lambda^0). \qquad (4.2.15)$$

Notice, for $\lambda^0, \ \lambda^1 > 0$,

$$\begin{aligned}
\varphi(\lambda^1, \lambda^0) &= \lambda^1 \log \frac{\lambda^1}{\lambda^0} - (\lambda^1 - \lambda^0) \\
&\geq \lambda^1 \log \frac{\lambda^1}{\lambda^0} + \lambda^1 \log \frac{\lambda^1}{\lambda^0} = 0,
\end{aligned} \qquad (4.2.16)$$

and similarly,

$$\begin{aligned}
\varphi(\lambda^0, \lambda^1) &= -\lambda^0 \log \frac{\lambda^1}{\lambda^0} - (\lambda^0 - \lambda^1) \\
&\geq -(\lambda^1 - \lambda^0) + (\lambda^1 - \lambda^0) = 0.
\end{aligned} \qquad (4.2.17)$$

The third condition of Theorem 1.1.1 is an easy consequence of 4.2.16 and 4.2.17 using the same steps as in 4.1.18. Thus by Theorem 1.1.1, L is guaranteed to escape I_0 in finite time P_π-a.s., and since $\pi_t = \Phi(L_t)$, where Φ is bijective, then the same holds true for the *a posteriori* probability. A difficulty however, is that the π-boundary of I_0 is not simply its endpoints as in the diffusion case, and worse, it is stochastic unless u_t is deterministic for all $t \geq 0$. Therefore, although conditions have been given under which $\{\pi_t : t \geq 0\}$ can be guaranteed to escape intervals, the π-boundary of the process is difficult to work with in the general case.

Avoiding this difficulty for the moment, it is desired to compute the characteristic operator for $\{\pi_t : t \geq 0\}$ on $\hat{\mathcal{D}}(U)$. So, assuming $r \in \hat{\mathcal{D}}(U)$, an application of the

generalized Itô rule gives,

$$
r(\pi_t) - r(\pi_0) =
$$
$$
\int_0^t (\hat{\lambda}_s^1 - \hat{\lambda}_s^0) \Big[-\pi_s (1 - \pi_s) r'(\pi_s) + (u_s + \pi_s) \big[r\big(\frac{u_s + 1}{u_s + \pi_s} \pi_s \big) - r(\pi_s) \big] \Big] ds
$$
$$
+ \int_0^t \big[r\big(\frac{u_{s-} + 1}{u_{s-} + \pi_{s-}} \pi_{s-} \big) - r(\pi_{s-}) \big] d\bar{m}_t,
$$

$$(4.2.18)$$

using 4.2.11 and 4.2.12.

Now consider the case where $\lambda_t^1 = \lambda^1$, $\lambda_t^0 = \lambda^0$, $t \geq 0$, are deterministic constants. Assume that these constants satisfy $\lambda^1 > \lambda^0 > 0$, without loss of generality. Thus (A1)″ and (A2)″ hold. Also suppose that the cost-rate function is given as $c(\pi) = c(\lambda^1 - \lambda^0)$, a positive deterministic constant; thus $E_\pi[\int_0^\infty c(\pi_s) ds] = \infty$ and therefore condition (T) holds. The terminal cost is $e(\pi)$, as usual.

In view of 4.2.9 with

$$
u_t = u = \frac{\lambda^0}{\lambda^1 - \lambda^0},
$$

it is clear that $\partial_\pi^- I_0 = \{a\}$, i.e., $\{\pi_t : t \geq 0\}$ will exit I_0 continuously on the right. However, since $\{\pi_t : t \geq 0\}$ will exit I_0 on the left only by jumping, then it follows that $\partial_\pi^+ I_0 = [b, \frac{u+1}{u+b} b)$. To see this, consider that whenever $\pi_t \in [b - \frac{b(1-b)}{u+b}, b)$, it has the potential of getting into $\partial_\pi^+ I_0$ because its jump size is $\frac{\pi_{t-}(1-\pi_{t-})}{u+\pi_{t-}}$ at that time. Letting $\Sigma \pi_{t-}$ denote the new state arrived at by jumping from π_{t-}, i.e.,

$$
\Sigma \pi_{t-} = \pi_{t-} + \frac{\pi_{t-}(1 - \pi_{t-})}{u + \pi_{t-}} = \frac{u + 1}{u + \pi_{t-}} \pi_{t-},
$$

yields,

$$
\partial_\pi I_0 = \{a, [b, \Sigma b)\},
$$

as the π-boundary of I_0, for I_0 as defined above.

From 4.2.18, the characteristic operator of $\{\pi_t : t \geq 0\}$ is found to be,

$$
Ur(\pi) = (\lambda^1 - \lambda^0) \big[-\pi(1-\pi) r'(\pi) + (u+\pi) \big[r\big(\frac{u+1}{u+\pi} \pi \big) - r(\pi) \big] \big] \quad \forall \pi \in (0,1). \quad (4.2.19)
$$

The application of the results in sections 3.1 and 3.2 begins by setting up the problem specified by conditions (C1) and (C2), which is:

$$\begin{cases} Ur(\pi) = -c(\lambda^1 - \lambda^0) & \forall \pi \in (0,1); \\ r(\pi) = e(\pi) & \pi \in \{a, [b, \Sigma b)\}, \end{cases}$$

choosing a particular form for the cost rate constant without loss of generality (see diffusion case in section 4.1). Substituting 4.2.19 into the above and dividing by $(\lambda^1 - \lambda^0) > 0$ yields

$$-\pi(1-\pi)r'(\pi) + (u+\pi)\left[r(\frac{u+1}{u+\pi}\pi) - r(\pi)\right] = -c \quad \forall \pi \in (0,1)$$

$$r(\pi) = e(\pi) \quad \pi \in \{a, [b, \Sigma b)\}, \tag{4.2.20}$$

which is a family of functional advance differential equations, indexed by the boundary points $\{a, b\}$. Motivated by Theorem 3.2.1, it is required that a solution to 4.2.20 be continuous for all $\pi \in [a, \Sigma b)$. Letting $\tilde{U}^\pi$ represent the operator in the equivalent problem, it was shown in Theorem 1.1.2 that for every $0 < a < b < 1$, that there exists a unique solution to,

$$\tilde{U}^\pi R(\pi) = -c \quad \forall \pi \in (0,b)$$

$$R(\pi) = e(\pi) \quad \forall \pi \in \{a, [b, \Sigma b)\}$$

with $R(\pi)$ continuous for all $\pi \in (0,b)$. Also, in the theorem to follow it is shown that there exists a right–continuous function, $d(\pi)$, satisfying,

$$\tilde{U}^\pi d(\pi) = -c \quad \forall \pi \in (0,1)$$

$$d(\pi) = 0 \quad \forall \pi \in [b, \Sigma b)$$

$$d(\pi) < 0 \quad \forall \pi \geq \Sigma b,$$

for any $b \in (0,1)$. So, take $D(\pi) = d(\pi) + e(\pi)$, $\pi \in [b,1)$, and construct,

$$r(\pi) = R(\pi)\,1_{\{\pi < b\}} + D(\pi)\,1_{\{\pi \geq b\}}.$$

This gives a family of right–continuous functions, $\Re = \{r(\cdot\,; a, b) : 0 < a < b < 1\}$ satisfying,

$$\tilde{U}^\pi r(\pi) = -c \quad \forall \pi \in (0,b)$$

$$r(\pi) = e(\pi) \quad \forall \pi \in \{a, [b, \Sigma b)\},$$

which are strictly continuous for all $\pi < b$, and all of which furnish the conditions (C1) and (C2). In addition, by the corollary to Theorem 1.2.1, every $r \in \Re$ also enjoys, $r \in C^{2+}(E)$, with $E = (0, b)$, and as a result, by the corollary to Theorem 3.2.2 it follows that every $r \in \Re$ is concave on E. Recall also from the corollary to Theorem 1.2.1 that, $r'(\pi)$ is continuous for all $\pi \in (0, b)$, with the possible exception of $\pi = \Sigma^{-1}b$. This implies,

$$r'(\pi^-) \geq r'(\pi) \qquad \pi = \Sigma^{-1}b, \ \forall r \in \Re, \tag{4.2.21}$$

since every $r \in \Re$ is concave. Solving for the derivative in 4.2.20,

$$r'(\pi) = \frac{c + (u + \pi)[r(\frac{u+1}{u+\pi}\pi) - r(\pi)]}{\pi(1 - \pi)}, \tag{4.2.22}$$

and evaluating it at $B^{\pm}$, with $B = \Sigma^{-1}b$, yields,

$$r'(B^{\pm}) = \frac{c + (u + B)[r(b^{\pm}) - r(B)]}{B(1 - B)}, \tag{4.2.23}$$

with the immediate consequence,

$$r(b^-) \geq r(b), \qquad \forall r \in \Re \tag{4.2.24}$$

in view of (4.2.21). Now if any $r \in \Re$ satisfies 4.2.24 with strict inequality, then obviously such a member violates condition (C3). As a result, one is only interested in the subfamily,

$$\Re_1 = \{r \in \Re : r(b^-; a, b) = e(b)\}. \tag{4.2.25}$$

Again, the corollary to Theorem 1.2.1 has something to say. If $r \in \Re_1$, then $r(\pi)$ is twice continuous differentiable at $\pi = a$ (unless $b = \frac{u+1}{u+a}a$) and therefore, in order to satisfy condition (C3), it is necessary that $r(\pi)$ satisfy,

$$r'(a) = e'(a) \qquad r \in \Re_1. \tag{4.2.26}$$

Thus, if there exists $r \in \Re_2 = \{r \in \Re_1 : r'(a; a, b) = e'(a)\}$, then r satisfies the conditions (C1), (C2), and (C3), and in addition, is concave on $[(a, b)]_\pi$. In this case, $r(\pi)$ is the subrisk being sought.

An interesting subtlety has arisen here however. Contrasted with the subrisk problem in the diffusion case, it appears that the problem here is 'overdetermined'. Consider the diffusion case. There, the two free constants of the *second order differential equation* were chosen to satisfy the boundary condition (C2). Then, the boundary points '*a*' and '*b*' were chosen as the solution to a system of two equations in two unknowns whose unique solution was guaranteed to match the derivations at the boundary, and thence yield condition (C3). Here, the single free constant of the *first order differential equation* is chosen to satisfy the boundary condition at $\pi = a$, and this leaves only '*a*' and '*b*' with which to satisfy both the boundary condition at $\pi = b$ and the condition on both derivatives to obtain condition (C3). The pathway out of this apparent difficulty is provided by the following lemma.

Lemma 4.2.1 *Let $r \in \Re$. Then $r'(b^-) > e'(b)$.*

Proof:

From 4.2.22,

$$
\begin{aligned}
r'(b^-) - \frac{c}{b(1-b)} &= \frac{u+b}{b(1-b)} e\left(\frac{u+1}{u+b}b\right) - \frac{u+b}{b(1-b)} r(b^-) \\
&= \frac{u\,e(b)}{b(1-b)} - \frac{u+b}{b(1-b)} r(b^-) \\
&= -\frac{b\,e(b)}{b(1-b)} + \frac{u+b}{b(1-b)}[e(b) - r(b^-)] \qquad (4.2.27) \\
&= -\frac{e(b)}{1-b} + \frac{u+b}{b(1-b)}[r(b) - r(b^-)] \\
&= e'(b) + \frac{u+b}{b(1-b)}[r(b) - r(b^-)].
\end{aligned}
$$

Since $r \in \Re$, it is clear from the last line and 4.2.24, that

$$
r'(b^-) \geq \frac{c}{b(1-b)} + e'(b) > e'(b) \qquad (4.2.28)
$$

since $c > 0$ is assumed. ∎

Corollary: If $r \in \Re_1$, then,

$$r'(b^-) = \frac{c}{b(1-b)} + e'(b). \tag{4.2.29}$$

Proof: Immediate from 4.2.27. ∎

Hence, if there exists $r \in \Re_2$, then it indeed satisfies the conditions (C1-3). In summary then, the problem to solve in the search for the subrisk is to prove the existence of *any* pair a_*, b_* satisfying,

$$\begin{aligned} r(b_*^-; a_*, b_*) &= e(b_*) \\ r'(a_*; a_*, b_*) &= e'(a_*) \end{aligned} \tag{4.2.30}$$

where $r \in \Re$. Recall that the uniqueness of the pair will follow from its existence in view of Theorem 3.1.4. Be aware that 4.2.30 is a well–posed problem because it has been shown that $\Re$ is not empty, i.e., for any $0 < a < b < 1$, there exists a solution to 4.2.20. Finding an a_*, b_* to satisfy 4.2.30 is somewhat involved, and so is given in Appendix II. Based upon their existence, the following theorem has been shown.

Theorem 4.2.1

In the problem of sequential hypothesis testing, based on observations of the homogeneous Poisson process,

$$n_t = (\theta \lambda^1 + 1 - \theta \lambda^0)\, t + m_t \qquad \lambda^1 > \lambda^0, \; t \geq 0,$$

with running cost,

$$E_\pi\Big[\int_0^\tau c(\pi_s)ds\Big] = E_\pi[\, c(\lambda^1 - \lambda^0)\tau],$$

there exists a Bayesian rule, $u^* = (\tau^*, \delta^*)$, which is a threshold policy given by,

$$\tau^* = \inf\{\, t \geq 0 : \pi_t \notin (a_*, b_*)\,\}$$

$$\delta_* = \begin{cases} 1 & \text{if } \pi_\tau \geq b_* \\ 0 & \text{if } \pi_\tau \leq a_*. \end{cases}$$

The Bayes risk is given by,

$$\rho(\pi) = \begin{cases} e(\pi) & \pi \notin (a_*, b_*) \\ r_*(\pi) & \pi \in (a_*, b_*), \end{cases}$$

and the subrisk, $r_*(\pi) = r(\pi; , a_*, b_*)$ is given by,

$$r(\pi; a_*, b_*) = \begin{cases} R(\pi; , a_*, b_*) & \pi < b_* \\ D(\pi; b_*) & \pi \geq b_* \end{cases}$$

with,

$$D(\pi; b) = e(\pi) + d(\pi; b)$$

$$d(\pi; b) = C(\lambda^1(1 - \pi) + \lambda^0\pi)(1 + N_b(\pi))$$

where,

$$x(\pi) = \log\left[\frac{\pi}{1 - \pi}\right], \quad \tilde{x} = \frac{x}{\log\frac{\lambda^1}{\lambda^0}}, \quad C = c\,\frac{\lambda^1 - \lambda^0}{\lambda^0\lambda^1},$$

and $N_b(\pi)$ is the integer such that $\tilde{x}(b) - \tilde{x}(\pi) - 1 \leq N_b(\pi) < \tilde{x}(b) - \tilde{x}(\pi)$. The definition of the subrisk is completed by specifying,

$$R(\pi; a, b) = d(\pi; b) + \bar{M}(\pi; b) + K(a, b)\,\bar{S}(\pi; b);$$

$$\bar{M}(\pi; b) = \lambda^1(1 - \pi)\,M_0(\pi; b) + \lambda^0\pi\,M_1(\pi; b);$$

$$\bar{S}(\pi; b) = \lambda^1(1 - \pi)\,S_0(\pi; b) + \lambda^0\pi\,S_1(\pi; b),$$

with M_i, S_i, $i = 0, 1$ defined by,

$$S_i(\pi; b) = e^{-\nu_i\tilde{x}(\pi)}\sum_{n=0}^{N_b(\pi)}\frac{(-1)^n}{n!}[(\tilde{x}(b) - \tilde{x}(\pi) - n)\nu_i e^{-\nu_i}]^n;$$

$$M_i(\pi; b) = -Ce^{\nu_i(\tilde{x}(b) - \tilde{x}(\pi) - 1)}\sum_{n=0}^{N_b(\pi)-1}e^{-\nu_i n}\sum_{m=0}^{n}\frac{(-1)^m}{m!}[(\tilde{x}(b) - \tilde{x}(\pi) - n - 1)\nu_i]^m;$$

where,

$$\nu_i = \frac{\lambda^i\log\frac{\lambda^1}{\lambda^0}}{\lambda^1 - \lambda^0}; \qquad i = 0, 1.$$

The constant $K(a, b)$ is defined via,

$$K(a, b) = \frac{e(a) - d(a; b) - \bar{M}(a; b)}{\bar{S}(a; b)}.$$

As a final note, be aware that the 'empty sum equals zero', and '$0^0 = 1$' conventions are used.

Proof: Except for the explicit form given for the subrisk, the proof is contained in the preceding discussion. As for the subrisk, in view of Theorem 1.2.1 it is only necessary to show that it satisfies 4.2.20 and is continuous for all $\pi < b$. The semicolons are dropped here for notational convenience. First note that $r(a) = R(a) = e(a)$, due to the definition of $K(a, b)$. Next note that,

$$d(\pi) = C(\frac{\lambda^1 - \lambda^0}{\lambda^1 \lambda^0}) (\lambda^1 - (\lambda^1 - \lambda^0)\pi) (1 - 1) = 0, \quad \forall \pi \in [b, \frac{u+1}{u+b}b)$$

and so, $D(\pi) = e(\pi)$ on this set. Hence, the boundary conditions of 4.2.20 have been satisfied. Also observe that,

$$-\pi(1 - \pi) \, d'(\pi) + (u + \pi)[d(\frac{u+1}{u+\pi} \pi) - d(\pi)] = -c \qquad \forall \pi \in (0, 1)$$

where the derivative is taken from the right, as usual. Next, one can show, albeit tedious, that,

$$U^\pi \bar{M} \equiv U^\pi \bar{S} \equiv 0. \tag{4.2.31}$$

Thus far, have argued that the subrisk satisfies 4.2.20. It only remains to demonstrate the continuity on $\pi < b$. Clearly, $R(\pi)$ is continuous for all π except possibly at those $\pi < b$ where,

$$N_b(\pi) - N_b(\pi^-) = -1. \tag{4.2.32}$$

It is not too difficult to see that $S_i(\pi)$, $i = 0, 1$, is continuous at these points. On the other hand, $d(\pi)$ is clearly discontinuous at these points. A simple calculation shows that,

$$d(\pi) - d(\pi^-) = -C(\lambda^1(1 - \pi) + \lambda^0 \pi),$$

at a discontinuity. On the other hand at the same discontinuity it follows that,

$$M_i(\pi) - M_i(\pi^-) = C \qquad i = 0, 1.$$

From this it is easy to see that $\bar{M}(\pi)$ is also discontinuous, but of the same magnitude and opposite sign as $d(\pi)$. Therefore, the sum of the two is continuous. The theorem has been shown. ∎

Given the form of the risk in the theorem, it is not hard to see why explicit results are few and far between in the sequential analysis of jump-type processes. Similar results as above have been obtained for the well-known Wald problem in dealing with Poisson processes [DVORETZKY &c].

Unlike the diffusion formulation in section 4.1, there is no easy extension of the time-homogeneous result to the more general case. An obvious reason is that in the diffusion case the drift factors out of $d(\pi)$, whereas in the present case, the counting rates (nor even u) do not factor out. As a result, in general when the rates are stochastic, the state space of the process must be enlarged to include the rate. This therefore leads to a partial functional differential equation for the risk in the two variables, π, and u (see 4.2.7). In addition, since the π-boundary is also stochastic in this case, one is forced to consider the (π, u)-boundary of the (π_t, u_t) process. This has the immediate consequence that in general, one must extend the notion of threshold policies from intervals to include open sets. The entire task is clearly nontrivial.

This concludes the section and chapter.

APPENDIX I

Suppose one is given a measurable space $(\Omega, \mathcal{F})$ upon which two distinguished probability measures P_i, $i = 0, 1$, are defined. Also suppose one is given a right-continuous subfiltration $\{ \mathcal{O}_t : t \geq 0 \}$ for the natural filtration $\mathcal{F}_t \uparrow \mathcal{F}_\infty = \mathcal{F}$. Next define the family of probability measures, $\{ P_\pi : 0 \leq \pi \leq 1 \}$, via,

$$P_\pi\{A\} = \pi P_1\{A\} + (1 - \pi)P_0\{A\} \qquad \forall A \in \mathcal{F},\ \pi \in [0, 1].$$

Finally, suppose there is an $\mathcal{F}$-measurable random variable $\theta : \Omega \longrightarrow \{0, 1\}$ such that,

$$P_\pi\{\theta = 1\} = \pi \quad , \quad P_\pi\{\theta = 0\} = 1 - \pi, \qquad (A.1.1)$$

i.e., $\theta = 1$ with prior probability π and vice versa.

Given the above set-up, consider first the problem of expressing $E_\pi[\gamma | \mathcal{O}_t]$ in terms of P_i-conditional expectations, $i = 0, 1$, where γ is some $\mathcal{F}$-measurable random variable and $\pi \in [0, 1]$. Now if $\pi = 0, 1$, the problem is trivial, so suppose $\pi \in (0, 1)$. Noticing that $P_i \ll P_\pi$ for $i = 0, 1$, $\pi \in (0, 1)$, it follows that one may define the Radon-Nikodym derivatives,

$$L^i(\omega) = \frac{dP_i}{dP_\pi}(\omega) \qquad \omega \in \Omega,\ i = 0, 1,\ \pi \in (0, 1),$$

and then there holds,

$$E_\pi[L^i | \mathcal{O}_t] = \frac{dP_i^t}{dP_\pi^t} = L_t^i \qquad i = 0, 1, \qquad (A.1.2)$$

where P_i^t, $i = 0, 1$, P_π^t are the $\mathcal{O}_t$-restrictions of the respective measures. Given this set-up, one can prove the following theorem.

Theorem A.1.1 *Let $\pi \in [0, 1]$. Then,*

$$E_\pi[\gamma | \mathcal{O}_t] = \pi L_t^1 E_1[\gamma | \mathcal{O}_t] + (1 - \pi)L_t^0 E_0[\gamma | \mathcal{O}_t] \quad P_\pi\text{-a.s.}.$$

Proof: Let $A \in \mathcal{O}_t$ and compute,

$$\int_A E_\pi[\gamma|\mathcal{O}_t]\, dP_\pi = \int_A \gamma\, dP_\pi$$

$$= \int_A \gamma \frac{dP_\pi}{dP_\pi}\, dP_\pi \qquad (A.1.3)$$

$$= \int_A \gamma \frac{\pi dP_1 + (1-\pi)dP_0}{dP_\pi}\, dP_\pi,$$

using the Radon-Nikodym theorem. Continuing from the last line,

$$\int_A E_\pi[\gamma|\mathcal{O}_t]\, dP_\pi = \pi \int_A \gamma\, dP_1 + (1-\pi)\int_A \gamma\, dP_0$$

$$= \pi \int_A E_1[\gamma|\mathcal{O}_t]\, dP_1 + (1-\pi)\int_A E_0[\gamma|\mathcal{O}_t]\, dP_0. \qquad (A.1.4)$$

Now for $i = 0, 1$ compute,

$$\int_A E_i[\gamma|\mathcal{O}_t]\, dP_\pi = \int_A E_i[\gamma|\mathcal{O}_t]\, dP_\pi^t = \int_A E_i[\gamma|\mathcal{O}_t]\frac{dP_i^t}{dP_\pi^t}\, dP_\pi^t. \qquad (A.1.5)$$

Combining A.1.2–5 yields,

$$\int_A E_\pi[\gamma|\mathcal{O}_t]\, dP_\pi = \pi \int_A E_1[\gamma|\mathcal{O}_t]L_t^1\, dP_\pi + (1-\pi)\int_A E_0[\gamma|\mathcal{O}_t]L_t^0\, dP_\pi,$$

and hence,

$$\int_A E_\pi[\gamma|\mathcal{O}_t]\, dP_\pi = \int_A \{\pi L_t^1 E_1[\gamma|\mathcal{O}_t] + (1-\pi)L_t^0 E_0[\gamma|\mathcal{O}_t]\}\, dP_\pi.$$

Since $A \in \mathcal{O}_t$ is arbitrary, it follows that,

$$E_\pi[\gamma|\mathcal{O}_t] = \pi L_t^1 E[\gamma|\mathcal{O}_t] + (1-\pi)L_t^0 E_0[\gamma|\mathcal{O}_t], \qquad (A.1.6)$$

except on $\mathcal{O}_t$ sets of P_π-measure zero. ∎

Now suppose that one takes $\gamma = \theta\gamma'$, θ the binary random variable defined in the beginning, and γ' any $\mathcal{F}$-measurable random variable. Then note,

$$\int_A E_i[\theta\gamma'|\mathcal{O}_t]\, dP_i = \int_A \theta\gamma'\, dP_i = \int_{A\cap\{\theta=1\}} \gamma'\, dP_i \quad i = 0, 1,$$

and hence,

$$\int_A E_i[\theta\gamma'|\mathcal{O}_t]\,dP_i = \begin{cases} 0 & i = 0 \\ \int_A \gamma'\,dP_i & i = 1. \end{cases}$$

Thus, for this choice of γ A.1.6 becomes,

$$E_\pi[\theta\gamma'|\mathcal{O}_t] = \pi L_t^1 E_1[\gamma'|\mathcal{O}_t]. \qquad (A.1.7)$$

In the remainder of this appendix, A.1.7 will be used to connect the *a posteriori* probability process to the likelihood ratio, where the *a posteriori* probability is defined as,

$$\pi_t = P_\pi\{\theta = 1|\mathcal{O}_t\} \qquad \pi \in [0,1], \qquad (A.1.8)$$

and the likelihood ratio is given by,

$$\Lambda_t(\omega) = \frac{dP_1^t}{dP_0^t}(\omega) = E_0\left[\frac{dP_1}{dP_0}\Big|\mathcal{O}_t\right] \quad \forall \omega \in \Omega, P_0 - a.s., \qquad (A.1.9)$$

assuming of course that $P_1 \ll P_0$. Letting $\gamma' \equiv 1$ in A.1.7 one immediately obtains,

$$\pi_t = P_\pi\{\theta = 1|\mathcal{O}_t\} = E_\pi[\theta|\mathcal{O}_t] = \pi\frac{dP_1^t}{dP_\pi^t} \quad \pi \in (0,1). \qquad (A.1.10)$$

Using the Radon-Nikodym Theorem successively and observing that $P_1^t \ll P_0^t$, one obtains the following sequence of steps,

$$\begin{aligned} \pi_t = \pi\frac{dP_1^t}{\pi dP_1^t + (1-\pi)dP_0^t} &= \pi\frac{\frac{dP_1^t}{dP_0^t}}{1 - \pi + \pi\frac{dP_1^t}{dP_0^t}} \\ &= \frac{\frac{\pi}{1-\pi}\Lambda_t}{1 + \frac{\pi}{1-\pi}\Lambda_t} \\ &= \phi(\Lambda_t), \end{aligned} \qquad (A.1.11)$$

where the last line serves to define the mapping $\phi : [0,\infty] \longrightarrow [0,1]$. From A.1.11 one obtains,

$$\Lambda_t = \phi^{-1}(\pi_t) = \frac{1-\pi}{\pi}\frac{1-\pi_t}{\pi_t}. \qquad (A.1.12)$$

Finally, for $\pi = 1$ note that

$$\pi_t = E_1[\theta|\mathcal{O}_t] = \pi = 1,$$

while for $\pi = 0$,

$$\pi_t = E_0[\theta|\mathcal{O}_t] = 0 \cdot \Lambda_t = 0.$$

APPENDIX II

In this appendix, it is shown that there exists a unique solution (a_*, b_*) to the system of equations,

$$r(b^-; a, b) = e(b); \qquad\qquad (A.2.1.a)$$

$$r'(a; a, b) = e'(a), \qquad\qquad (A.2.1.b)$$

with $r(\pi; a, b)$ and $e(\pi)$ as given in Theorem 4.2.1 (see also 4.2.30). First suppose that a solution (a_*, b_*) to A.3.1 exists, then from Theorem 1.2.1 and its corollary it follows that $r_*(\pi) = r(\pi; a_*, b_*) \in C^2(0, b_*)$. From this and the corollary to Theorem 3.2.1 it follows that r_* is concave on $(0, b_*)$. This implies, in view of Theorem 3.2.2, Lemma 4.2.1, and A.2.1 that the conditions (C1), (C2), and (C3) hold. Finally, invoking Theorem 3.1.4 shows that the solution is unique. Thus, it is only necessary to show that at least one solution to A.2.1 exists.

The approach to accomplish this is to show that for every $a \in [0, 1]$ there exists a $b_1(a) \in [0, 1]$ such that A.2.1.a becomes,

$$r(b_1^-(a); a, b_1(a)) = e(b_1(a)). \qquad\qquad (A.2.2)$$

In addition, it will be shown that the mapping $a \mapsto b_1(a)$ is continuous. Following that it will be demonstrated using A.2.1.b that there exists a continuous mapping, $b_2(a)$, satisfying,

$$r'(a; a, b_2(a)) = e'(a), \qquad\qquad (A.2.3)$$

with $b_2(0) = 1$, and such that the equation $b_2(a) = a$ has a solution. Consequently, the two curves $b_1(a), b_2(a)$ must cross, i.e., there exists an a_0 satisfying $b_1(a_0) = b_2(a_0)$. Choosing $b_* = b_1(a_0)$, and $a_* = a_0$ then yields a pair satisfying A.2.1.

To begin, from Theorem 4.2.1 there follows,

$$r(\pi; a, b) = d(\pi; b) + \bar{M}(\pi; b) + K(a, b)\bar{S}(\pi; b) \quad \forall \pi < b. \qquad\qquad (A.2.4)$$

Letting $\pi \uparrow b$ and noting $N_b(b^-) = 0$, after some algebra one obtains,

$$r(b^-; a, b) = C(\lambda^1(1 - b) + \lambda^0 b) + K(a, b)\left(\frac{1-b}{b}\right)^u (1 - b)(\lambda^0 + \lambda^1), \qquad (A.2.5)$$

where by definition $K(a, b)$ is

$$K(a, b) = \frac{e(a) - [d(a; b) + \bar{M}(a; b)]}{\bar{S}(a; b)}. \qquad (A.2.6)$$

In the proof of Theorem 4.2.1, it was shown that $\bar{S}(\pi; b)$ and $[d(\pi; b) + \bar{M}(\pi; b)]$ are continuous for all $\pi < b$. Thus $K(a, b)$ is at least continuous in its first argument and therefore it follows that the mapping $a \mapsto b$ defined implicitly by A.2.2 and A.2.5 is continuous. Now, for any $a \in [0, 1]$, suppose $b = b_1(a)$ is chosen so that $a \in [\Sigma^{-1}b, b)$. Then from A.2.6 there follows,

$$K(a, b) = \frac{e(a) - C(\lambda^1(1 - a) + \lambda^0 a)}{(1 - a)(\lambda^0 + \lambda^1)}\left(\frac{a}{1 - a}\right)^u. \qquad (A.2.7)$$

After some simplification, using A.2.5-7 one obtains,

$$r(b^-; a, b) = C(\lambda^1(1 - b) + \lambda^0 b) + [e(a) - C(\lambda^1(1 - a) + \lambda^0 a)]\left(\frac{1-b}{1-a}\right)\left(\frac{a(1-b)}{(1-a)b}\right)^u. \qquad (A.2.8)$$

From A.2.8 it is clear then that two solutions to A.2.2 are given by $b_1(0) = 0$, and $b_1(1) = 1$. Thus, it has been shown that a continuous curve connects the points $(0, 0)$ and $(1, 1)$ in the (b, a)-plane.

The second half of the argument deals with A.2.1.b. First, suppose that for any $a \in [0, 1]$, the inequality $a < \Sigma^{-1}b$ is respected by $b = b_2(a)$. Then from A.2.4, one obtains the (right) derivative,

$$r'(\pi; a, b) = d'(\pi; b) + \bar{M}'(\pi; b) + K(a, b)\bar{S}'(\pi; b) \quad \pi < \Sigma^{-1}b. \qquad (A.2.9)$$

Recall from Theorem 4.2.1 that, $U\bar{M}(\pi; b) \equiv U\bar{S}(\pi; b) \equiv 0$, and so,

$$r'(\pi; a, b) = d'(\pi; b) + \frac{u + \pi}{\pi(1 - \pi)}\left[\bar{M}(\Sigma\pi; b) - \bar{M}(\pi; b) + K(a, b)\left(\bar{S}(\Sigma\pi; b) - \bar{S}(\pi; b)\right)\right]. \qquad (A.2.10)$$

Substituting $\pi = a$ into A.2.10 and the result into A.2.1.b yields,

$$\bar{M}(\Sigma a; b) - \bar{M}(a; b) + K(a, b)\Big(\bar{S}(\Sigma a; b) - \bar{S}(a; b)\Big) = \frac{a(1-a)}{u+a}\Big(e'(a) - d'(a; b)\Big).$$

$$(A.2.11)$$

The last line can be simplified in two ways. First, a direct computation of the right derivative $d'(\pi; b)$ yields,

$$d'(\pi; b) = -C(\lambda^1 - \lambda^0)(1 + N_b(\pi)).$$

$$(A.2.12)$$

Also, from A.2.2 and A.2.5 there holds,

$$K(a, b) = \frac{e(b) - C\big(\lambda^1(1-b) + \lambda^0 b\big)}{(1-b)(\lambda^0 + \lambda^1)}\left(\frac{b}{1-b}\right)^u.$$

$$(A.2.13)$$

Combining A.2.11-13 with a little algebra yields an equation whose left-hand side is,

$$(1-b)^{u+1}\big[\bar{M}(\Sigma a; b) - \bar{M}(a; b)\big] + \frac{e(b) - C\big(\lambda^1(1-b) + \lambda^0 b\big)}{\lambda^0 + \lambda^1}b^u\big[\bar{S}(\Sigma a; b) - \bar{S}(a; b)\big],$$

$$(A.2.14)$$

and whose right-hand side is,

$$\frac{a(1-a)}{u+a}e'(a)(1-b)^{u+1} + \frac{a(1-a)}{u+a}C(\lambda^1 - \lambda^0)(1-b)^{u+1}(1 + N_b(a)).$$

$$(A.2.15)$$

Now notice that $(a, b) = (0, 1)$ solves both sides (recall $\Sigma 0 = 0$). The only troublesome term is in A.3.15, $a(1-b)^{u+1}N_b(a)$, which goes to zero since,

$$\begin{aligned} 0 \le a(1-b)^{u+1}N_b(a) &< a(1-b)^{u+1}\big(\tilde{x}(b) - \tilde{x}(a)\big) \\ &= \tilde{a}(1-b)^{u+1}\log\left(\frac{b(1-a)}{(1-b)a}\right) \\ &\le \tilde{a}(1-b)^{u+1}\left(\frac{b(1-a)}{(1-b)a} - 1\right) \\ &= \tilde{b}(1-a)(1-b)^u - \tilde{a}(1-b)^{u+1} \xrightarrow{a\downarrow 0, b\uparrow 1} 0. \end{aligned}$$

$$(A.2.16)$$

Thus far have shown $b_2(0) = 1$, and from A.2.10 it is clear that $a \mapsto b_2(a)$ is continuous, arguing as before. The last thing to show is that there exists an a_0 satisfying $b_2(a_0) = a_0$, i.e., that the curve $b_2(a)$ intercepts the line $b = a$ in the (b, a)-plane. For

if this is shown, the two curves $b_1(a)$ and $b_2(a)$ must cross, yielding a solution to the problem. To show $b_2(a_0) = a_0$ for some a_0, consider A.2.4 for all b in the vicinity of a. In fact, for any $a \in [0,1]$, choose $b = b_2(a)$ to satisfy $a \in [\Sigma^{-1}b, b)$, then from A.2.4

$$r(\pi; a, b) = d(\pi; b) + K(a,b)\bar{S}(\pi; b)$$
$$= C\big(\lambda^1(1-\pi) + \lambda^0\pi\big) + K(a,b)\left(\frac{1-\pi}{\pi}\right)^u (1-\pi)(\lambda^0 + \lambda^1), \qquad (A.2.17)$$

and so,

$$r'(\pi; a, b) = -C(\lambda^1 - \lambda^0) - K(a,b)\left(\frac{1-\pi}{\pi}\right)^u \left(\frac{u+\pi}{\pi}\right)(\lambda^0 + \lambda^1). \qquad (A.2.18)$$

Substituting for $K(a,b)$ from A.2.13 under the condition $a \in [\Sigma^{-1}b, b)$ yields,

$$r'(\pi; a, b) = -C(\lambda^1 - \lambda^0) - \frac{e(b) - C\big(\lambda^1(1-b) + \lambda^0 b\big)}{1-b}\left(\frac{b(1-\pi)}{(1-b)\pi}\right)^u \left(\frac{u+\pi}{\pi}\right). \qquad (A.2.19)$$

Thus, setting $r'(a; a, b) = e'(a)$ and using A.2.19 one gets,

$$-a(1-b)C(\lambda^1 - \lambda^0) - (u+a)\Big[e(b) - C\big(\lambda^1(1-b) + \lambda^0 b\big)\Big]\left(\frac{b(1-a)}{(1-b)a}\right)^u = a(1-b)e'(a). \qquad (A.2.20)$$

Evaluating A.2.20 on the line $b = a$ gives,

$$-a(1-a)C(\lambda^1 - \lambda^0) - (u+a)\Big[e(a) - C\big(\lambda^1(1-a) + \lambda^0 a\big)\Big] = a(1-a)e'(a). \quad (A.2.21)$$

If one assumes that $e(\pi)$ is symmetric and normalized, which can be done without loss of generality according the results of section 2.3, then after some algebraic simplification A.2.21 can be rewritten,

$$a(u+1) - c = 0, \qquad (A.2.22)$$

i.e., $a = \frac{c(\lambda^1 - \lambda^0)}{\lambda^1}$. So if $c < u + 1$, then a solution to A.2.1.b exists on the line $a = b$. In this case, there exists a solution (a_*, b_*) solving A.2.1, as argued previously. Notice that the condition $c < u + 1$ holds automatically for all $c \leq 1$. On the other hand, this condition can also always be obtained by the simple change of time scale $t \mapsto t/c$. This completes the argument.

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