DEDECTION OF GAUSSIAN PROCESSES OBSERVED
THROUGH MEMORYLESS NONLINEARITIES

by

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A Decision Directed Approach to the Target Discrimination Problem.

ABSTRACT

Title of Thesis: Decision Directed Approaches to the Target Discrimination Problem.

Ramesh Ragothama Rao, Master of Science 1982

Thesis directed by: A. Ephremides, Professor,
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A radar detection problem is studied, in which nature can be in one of three states, corresponding to the three 'targets', ships, chaff and sea clutter. A Neyman-Pearson type of ternary detector is developed, and the optimum decision rule is determined assuming that the conditional distribution functions of the observations are known. It is shown that if the a-priori description of the conditional distribution functions is incomplete, then problems in simultaneous estimation and detection arise.

Then in a simplified setting we develop two theorems on parameter estimation under uncertainty. They are valid under different conditions, but essentially in each of the two, we assume that there exist estimates which converge in quadratic mean to the unknown parameter, when the identity of the underlying target is known; then we propose new estimates (in terms of the old) which converge in q.m. to the unknown parameter, even when there is uncertainty about the identity of the underlying target.

ABSTRACT

Title of Thesis: Detection of Gaussian Processes Observed through Memoryless Nonlinearities

Georges P. Panayotopoulos, Master of Science, 1979

Thesis directed by: John S. Baras
Associate Professor
Electrical Engineering Department

This work deals with the detection of nonGaussian processes which can be modelled as the nonlinear outputs of linear dynamical Systems driven by White Gaussian noise. Our approach is based on the idea to find some auxiliary process θ_t and a nonlinear transformation $F(t,\theta_t,Y_y)=X_t$ so that X_t to be Gaussian.

To the memory of my Mother Antigoni

ACKNOWLEDGEMENTS

Much of the time it was a fascinating experience the exploration of the stochastic world. For this reason I owe thousands of thanks and my gratitude to my advisor Dr. J. Baras for his help and his understanding of my various problems. He helped me in more than one way, not only as a scientist but also he showed very much understanding as a man.

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INTRODUCTION

The present work was written as a thesis towards the Master's degree at the University of Maryland.

It deals with the nonlinear detection problem that is, the detection of nonGaussian processes which can be modelled as the nonlinear outputs of linear dynamical systems driven by White Gaussian Noise.

In chapter 1 we give an introduction to the signal Detection problem and we emphasize the so called Signal in Noise case.

In chapter 2 we deal with our problem. Specific 2.1 deals with the case when we have Rayleigh processes. 2.5 examines the problem for chi-square with four degrees of freedom and 2.6 the case for lognormal processes.

Our approach is based on the idea to find some auxiliary process θ_t and a nonlinear transformation $F(t,\theta_t,Y_t)=X_t$ so that X_t to be Gaussian.

In chapter 3 we search one way to approach the estimation problem. It is based on the fact that when we know the covariance of a nonlinear transformation of a Gaussian process, sometimes it is possible to recover the covariance of the Gaussian process.

CHAPTER 1 SIGNAL DETECTION

(1.1) DESCRIPTION OF THE PROBLEM

Two are the problems with which science deals. The first problem which would be named the "prediction" problem is the following:

Based upon some data which have to do with the past and present someone has to predict the future. This should be possible provided causality holds in nature.

The other problem, which would be called the "inverse" problem is to find the source of the given data.

I would say that the detection problem belongs to the category of the inverse problems.

Briefly, the detection problem could be described as follows:

We make observations Y_t in the time interval $[0,T]$. Based on these observations one has to decide and accept one of the following hypotheses

H_0 : The observation process is y_{0t}

H_1 : The observation process is y_{1t}

Of course the decision will be taken based upon some criterion of optimality. In general our purpose is to minimize some cost function in the long run.

Each time, we choose a cost function which depends upon the a priori information we have for the problem.

In accordance with the chosen cost function we have the following 3 basic decision-making strategies.

1. The Bayes criterion

2. The minimax criterion

3. The Neyman-Pearson criterion

The observation process Y_t is a random element in a Hilbert space H.

For instance if we make only one observation at only one time instant then we shall have a random element in $\mathbb{R}$ that is a random variable.

If we make n observations at the time instants $t_1, t_2, \ldots, t_m$ then Y_t is a random element in $\mathbb{R}^m$ that is an m-dimensional random vector.

Finally Y_t may be a random process that is a random element in $\mathbb{L}_2(I)$ where $I=[o,T]$.

In any case there will be a decision surface D in H which will divide the space into two regions R_0, R_1.

For a specific experiment Y_t will belong either to R_0 or to R_1 and we shall decide for H_0 or H_1 respectively.

The decision surface is determined from the decision strategy we use.

The above is equivalent to the following. In all three strategies one computes a quantity which is called the likelihood ratio (LR) and then compares the LR with a threshold value v. If the LR is greater than v we accept H_1 otherwise we accept H_0.

The strategy has to do with the computation of the threshold value v.

So the pure theoretical detection problem deals with the computation of the LR which in the measure theory foundation of probability is not anything else but the Radon-Nicodym derivative of P_1 with respect to P_0 that is $\frac{dP_1}{dP_0}$ where Pi is the probability measure induced by the observation process Y_t under hypothesis Hi.

Now we shall state briefly each criterion.

(1.2) BAYES CRITERION

In this case we know a priori the following information

First the a priori probability ξ to occur H_0. So $1-\xi$ is the probability H_1 to happen.

Second the so called cost matrix

$$C=\begin{bmatrix} C_{00} & C_{01} \\ C_{10} & C_{11} \end{bmatrix}$$

where C_{i_j} is the cost we have to pay when we [illegible]ept H_i while H_j is true.

The cost function we have to minimize is $F(D)=\xi$

$$F(D)=\xi[C_{10}Q_0+C_{00}(1-Q_0)]+(1-\xi)[C_{01}Q_1+C_{11}(1-Q_1)]$$

where $Q_0=P_0\{w:y_t(w)\in R_1\}$

$Q_1=P_1\{w:y_t(w)\in R_0\}$

Q_0, which is called the error of the first kind, is the probability to accept H_1 when H_0 is true. The radar engineers call it the false alarm probability.

Q_1 is the probability to accept H_0 when H_1 is true. $1-Q_1=Q_d$ is called the detection probability.

The cost function F(D) is called the average risk. The

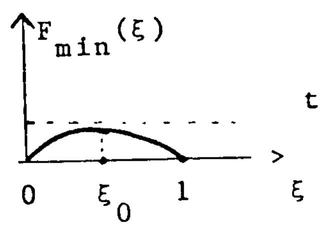
The minimization of the average risk gives rise to the following result.

The threshold value $\Lambda_o = \frac{\xi(C_{10}-C_{00})}{(1-\xi)(C_{01}-C_{11})}$

(1.3) THE MINIMAX CRITERION

In this case we have no information about the a priori probability ξ

Then we compute the Bayes risk $F_{min}(\xi)$ for different ξ



the minimax risk

e minimax strategy is the Bayes strategy for $\xi=\xi_0$ when the Bayes risk is maximum.

(1.4) NEYMAN-PEARSON CRITERION

In this case, which is used when we have to do with radar problems, ξ is not meaningful. On the other hand it is difficult to estimate the C_{ij}.

In this case it is meaningfull to preassign a false-alarm probability Q_0 we can afford and they try to maximize the detection probability Qd.

After this brief discussion of the 3 criteria we return to our subject.

It is usual to use the term signal Detection when the observation process is a random process. On the other hand we use the term "Hypotheses testing" when Y_t is a random element in $\mathbb{R}^m$.

The most studied case in Signal Detection theory is the case when

H_0 : The observation process is pure White Gaussian Noise (WGN)

H_1 : The observation process is a distorted by additive WGN version of some signal process

In the following, this case will be referred as the "Signal in Noise" case.

(1.5) THE SIGNAL IN NOISE CASE

In general the classical approach to the Signal Detection problem was through the Karhunen-Loève expansion or biorthogonal expansion (see Appendix 1).

The modern approach is based on martingale theory, more specifically on some results due to Girsanov and Kunita-Watanabe.

The first to introduce this approach in the detection theory was Thomas Kailath who in 1968 wrote a famous paper "General Likelihood ratio formula for random signals in Gaussian Noise"

We are going to state this formula.

If we have observations $\{\dot{X}_t, t \in [o,T]=I\}$ such that

H_0 : $\dot{X}_t=\dot{W}_t$

H_1 : $\dot{X}_t=Z_t+\dot{W}_t$

A more consistent mathematical rephrasal of this would be

$H_0 : X_t = W_t + \int_0^t Z_\tau d\tau$

$H_1 : X_t = W_t$

where W_t is the Standard Wiener process and Z_t is a random process which satisfy the following condition

$$E \int_0^T |Z_t| dt < \infty$$

This condition is sufficient for the proof of the theorem but not necessary.

Another basic assumption is that the future noise is independent of the past signal that is we can assume only one sided dependence between the signal and the noise

$E[\dot{W}_t Z_{t-s}] = 0 \quad , \ s > 0$

Then

$$LR = \frac{dP_1}{dP_0} = \exp\ [\textstyle{⨍}_I \ \hat{Z}_t dX_t - \frac{1}{2} \int_I \ \hat{X}_t^2 dt]$$

$\hat{Z}_t$ is the causal least-squares estimate of Z_t based on data $\{\dot{X}_\tau, \tau \leq t\}$ that is

$\hat{Z}_t = E\{Z_t | \dot{X}_\tau, \tau \leq t\}$

The symbol $⨍$ represents the Ito integral. In the Appendix I we give some properties of the Ito integral.

Now we shall make some comments on the above formula.

Since in this formula we have to know the least squares estimate of Z_t which is very difficult to obtain in the general case one could argue that this formula doesn't offer much in the computation of the LR. This is partly true. As Dr. Kailath has stressed in his paper the major value of this formula lies in its physical interpretation. That is in the fact that we gain physical insight into the structure of the likelihood ratio computer. This receiver must be of the "Correlator - Estimator" form.

On the other hand this result is a unified formula for results which existed for some specific cases like the case when Z_t is a known signal or a Gaussian process, etc.

For a proof of this theorem see Appendix II.

In the following we shall give some known results for the LR for the case of a Gaussian signal.

(1.6) GAUSSIAN SIGNAL IN WGN

Someone has to decide between the hypotheses

$$H_0 : \dot{X}_t = \dot{W}_t$$

$$H_1 : \dot{X}_t = \dot{W}_t + Z_t \qquad t \in I=[0,T]$$

with Z_t Gaussian signal of Zero mean and covariance $K(t,s)$. Also we assume that all the conditions for the general Signal-in-Noise problem are satisfied.

The classical approach through the Karhunen-Loève expansion gives rise to the following formula.

$$LR = \exp\,(\Lambda - B) \text{ where}$$

$$2\Lambda = \iint_{II} \dot{X}_t\, H\,(t,s;1)\, \dot{X}_s\, dtds$$

$$2B = \int_0^1 d\tau \int_I H(t,t;\tau)dt$$

where $H(t,s;u)$ is the resolvent kernel for the operator $k(t,s)$ that is the solution of the resolvent Fredholm integal equation

$$H(t,s;u) + u \int_I H(t,\tau;u)\, K(\tau,s)d\tau = K(t,s)$$

which has a unique solution provided that $-\frac{1}{u} \notin \sigma(K)$, $\sigma(K)$ is the spectrum of the operator K.

Price gave an interesting interpretation for Λ. As we can see $2\Lambda=\int_I \dot{X}_t \, Z_{et} \, dt$ where

$$Z_{et} \triangleq \int_I H(t,s;1) \, \dot{X}_s ds$$

We shall demonstrate that $H(t,s;1)$ can be interpreted as the smoothing filter for Z_t given data $\{\dot{X}_\tau, \; t \in I\}$

Really the smoothed estimate of Z_t is

$Z_{et}=E\{Z_t | \dot{X}_\tau, \; t \in I\}$ but since Z_t is a Gaussian signal the least-squares estimate will be the linear one. So,

$$Z_{et}=\int_I G(t,s)\dot{X}_s ds$$

by using the projection theorem

$$E[(Z_t-Z_{et})\dot{X}_s]=0 \Rightarrow$$

$$G(t,s)+ \int G(t,\tau) \, K(\tau,s)ds = K(t,s)$$

But since the resolvent Fredholm equation has a unique solution $H(t,s;1)=G(t,s)$

Dr. Kailath was able to prove that his formula is equivalent to the one for the Gaussian case. He used in order to prove this, the following formula which is equivalent to the above result.

$$LR=\exp(\Lambda-B)= \prod_1^\infty (1+\lambda_i)^{-\frac{1}{2}} \exp[\tfrac{1}{2} \textstyle\sum X_i^2 \frac{\lambda_i}{1+\lambda_i}]$$

with $X_i \triangleq \int_0^T \psi_i(t)\dot{X}_t dt$, $\int_0^T K(t,s)\psi_i(s)ds = \lambda_i\psi_i(t)$

This is easily proved if we take into account the fact when λ_i, ψ_i are the eigenvalues and eigenfunctions of $K(t,s)$ respectively then the Fredholm integral equation

$$H(t,s) + \int_0^T H(t,\tau)\, K(\tau,s)d\tau = K(t,s)$$ or in the operator

form

$H + HK=K$ has a solution which can be written

$$H(t,s) = \sum_1^\infty \frac{\lambda_i}{1+\lambda_i} \psi_i(t)\Psi_i(s)$$

Now we shall discuss another detection problem which will be proved to be equivalent to the Signal in Noise problem.

In this case we have to decide between the hypothese

(D3) H_0 : $\dot{X}_t$ is WGN with covariance δ (t-s)

H_1 : $\dot{X}_t$ is a Gaussian process with covariance $R(t,s)$

We shall assume that the problem is nonsingular that is the measure P_1 is not singular with respect to P_0 that is P_1 is equivalent to P_0.

It has been proved by Shepp and others that D3 is nonsingular iff

a) $R(t,s) = \delta(t-s)+K(t,s)$

b) K is square integrable function on $(0,T) \times (0,T)$

$$K \in \mathbb{L}_2(I \times I)$$

c) -1 is not an eigenvalue of K , $-1 \notin \sigma(K)$

Now assuming the conditions for nonsingularity a, b, c, hold we are going to prove the so called representation theorem for second order processes.

(1.7) REPRESENTATION THEOREM

Assume that $\dot{X}_t$ is Gaussian and a, b, c, hold, then

$\dot{X}_t$ can be represented as

$$\dot{X}_t = \dot{\nu}_t + Z_t$$

where ν_t is the standard wiener process so $\dot{\nu}_t$ is WGN and Z_t is Gaussian with covariance $K(t,s)$

The proof will be based on the so called resolvent identity property. This property states that the Fredholm equation

$$H(t,s) + \int_o^T H(t,\tau)K(\tau,s)d\tau = K(t,s) \text{ or } H+HK=K$$ and the Wiener-Hopf equation

$$h(t,s) + \int_o^t h(t,\tau)K(\tau,s)d\tau = K(t,s) \text{ or } h+hK=K$$

$s \leq t \leq T$ and $h(t,s)=o$ for $t<s$

have unique solutions in $\mathbb{L}_2(I \times I)$ and they are related

$$H(t,s) = h(t,s) + h^*(t,s) - \int_o^T h^*(t,\tau)h(\tau,s)d\tau$$

with $h^*(t,s) = h(s,t)$ the adjoint filter.

The above resolvent identity is very difficult to be proved and involves some concepts like the abstract triangular integral.

Using this property the proof of the representation theorem follows very easily.

We are going to sketch the proof in the operator form.

We define a Voltera operator $u(t,s,)$ through the Voltera equation.

$$u(t,s) + \int_o^t u(t,\tau)h(\tau,s) = h(t,s)$$ or in an operator form $u+uh=h$

From the Fredholm integral equation

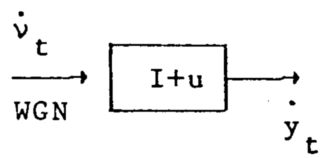
$H+HK=K \Leftrightarrow (I+K) = (I-H)^{-1}$ and the resolvent identity

$I-H=(I-h^*)(I-h)$

We obtain $I+K=(I+u)(I+u^*)$

u is a Voltera kernel that is $u(t,s)=0, s>t$

Now if we pass WGN $\dot{\nu}_t$ through the operator $I+u$ and say $\dot{y}_t$ the output



$\dot{y}_t$ will be Gaussian and

$$\dot{y}_t = \dot{\nu}_t + \int_0^t u(t,\tau)\,\dot{\nu}_\tau d\tau = \dot{\nu}_t + Z_t$$

The covariance of $\dot{y}_t$ will be

$$R_{\dot{y}} = I + u + u* + uu* = I + K$$

so $\dot{X}_t$, $\dot{y}_t$ have the same statistics and

$\dot{X}_t = \dot{y}_t$ almost surely that is

$\dot{X}_t = \dot{\nu}_t + Z_t$ Q.E.D

After this the problem (D3) can be formulated as follows:

$$H_1 : \dot{X}_t = \dot{\nu}_t + Z_t$$

$$H_0 : \dot{X}_t = \dot{\nu}_t$$

which is the Gaussian Signal in Noise problem.

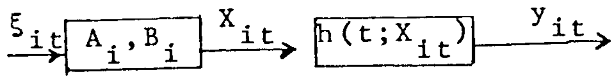
CHAPTER 2 THE NONLINEAR DETECTION PROBLEM

(2.1) DESCRIPTION OF THE PROBLEM

Basically one has to decide between two random processes y_{1t}, y_{0t} which are modelled as a nonlinear transformation of the outputs of two linear dynamical systems driven by WGN

H_0 : The observation process is r_{0t}

H_1 : The observation process is r_{1t}



$$\dot{X}_{it} = A_i(t;\theta_i)\, X_{it} + B_i\,(t;\theta_i)\, \xi_{ti} \qquad i=0,\ell$$

$$y_{it} = h(t;x_{it})$$

The observation process may be a pure or a distorted version of the signal process.

Also we want to estimate the perameter θ_i provided that we know the a priori probability density $f_i(\theta_i)$.

The problem is difficult because y_{it} are not Gaussian but they are nonlinear transformations of Gaussian signals.

We are going to approach this problem in the following

We shall generate some proper vector random process and using some proper nonlinear transformation we shall go to a detection problem which will deal with Gaussian random vector processes.

So the idea is:

Given a process y_t nonGaussian, find some vector process θ_t and a nonlinear transformation such that $F(t,y_{it},\theta_t)=X_{it}$ to be Gaussian vector processes.

In the following we shall examine some specfic cases in order to illustrate this approach.

(2.2) DETECTION OF 2 RAYLEIGH PROCESSES

In this case the Signal processes are Rayleigh with known covariances.

It is known that if Y is a Rayleigh random variable then if θ is uniformly distributed in $(-\pi,\pi]$ the random vector

$$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} Y\cos\theta \\ Y\sin\theta \end{bmatrix}$$

is normal and X_1, X_2 are uncorrelated.

So if y_{it} are the two signal processes we must generate a stationary process ϕ_t independent of y_{it} so that the first order probability density of ϕ_t to be the uniform density in the interval $(-\pi,\pi)$.

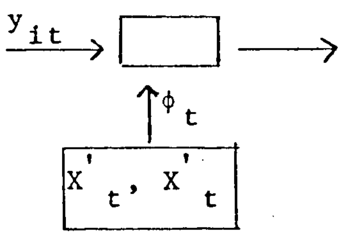
Then we take the random vector processes

$$X_t^{(i)} = \begin{bmatrix} x_{1t}^{(i)} \\ x_{2t}^{(i)} \end{bmatrix} = \begin{bmatrix} y_{it} \cos\theta_t \\ y_{it} \operatorname{sig}\theta_t \end{bmatrix} \qquad i=0,1$$

This process will be normal and the problem is to determine the statistics of $X_t^{(i)}$

The idea is to generate the process ϕ_t through two other Gaussian processes X'_{it}, X'_{2t} by using the transformation

$$\bar{\phi}_t = \tan^{-1} \frac{X_{it}}{X_{it}}$$



$$X_t^{(i)} = \begin{bmatrix} X_{1t}^{(i)} \\ X_{2t}^{(i)} \end{bmatrix} = \begin{bmatrix} y_{it} \\ y_{it} \sin\phi_t \end{bmatrix}$$

So $X'_t = \begin{bmatrix} X'_{1t} \\ X'_{2t} \end{bmatrix}$

X'_{1t}, X'_{2t} Gaussian and $E\begin{bmatrix} X_{t+\tau} X_t^T \end{bmatrix} = \begin{bmatrix} R_g(\tau) & 0 \\ 0 & R_g(\tau) \end{bmatrix}$

that is X'_{1t}, X'_{2t} uncorrelated in

Then as we prove in the appendix III the statistics of $X_t^{(i)}$ are

$$E\begin{bmatrix}X_t^{(i)}\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}, \quad R_x^{(i)}(\tau) = E\begin{bmatrix}X_{t+\tau}^{(i)} \; X_t^{(i)T}\end{bmatrix} =$$

$$\begin{bmatrix}R_i(\tau) & 0\\ 0 & R_i(\tau)\end{bmatrix} \quad i = 0,1$$

$$R_i(\tau) = \frac{1-R_g^2(\tau)}{2\,R_g(\tau)}\left[\Pi_1\left(-R_g^2(\tau)\,;\,R_g(\tau)\right) - K(R_g(\tau))\right] R_{yi}(\tau)$$

where K and Π_1 are the complete elliptic integral functions of the first and third kind respectively.

In the following we shall examine the detection problem for two cases

(A) when the observation process is noiseless

(B) when the observation process is destorted by additive WGN version of the signal process.

(2.3) THE DETECTION PROBLEM FOR NOISELESS OBSERVATIONS

We have to decide between the hypotheses

H_i : The observation process is the Gaussian vector process $X_{1t}^{(i)} = [X_{1t}^{(i)}, X_{2t}^{(i)}]^T$ with zero mean and covariance

$$R_{xi}(\tau) = \begin{bmatrix}R_i(\tau) & 0\\ 0 & R_i(\tau)\end{bmatrix} \qquad i = 0,1$$

In order to obtain a formula for the Radon-Nicodym derivative $\frac{dP_1}{dP_0}$ we shall apply a theorem the proof of which can be found in the book "Gaussian random processes" by Ibragimov and Rozanov page 88 theor. 8.

We shall state this theorem

Let P_0, P_1 to be two Gaussian measures generated by the

Gaussian processes X_{0t}, X_{1t} of zero mean and covariances $R_0(\tau)$, $R_1(\tau)$ respectively.

Then these measures are equivalent iff the difference

$$b(s,t)=R_0(s,t)-R_1(s,t)$$

is representable as

$$(1) \qquad b(s,t) = \iint e^{-i(\lambda s-\mu t)}\Psi(\lambda,\mu)F_0(d\lambda)F_1(d\mu)$$

for $s,t \in I = [0,T]$

where the function $\Psi(\lambda,\mu)$ determined from the integral equation (1) is such that

$$\iint |\Psi(\lambda,\mu)|^2 F_0(d\lambda)\ F_1(d\mu) < \infty$$

with

$$\Psi(\lambda,\mu) \in \mathbb{L}_2\ [I \times I\ ,\ F_0 \times F_1]$$

Then the density function $\dfrac{P_1(d\omega)}{P_0(d\omega)}$

$$\frac{P_1(d\omega)}{P_0(d\omega)} = D\ \exp\ [-\tfrac{1}{2}\iint\Psi(\lambda,\mu)\Psi(d\lambda,d\mu)] = P(\omega)$$

where D is a normalizing multiplier and the stochastic measure $\Psi(d\lambda,d\mu)$ is defined through the relation

$$\Psi(\Delta_1 x\Delta_2)=\Phi_0(\Delta_1)\ \overline{\Phi_0(\Delta_2)} - F_0(\Delta_1 \cap \Delta_2)$$

where F_0 is the spectral measure corresponding to X_{0t} and Φ_0 the stochastic measure which corresponds to the same process.

A simple generalization for the case when X_t are 2-dimensional random vector processes with uncorrelated components and all components to have the same covariance gives rise to the formula

$$\frac{P_1(d\omega)}{P_0(d\omega)} = D \exp\left[-\iint \Psi(\lambda,\mu)\, \Psi(d\lambda,d\mu)\right]$$

In the same book theorem 11 page 34 gives a simpler condition for the equivalence.

Upon this theorem P_0, P_1 are equivalent iff $b(s,t)$ $s,t \in I$ can be extended to a square integrable function on the whole plane whose Fourier transform

$$\phi(\lambda,\mu) = \frac{1}{4\pi^2} \iint e^{i(\lambda s-\mu t)} b(s,t)\,ds\,dt$$

satisfies the condition

$$\iint \frac{|\phi(\lambda,\mu)|^2}{f_0(\lambda) f_1(t)}\, d\lambda\, d\mu < \infty$$

where $f_i(\lambda) = \dfrac{F_i(d\lambda)}{d\lambda}$

The above theorem although it states clearly how to find $\frac{dP_1}{dP_0}$ is very difficult to be applied because in order to determine $\Psi(\lambda,\mu)$ we have to solve a difficult integral equation.

If we apply the Karhunen Loéve expansion approach to the above problem we shall have the following.

For simplicity we assume one-dimensional random processes. The generalization into two dimensional will be straight forward.

The observation process X_t under hypothesis H_0 can be expanded as

$X_{0t} = \sum_{i=0}^{\infty} \alpha_i \phi_i(t)$ where α_i are orthogonal Gaussian random variables and $\phi_i(t)$ are orthonormal functions with

$$E_0(\alpha_i\alpha_j)=\delta_{ij}\lambda_i \quad , \quad R_0(t,s)=\sum_{\tau=0}^{\infty}\lambda_i\varphi_i(t)\phi_i(S)$$

and

λ_i , φ_i are the eigenvalues and eigenfunctions of $R_0(t,s)$ that is

$$\int_0^T R_0(t,s)\ \varphi_i(s)ds = \lambda_i\varphi_i(t)$$

Similarly the observation process X_t under hypothesis H_1 can be written

$X_{1t} = \sum_{j=0}^{\infty} b_j\ \Psi_j(t)$ where b_j are orthogonal Gaussian with zero mean and Ψ_j are orthonormal

$$E_1(b_ib_j) = \mu_i\ \delta_{ij} \quad , \quad R_1(t,s) = \sum_{i=0}^{\infty} \mu_i\ \Psi_i(t)\ \Psi_i(s)$$

and

μ_i , Ψ_i are the eigenvalues and eigenfunctions of $R_1(t,s)$ that is

$$\int_0^T R_1(t,s)\ \Psi_i(s)\ ds = \mu_i\ \Psi_i(t)$$

From the above expansion if we keep only the n terms we have the Gaussian vectors

$$\alpha = (\alpha_1,\alpha_2, \ \ , \alpha_n)^T \quad b = (b_1, b_2, \ \ , b_n)^T.$$

with probability densities

$$P_{on}(\alpha)= \prod_1^n (2\pi\lambda_k)^{-\frac{1}{2}} \exp\left[-\tfrac{1}{2}\sum_1^n \frac{\alpha_k^2}{\lambda_k}\right]$$

$$P_{in}(b)= \prod_1^n (2\pi\mu_k)^{-\frac{1}{2}} \exp\left[-\tfrac{1}{2}\sum_1^n \frac{b_k^2}{\mu_k}\right]$$

$$\text{So the LR} = \lim_{n\to\infty} \frac{P_{1n}(b)}{P_{on}(a)} = \prod_1^\infty \left(\frac{\lambda_k}{\mu_k}\right)^{+\frac{1}{2}} \exp\left[-\tfrac{1}{2}\sum\left(\frac{b_k^2}{\mu_k} - \frac{\alpha_k^2}{\lambda_k}\right)\right]$$

If we define

$$\Lambda = \sum_{i=0}^{\infty} \ln \left(\frac{\lambda_i}{\mu_i}\right) \quad , \quad B = \sum_{i=0}^{\infty} \left(\frac{b_i^2}{\mu_i} - \frac{\alpha_i^2}{\lambda_i}\right)$$

then

$$LR = \exp\left(\tfrac{1}{2}(\Lambda - B)\right)$$

So for the case of 2-dimensional processes with uncorrelated components and each component to have the same covariance function we shall have

$$\frac{dP_1}{dP_0} = \exp\left(\Lambda - B\right)$$

(2.4) THE DETECTION PROBLEM FOR DISTORTED OBSERVATIONS

If $\dot{\xi}_t$ is the observation process we have to decide between the hypotheses

$$\begin{aligned} H_0 &: d\xi_t = y_{0t}\,dt + dW_t \\ H_1 &: d\xi_t = y_{1t}\,dt + dW_t \end{aligned} \qquad \text{(D1)}$$

where y_{it} is Rayleigh with correlation $R_{yi}(\tau)$ and W_t is the standard Wiener process.

After applying the nonlinear transformation the observation process will be a 2-dimensional vector process $\dot{y}_t$ and the detection problem will be

$$\begin{aligned} H_0 &: d\eta_t = X_t^{(0)}\,dt + b_t\,dW_t \\ H_1 &: d\eta_t = X_t^{(1)}\,dt + b_t\,dWt \end{aligned} \qquad \text{(D2)}$$

where $X_t^{(\ )}$ is a two dimensional zero mean Gaussian Vector process with uncorrelated components and each component has covariance $R_i(\tau)$ known.

and

$$b_t = \begin{bmatrix} \cos\theta_t \\ \sin\theta_t \end{bmatrix}$$

We are going to establish the equivalence of D_1-D_2.

From the general formula for the likelihood ratio for the Signal in Noise case we shall have for D_1 the formula

$$(LR)_1 = \exp\left[\int_0^T (y_{1t}-y_{0t})\,d\xi_t - \tfrac{1}{2}\int_0^T (y_{1t}^2 - y_{0t}^2)\,dt\right]$$

(see thorem 7.6.4 "Statistics of random processesI" by Liptser-Shiryayev)

In the same book theorem 7.20 gives the LR for the problem D_2 when we have to do with one dimensional processes. The multidimensional analog of this theorem is the following

$$(LR)_2 = \exp\{\int_0^T [X_t^{(1)} - X_t^{(0)}]^T (b_t^T \cdot b_t)^+ \, d\eta_t - \tfrac{1}{2}\int_0^T [X_t^{(1)} - X_t^{(0)}]^T (b_t^T \cdot b_t)^+ [X_t^{(1)}+X_t^{(0)}]\,dt\}$$

in our case with the notation α^+ we mean either α^{-1} if $\alpha \neq 0$ or 0 if $\alpha=0$.

For our detection problem $b_t = \begin{bmatrix}\cos\theta_t \\ \sin\theta_t\end{bmatrix}$ so

$$(LR)_2 = \exp\{\int_0^T [X_t^{(1)}-X_t^{(0)}]^T d\eta_t - \tfrac{1}{2}\int_0^T [X_t^{(1)}-X_t^{(0)}]^T [X_t^{(1)}+X_t^{(0)}]\,dt\}$$

$$= \exp[\int_0^T (X_{1t}^{(1)}-X_{1t}^{(0)})\,d\eta_{1t} + (X_{2t}^{(1)} - X_{2t}^{(0)})\,d\eta_{2t} - \tfrac{1}{2}\int_0^T [({X_{1t}^{(1)}}^2 - {X_{1t}^{(0)}}^2) + ({X_{2t}^{(1)}}^2 - {X_{2t}^{(0)}}^2)]\,dt\}$$

If we take into account how $X_t^{(1)}$ is related with y_{it} we ...lly see that

$$(LR)_1 = (LR)_2 = LR$$

From the formula $(LR)_2$ we can see that in a formalistic way the LR can be written as

$$LR = LR_1 \cdot LR_2$$

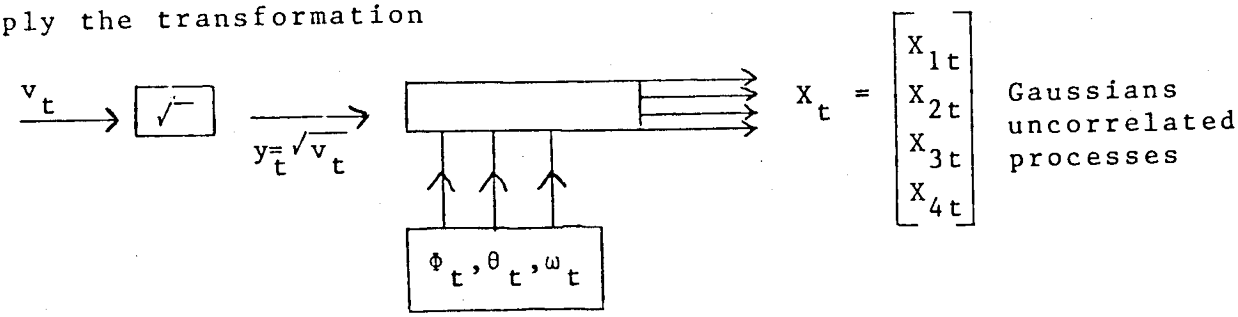
where LR_i corresponds to the detection problem

$$H_0 : d\xi_{it} = X_{0t}^{(i)}\,dt + d\zeta_t \qquad i = 1,2$$
$$H_1 : d\xi_{it} = X_{1t}^{(i)}\,dt + d\zeta_t$$

where $X_t^{(i)}$ is the ith component of the vector X_t..

(2.5) DETECTION PROBLEM FOR 2 CHI-QUAVE PROCESS WITH FOUR DEGREES OF FREEDOM

We generate 3 stationary processes ϕ_t, θ_t, ω_t and we apply the transformation



$$X_{1t} = y_t \cos\varphi_t \sin\theta_t \cos\omega_t$$
$$X_{2t} = y_t \sin\varphi_t \sin\theta_t \cos\omega_t$$
$$X_{3t} = y_t \cos\theta_t \cos\omega_t$$
$$X_{4t} = y_t \sin\omega_t$$

where φ_t must be uniformly distributed in $(-\pi,\pi)$ and θ_t, ω_t to have the following first order densities

$$f_\theta(\theta) = \tfrac{1}{2}\sin\theta \qquad \theta \in [0,\pi]$$
$$f_\omega(\omega) = \frac{2}{\pi}\cos^2\omega \qquad \omega \in (-\pi/2, \pi/2)$$

Under these assumptions we can prove that X_t is Gaussian with uncorrelated components.

Proof in Appendix IV.

Now one has to make lots of computations in order to determine the statistics of X_t.

The idea as in the Rayleigh case is to use 4 other Gaussian processes $X'_t = (X'_{1t}, X'_{2t}, X'_{3t}, X'_{4t})^T$ uncorrelated with the same covariance $R'(\tau)$ in order to generate φ_t, θ_t,

ω_t.

After that the detection problem is treated as in 2.3, 2.4.

(2.6) LOGNORMAL PROCESSES

We know that if X is lognormal r.v then $\ell n\ X$ is Gaussian.

In this case give Y_t we use the transformation $\ell n Y_t = X_t$.

Then X_t will be normal with covariance $R_{x(\tau)}$

$$Y_t = \exp(X_t) \Rightarrow E[Y_{t+\tau} Y_t] = R_y(\tau) = E\{\exp(X_{t+\tau})\ \exp(X_t)\}$$

$$= E\{\exp(X_{t+\tau} + X_t)\}$$

If we consider $X_{t+\tau} = X_1$ and $X_t = X_2$ then $X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$ is Gaussian random vector with let say zero mean and covariance

$$E(X\ X^T) = \begin{bmatrix} R_x(0) & R_x(\tau) \\ R_x(\tau) & R_x(0) \end{bmatrix} = B$$

So if someone considers the characteri[illegible]

$$\Phi(u) = E\ \{e^{i\ u^T x}\} = e^{-\frac{1}{2}\ u^T B u}$$

So $R_t(\tau) = \Phi(u)$ for $u = \begin{bmatrix} -1 \\ -j \end{bmatrix}$

[F]inally $R_y(\tau) = \exp\ \{R_x(0) + R_x(\tau)\}$

$$R_x(0) + R_x(\tau) = \ell n\ R_y(\tau)$$

$$R_x(\tau) = \ell n\ (R_y(\tau)) - R_x(0)$$

On the other hand $E(Y_t) = \exp\ (R_x(0)/2)$

So $\frac{R_x(0)}{2} = \ell n\ (M_y)$ where M_y is the mean of y_t $R_x(0) = 2\ell n(m_y)$

So finally $R_x(\tau) = \ell n(R_y(\tau)) - \ell n(m_y^2) \Longrightarrow$

$$R_x(\tau) = \ell n\left(\frac{R_y(\tau)}{M_y 2}\right)$$

In this way we have found the statistics of the Gaussian process and the detection problem is known.

CHAPTER 3 THE ESTIMATION PROBLEM

After the detection problem has been solved we want to estimate the parameter θ

$$\dot{X}_t = A\,(t;\theta)\ X_t + B(t;\theta)\ \dot{W}_t$$

$$y_t = h\,(t;X_t)$$

In the case when someone could recover the covariance of the state vector X_t, knowing the covariance of the process y_t we would be able to recover θ since

$$X_t = \Phi(t,0;\theta)\ X_0 + \int_0^t \Phi\ (t,s;\theta)\ B(s;\theta)\ dW_s$$

$$X_{t+\tau} = \Phi(t+\tau;0;\theta)\ X_0 + \int_0^t \Phi(t+\tau,\nu;\theta)\ B(\nu;\theta) dW_\nu$$

we assume X_0 Gaussian vector independent of $\dot{W}_t$ with covariance P_0

then

$$R_x(\tau) = E\ [X_{t+\tau}X_t^T] = \Phi(t+\tau,0;\theta)\ P_0\ \Phi(t,o;\theta) + \int_0^t \Phi\ (t+\tau,s,\theta)\ B(s;\theta)\ B^T(s;\theta)\ \Phi^T(t,s;\theta) ds$$

From these equations we could be able to compute θ.

So the basic problem is if we can recover $R_x(\tau)$ ... $R_y(\tau)$.

Recovery of the covariance $R_x(\tau)$. Let's consider the case when X is a two dimensional vector

Let $\xi = \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix}$ $\quad \eta = \begin{matrix} \eta_1 \\ \eta_2 \end{matrix}$ with

ξ_1, ξ_2 uncorrelated Gaussian r.v

η_1, η_2 uncorrelated Gaussian r.v

$E(\xi y^T) = \begin{bmatrix} \rho & o \\ o & \rho \end{bmatrix} = R$ and both ξ_i, η_i $N(0,1)$

Since X_1, X_2 are uncorrelated the space

$\mathbb{L}_2(\phi(x_1)\ \phi(x_2)\ dx_1 d x_2)$ with $\Phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$ has a complete base the polynomials $\{H_{mn}(x_1,x_2)\ m=o,\dots,\infty,\ n=o,\dots\infty\}$ with $H_{mn}(x_1,x_2)=H_m(x_1)\ H_n(x_2)$ and $H_i(x)$ to be the Hermite polynomial.

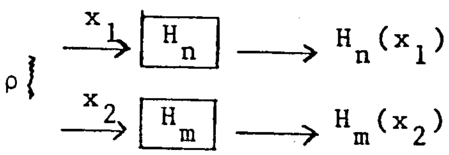
So if $f(\xi)$ belongs to the above space it can be written as

$$f(\xi) = \Sigma\Sigma \frac{\alpha_{mn}}{m!n!} H_{mn}(\xi_1,\xi_2)$$

with $\alpha_{mn} = < f(\xi), H_{mn}(\xi) > = E\ [f(\xi)\ H_{mn}(\xi)]$

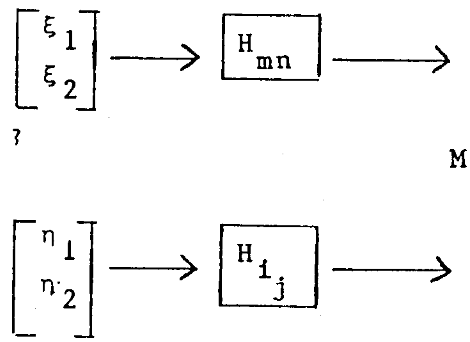
$f(\eta)$ has an identical expansion

From the one dimensional case it is known that $H_n(x_1)$, $H_m(x_2)$ with x_1, x_2 correlated are uncorrelated when $m \neq n$



$$M(\rho)=E\ [H_n(x_1)H_m(x_2)] = \rho^n n!\delta_{mn}$$

In the two dimensional case



$$H_{mn}(\xi_1,\xi_2) = H_m(\xi_1)\ H_n(\xi_2)$$

M(e)

$$H_{ij}(y_1,y_2) = H_i(y_1)\ .\ H_j(y_2)$$

$$E[H_{mn}(\xi_1\xi_2)\ H_{ij}(\eta_1\ \eta_2)] = \rho^{m+n}\ n!m!\delta_{mi}\delta_{nj}$$

So $M(\rho) = E(f(\xi)f(\eta)) = E\ \{\Sigma\Sigma\Sigma\Sigma \frac{\alpha_{mn}\ a_{ij}}{m!n!\xi!j!}\ .$

$. H_{mn}(\xi_1\xi_2)\ H_{ij}(\eta_1,\eta_2)\} = \Sigma\Sigma \frac{\alpha^2_{mn}}{(m!n!)^2} m!\ n!\ \ e^{m+y}$

$$M(\rho) = \Sigma\Sigma \frac{\alpha^2_{mn}}{m!n!} e^{m+n}$$

So one has to examine the function $M(\rho)$ and to find conditions in order to be invertible.

These will be the same as in the one dimensional case as we can see from the fact that the function M(e) has the same structure in both cases.

(For such conditions see [2])

APPENDX I "STOCHASTIC CALCULUS"

1. Karhunen-Loeve expansion.

Given a second order process X_t , $t\varepsilon[0,T]$ with covariance $R(t,s)$ we can have the biorthogonal expansion.

$$X_t = \sum_{m=0}^{\infty} \alpha_m \phi_m(t)$$

where

$$\alpha_m \text{ are orthogonal r.v}$$

and

$$\{\phi_m(t) , m=0,1,..,\} \text{ is an orthonormal set}$$

We shall show that

$$E(\alpha_m \alpha_j) = \delta_{mj} \lambda_m$$

$$R(t,s) = \sum_{o}^{\infty} \lambda_m \phi_m(t).\phi_m(s)$$

where λ_m, ϕ_m are the eigenvalues and eigenfunctions of $R(t,s)$ that is

$$\int_0^T R(t,s) \phi_i(s)ds = \lambda_i \phi_i(t)$$

Really, if $E(\alpha_m \alpha_j) = \delta_{mj} \sigma_m^2 \Longrightarrow$

$$R(t,s) = E[x_t x_s] = E[\Sigma\Sigma \alpha_m \alpha_n \phi_m(t)\phi_n(s)]$$

$$= \Sigma \sigma_i^2 \phi_m(t) \phi_m(s)$$

But also, if we assume that the set of the eigenfunctions is complete in $\mathbb{L}_2(I)$ for t fixed we have

$$< R(t,.) , \phi_i(.) > = \lambda_i \phi_i(t) \Longrightarrow$$

$$R(t,s) = \sum_{o}^{\infty} (\lambda_i \phi_i(t))\phi_i(s)$$

So $\sigma_i^2 = \lambda_i$

For more details see E Wong, Stochastic process in information and Dynomical Systems, McGraw Hill 71.

2. Martingales

Let P to be a probability measure defined on a σ-Aloebra F of subsets of the basic space Ω. In other words consider the probability space (Ω, F, P).

Let also $[M_t, t \in I]$ a random process and $\{F_t, t\ I\}$ an increasing family of σ-subalgebras.

Then $\{M_t, F_t, P\}$ will be called a martingale iff

$$E(dM_t | F_t) = 0 \text{ or } E(M_t | F_s) = M_s \ , s<t$$

Local martingales

$\tau(\omega) \geq 0$ is a stopping time iff $\{\omega : \tau(\omega) \leq t\} \varepsilon F_t$

If M_t is a martingale then the stopped process $M(t \wedge \tau)$ is also martingale where $t \wedge \tau = \min(t, \tau)$

If $\tau_1, \tau_2, \ldots, \tau_n, \ldots$ is an increasing stopping times and the stopped processes are martingales then M_t is called a local martingale.

Local Semi-Martingales

X_t is called a local Semi-Martingale if $X_t = M_t + B_t$ where

M_t is a local martingale and

B_t is a process of bounded variation

3. Quadratic Variation of a Martingale

Let $\{M_t, F_t, P\}$ to be a sample-continuous second order martingale. Then if $X_t = M_t^2$ we define a process $B_t \equiv \langle M \rangle_t$ by the relation

$$\begin{aligned} dB_t &= E(dX_t | F_t) = E[M_{t+dt}^2 - M_t^2 | F_t] \\ &= E[(M_{t+dt} - M_t)^2 | F_t] + 2E[(M_{t+dt} - M_t) M_t | F_t] \\ &= E((dM_t)^2 | F_t) \geq 0 \end{aligned}$$

The process B_t which is nondecreasing will be denoted by $\langle M\rangle_t$ and will be called the quadratic variation of M_t.

It is interesting to observe that the process

$$m_t = M_t^2 - \langle M\rangle_t \text{ is a martingale.}$$

4. Wiener Process

The n-dimensional vector process (W_t, F_t, P) will be called standard Wiener Process iff

i) It is sample-continuous

ii) The components W_{it} are martingales that is

$$E(W_t | F_s) = W_s \quad , \; s<t$$

iii) $E[(W_t - W_s)(W_t - W_s)^T] = I \cdot (t-s)$

where I is the nxn Identity matrix.

The above used definition usually is called the Doob Levy eorem.

Another useful version for the definition of a Wiener process is

(W_t, F_t, P) is Wiener iff

i) It is sample continuous

ii) It is a martingale

iii) Its guadratic variation is t $, \langle W\rangle_t = t$

5. Ito Stochastic Integral

If $\{M_t, F_t, P\}$ is a martingale and Φ_t is a process adapted to $\{F_t, t\varepsilon I\}$ such that

$\int_0^T \phi_t^\tau \, d\langle M\rangle_t < \infty$ almost surely we define the Ito integral

$$\int_0^T \phi_t \, dM_t \triangleq \lim \Sigma \phi_{t\nu} \left[M_{t_{\nu+1}}^{(n)} - M_{\nu t}^{(n)} \right]$$ where the limit

is taken in the quadratic sense.

The calculus of the Ito integral is somehow different from the ordinary one. For instance observe that

$$\int_0^t W_s dW_s = \frac{1}{2} W_t^2 - \frac{1}{2}t$$ that is there exist the term $-\frac{1}{2}t$ which would not be exist in an ordinary integral.

This definition has been generalized for the case when M_t is a local semi-martingale.

Itos's differentiation rule.

Let X_t to be an Ito process that is $dx_t = \alpha_t dt + b_t dW_t$ or in a much more consistent mathematical rephrasal

$$X_t = X_0 + \int_0^t \alpha_s dS + \int_0^t b_s dW_s$$

Now define a new process $Y_t = f(t,X_t)$ where the function $f(t,x)$ is assumed to be twice continuously differentiable with respect to x and once with respect to t.

Then the process Y_t is also an Ito process with

$$dY_t = \dot{f}(t,x_t)dt + f_x(t,X_t)dX_t + \frac{1}{2} b_t^T f_{xx}(t,X_t)dt$$

Kunita and Watanabe have generalized the above rule for the case of a local semimartingale in their famous paper "On square integrable martingales".

We shall state the rule.

Let X_t a vector process whose components are continuous local semimartingales. Define $Y_t = f(t,X_t)$. f satisfies the previous conditions. Then

$$dY_t = \dot{f}(t,X_t)dt + f_x(t,X_t)dX_t + \frac{1}{2} \operatorname{tr} [f_{xx}(t,X_t)d\langle M,M\rangle_t]$$

where f_x is 1xn vector and represents the tronspose of the grandient of f

f_{xx} is nxn with $f_{xx}(i,j) = \frac{\partial f}{\partial x_j \partial x_i}$

$d \langle M,M \rangle_t = \{ d \langle M_i , M_j \rangle_t \}$

A useful result which is proved by using Ito's rule is

$$X_t = \int_0^t \alpha_s dW_s \quad , \quad y_t = \int_0^t b_s \, d\, W_s$$

then $\langle x,y \rangle_t = \int_0^t \alpha_s b_s d_s$

APPENDIX II PROOF OF THE GENERALIZED LR FORMULA

We have to decide between the hypotheses

$$H_0 : \dot{X}_t = \dot{W}_t$$
$$H_1 : \dot{X}_t = \dot{W}_t + Z_t \qquad \text{(D1)}$$

where W_t is the standard Wiener process.

We consider the "new information" or "innovation" process

$$\nu_t - X_t - \int_0^t \hat{Z}_s d_s \quad \text{or} \quad d\nu_t = dX_t - \hat{Z}_t dt$$

where $\hat{Z}_t = E\, [Z_t \mid \dot{X}_\tau , \ \tau \leq t \ , \ H_1]$ and

$$F_t = \sigma \ (X_\tau , \ \tau \leq t) \text{ so}$$

$$\hat{Z}_t = E(Z_t | F_t)$$

We shall prove that ν_t is the standard Wiener process. Really

i) ν_t is sample continuous

ii) ν_t is a martingale since

$$E[d\nu_t \mid F_t] = E[(Z_t - \hat{Z}_t)dt + dW_t \mid F_t] =$$
$$= E(Z_t \mid F_t)dt - \hat{Z}_t dt + E(dW_t \mid F_t) = 0$$

since $\hat{Z}_t$ is F_t measurable and (W_t, F_t) is a martingale

iii) $\nu_t = W_t + \int_0^t (Z_\tau - \hat{Z}_\tau) d_\tau = W_t + B_t$

Obviously B_t is of bounded variation since it is absolutely continuous as an indefinite integral. Consequently B_t doesn't contribute anything to the quadratic variation of ν_t. That is

$$\langle \nu \rangle_t = \langle W \rangle_t = t$$

Thus we have completed our proof.

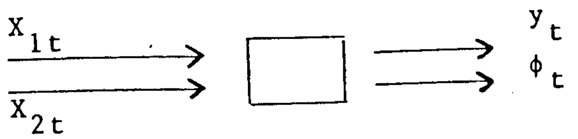
After that problem (D1) is equivalent to:

$$H_0 : \dot{X}_t = \dot{\nu}_t$$

$$H_1 : \dot{X}_t = \dot{\nu}_t + \hat{Z}_t$$

Now by applying Girsanov's theorem we obtain the LR formula

$$\frac{dP_1}{dP_0} = \exp \left[\int_0^T \hat{Z}_s ds - \tfrac{1}{2} \int_0^T \hat{Z}_s^2 \, ds \right]$$



$X_{it} \Lambda N(0,1)$

$$y_t = \sqrt{X_{1t}^2 + X_{2t}^2} \qquad \phi_t = \tan^{-1} \frac{X_{2t}}{X_{1t}}$$

where X_{1t}, X_{2t} zero mean Gaussian processes with covariance $R(\tau)$.

We are going to compute the covariance of $W_t = \cos\phi_t$

$$R_w(\tau) = E\left[\cos\phi_{t+\tau}\ \cos\phi_t\right]$$

We shall use the following notations:

$$X_1 = X_{1t}\ ,\ X_2 = X_{2t}\ ,\ X_3 = X_{1t+\tau} \quad X_4 = X_{2t+\tau}$$

$$Y_1 = \sqrt{X_1^2 + X_2^2} = g_1(X_1, X_2, X_3, X_4)$$

$$Y_2 = \sqrt{X_3^2 + X_4^2} = g_2(X_1, X_2, X_3, X_4)$$

$$Y_3 = \tan^{-1} \frac{(X_2)}{X_1} = g_3(X_1, X_2, X_3, X_4) = \phi_1$$

$$Y_4 = \tan^{-1} \frac{(X_4)}{X_3} = \phi_2 = g_4(X_1, X_2, X_3, X_4)$$

So if we consider the r. vector $X = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix}$ this is Gaussian

zero mean with covariance

$$R_x = E\left[X.X^T\right] = \begin{bmatrix} I & R \\ R & I \end{bmatrix}$$ where I is 2x2 identity matrix

and

$$R = \begin{bmatrix} \rho & 0 \\ 0 & \rho \end{bmatrix}$$ (ρ has taken the place of $R(\tau)$

From the r. vector X we have taken the random vector

$$Y = \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \end{bmatrix} = g(x) = \begin{bmatrix} g_1(x) \\ g_2(x) \\ g_3(x) \\ g_4(x) \end{bmatrix}$$

1st step Computation of the joint probability density

$$P(Y_1, Y_2, Y_3, Y_4) = P(Y)$$

$$P(Y) = \frac{1}{|\det J|} \cdot P_x(Y)$$

The Jacobian of g is
$$J = \begin{vmatrix} g_{1x_1} & g_{1x_2} & g_{1x_3} & g_{1x_4} \\ g_{2x_1} & . & . & . \\ . & . & . & . \\ . & . & . & . \\ g_{4x_1} & . & . & . \end{vmatrix} =$$

$$= \begin{vmatrix} X_{1/y_1} & \frac{X_2}{X_1} & 0 & 0 \\ -X_{2/Y_1^2} & X_{1/Y_1^2} & 0 & 0 \\ 0 & 0 & X_{3/Y_2} & X_{4/Y_2} \\ 0 & 0 & -X_{4/Y_2^2} & X_{3/Y_2^2} \end{vmatrix} - \frac{1}{Y_1 Y_2}$$

On the other hand

$$P_x(x) = \frac{1}{(2\pi)4/2} \quad \frac{1}{(\det R_x)1/2} \quad \exp\left[-1/2(X^T R_x^{-1} X)\right]$$

But

$$R_x^{-1} = \begin{bmatrix} I & R \\ R & I \end{bmatrix}^{-1} = \begin{bmatrix} (I-R^2)^{-1} & -R(I-R^2)^{-1} \\ -R(I-R^2)^{-1} & (I-R^2)^{-1} \end{bmatrix} = \begin{bmatrix} A & B \\ B & A \end{bmatrix}$$

where
$$A = (I-R^2)^{-1} = \left[\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} \rho^2 & 0 \\ 0 & \rho^2 \end{bmatrix}\right]^{-1} = \begin{bmatrix} (1-\rho^2)^{-1} & 0 \\ 0 & (1-\rho^2)^{-1} \end{bmatrix}$$

$$B = \begin{bmatrix} -\frac{\rho}{1-\rho}2 & 0 \\ 0 & \frac{\rho}{1-\rho}2 \end{bmatrix}$$

Now if we put
$$X = \begin{bmatrix} X_1 \\ X_2 \\ -- \\ X_3 \\ X_4 \end{bmatrix} = \begin{bmatrix} X \\ \\ W \end{bmatrix}$$

we take

$$X^T . R_X^{-1} . X = X^T A\ X - 2 X^T B\ W + W^T A\ W =$$

$$= \frac{X_1^2+X_2^2}{1-\rho^2} + X_3^2+X_4^2 - 2 \frac{\rho}{1-\rho^2} (X_1X_3+X_2X_4)$$

$$= \frac{1}{1-\rho^2} (Y_1^2 + Y_2^2 - 2\rho Y_1 Y_2 \cos (\phi_1-\phi_2))$$

So finally

$$P_Y(y_1,y_2,\phi_1,\phi_2) = \frac{1}{(2\pi)^2} \frac{y_1y_2}{(1-\rho^2)} \exp\left[-\frac{1}{2} \frac{y_1^2+y_2^2}{1-\rho^2} + \frac{\rho}{1-\rho^2} y_1y_2\cos(\phi_1-\phi_2)\right]$$

<u>Some results</u>

$$P(y_1,y_2) = \int_0^{2\pi}\int_0^{2\pi} A \, . \, e^{x\cos(\phi_1-\phi_2)} d\phi_1 d\phi_2$$

$$= \int_{-2\pi}^{2\pi} (2\pi - |\phi|)\ A\ e^{x\cos\phi}\ d\phi = 4\pi A \int_0^{2\pi} e^{x\cos\phi} d\phi -$$

$$- 2A \int_0^{2\pi} \phi\ e^{x\cos\phi}\ d\phi = 4\pi A . 2\pi\ I_0(x) - 2A.\ 2\pi^2\ I_0(x)$$

$$= 4\pi^2\ A\ I_0(x)$$

$$\text{So } P(Y_1,Y_2) = \frac{y_1y_2}{1-\rho^2}\ e^{-\frac{y_1^2+y_2^2}{2(1-\rho^2)}}\ I_0\left(\frac{\rho y_1 y_2}{(1-\rho^2)}\right)$$

Computation of $E[\cos\phi_1\ \cos\phi_2]$

If we define $\Phi(\lambda_1,\lambda_2;\rho) = E[e^{j\lambda_1\phi_1+j\lambda_2\phi_2}]$ then

provided that $\Phi(1,1;\rho)=0$ we shall have

$E[\cos\phi_1\cos\phi_2] = 1/2\ Re\ [\Phi(1,-1;\rho)]$

we have $P(Y,\phi) = A.e^{x\cos(\phi_1-\phi_2)}$ where

$$A = \frac{1}{(2\pi)^2} \frac{y_1y_2}{1-\rho^2} \exp\left(-\frac{y_1^2+y_2^2}{2(1-\rho^2)}\right) \quad x = \frac{\rho y_1 y_2}{1-\rho^2}$$

$$\text{So } \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} A\ e^{j(\phi_1-\phi_2)}\ e^{x\cos(\phi_1-\phi_2)}\ d\phi_1 d\phi_2 =$$

$$= 2A \int_0^{2\pi} e^{j\phi} e^{x\cos\phi} (2\pi-\phi)\, d\phi =$$

$$= 4\pi A \int_0^{2\pi} e^{j\phi} e^{x\cos\phi} d\phi - 2A \int_0^{2\pi} \phi\, \rho^{x\cos\phi} d\phi$$

$$= 4\pi A\ (2\pi\ I_1(x) - 2A\ (2\pi^2\ I_1(x))$$

$$= 4\pi^2 A\ I_1(x) \quad \text{But } I_0'(x) = I_1(x)$$

So $E\left[\cos\phi_1\ \cos\phi_2\right] = \int_0^\infty \int_0^\infty \frac{y_1 y_2}{2(1-\rho^2)} \exp\left(\frac{-y_1^2 + y_2^2}{2(1-\rho^2)}\right) I_0'\left(\frac{\rho y_1 y_2}{1-\rho^2}\right) dy_1 dy_2$

So we want to compute the integral

$$\int_0^\infty A y_2 I_1(\alpha y_2)\ e^{-\gamma^2 y_2^2}\ dy_2$$

where $A = \frac{y_1}{2(1-\rho^2} \quad e^{-\frac{y_1^2}{2(1-\rho^2)}}$

$$\alpha = \frac{\rho y_1}{1-\rho^2} \quad , \quad \gamma^2 = \frac{1}{2(1-\rho^2)}$$

One can show that

$$\int^\infty y\ I_0'\ (\alpha y)\ e^{-\gamma^2 y^2}\ dy = \sqrt{\frac{\pi}{8}}\ \gamma^{-3} \alpha\ e^{-\frac{\alpha^2}{8\gamma^2}} \left[I_0(\frac{\alpha^2}{8\gamma^2}) + I_0'\ (\frac{\alpha^2}{8\gamma^2})\right]$$

to compute integrals of the form

$$\int_0^\infty y^2\ e^{-\gamma^2 y^2}\ I_0(\alpha y^2) dy \quad , \quad \int_0^\infty y^2\ e^{-\gamma^2 y^2}\ I_0'(\alpha y^2) dy$$

with γ as before and $\alpha = \frac{\rho^2}{4(1-\rho^2)}$

The first integral

$$\int_0^\infty y^2\ e^{-\gamma^2 y^2}\ I_0(\alpha y^2) dy = \frac{1}{2\pi} \int_0^{2\pi} d\theta \int_0^\infty y^2\ e^{-(\gamma^2-\alpha\cos\theta y^2}\ dy$$

$$= \frac{1}{2\pi} \int_0^\infty \frac{1}{2} \cdot \frac{1}{2} \left[\frac{\pi}{(\gamma^2-\alpha\cos\theta)^2 (\gamma^2-\alpha\cos\theta)}\right]^{\frac{1}{2}} d\theta =$$

$$= \frac{\sqrt{\pi}}{2\sqrt{\pi}} \quad (1-\rho^2)^{3/2} \int_0^{\pi} \frac{d\theta}{(1-\rho^2\cos^2\theta)(1-\rho^2\cos^2\theta)^{\frac{1}{2}}}$$

$$= \frac{1}{\sqrt{2\pi}} \quad (1-\rho^2)^{3/2} \int_{-1}^{1} \frac{dx}{\left[(1-x^2)(1-\rho^2x^2)\right]^{\frac{1}{2}} \; (1-\rho^2x^1} \quad -$$

$$= \frac{2}{\sqrt{2\pi}} \quad (1-\rho^2)^{3/2} \; \Pi_1 \; (-\rho^2;\rho)$$

where Π_1 is the elliptic integral function of 3[rd] kind

The second integral

$$\int_0^{\infty} y^2 \; e^{-\gamma^2 y^2} \; I_1(\alpha y^2)dy = \frac{1}{2\pi} \int_0^{2\pi} \cos\theta \int_0^{\infty} y^2 \; e^{-y^2(\gamma^2-\alpha\cos\theta)} dy$$

$$= \frac{(1-\rho^2)^{3/2}}{2\sqrt{\pi}} \cdot 2 \int_0^1 \frac{(2x^2-1)dx}{(1-\rho^2x^2) \; \left[(1-x^2)(1-\rho^2x^1)\right]^{\frac{1}{2}}}$$

$$= - \frac{2(1-\rho^2)^{3/2}}{\rho^2\sqrt{\pi}} \left[\int_0^1 \frac{dx}{\left[(1-x^2)(1-\rho^2x^2)\right]^{\frac{1}{2}}} \right.$$

$$\left. \frac{\rho^2}{} - 1) \int_0^1 \frac{dx}{(1-\rho^2x^2)\left[(1-x^1)(1-\rho^2x^2)\right]^{\frac{1}{2}}} \right.$$

$$= - \frac{2(1-\rho^2)^{3/2}}{\rho^2\sqrt{\pi}} \left[\mathbb{K}(\rho) + \frac{\rho^2-2}{2} \; \Pi_1 \; (-\rho^2;\rho) \right]$$

where $\mathbb{K}(\rho)$ is the elliptic integral function of the first kind

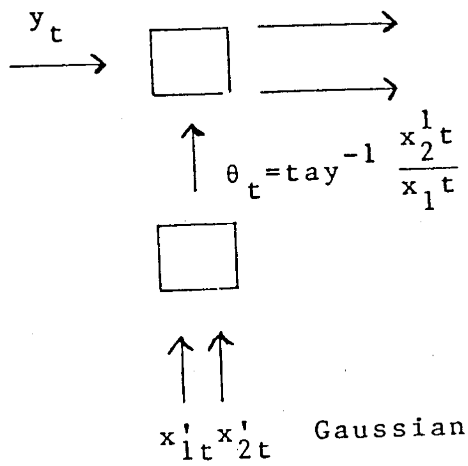
So finally we have

$$E\left[\cos\theta_1\cos\theta_2\right] = \frac{1-\rho^2}{2\rho} \left[\Pi_1 \; (-\rho^2;\rho) - \mathbb{K}(\rho) \right]$$

$$= E \left[\sin\theta_1\sin\theta_2\right]$$

$$E \left[2\sin\theta_1\cos\theta_2\right] = E \left[\sin(\theta_1-\theta_2)\right] = 0$$

So if we have a Rayleigh process y_t we apply the following nonlinear transformation



$$x_{1t} = y_t \cos\theta_t$$

$$x_{2t} = y_t \sin\theta_t$$

$$R_y(\tau) = E\left[y_{t+\tau} y_t\right]$$

Gaussian with correlation $R(\tau)$

Then the vector process $X_t = \begin{bmatrix} X_{1t} \\ X_{2t} \end{bmatrix}$ is zero mean Gaussian

h covariance $R_x(\tau) = \begin{bmatrix} R_1(\tau) & 0 \\ 0 & R_1(\tau) \end{bmatrix}$

with $R_1(\tau) = \dfrac{1-R^2(\tau)}{2R(\tau)} \left[\Pi\Pi_1 \; (-R^2(\tau)) - \mathbb{K} \; (R(\tau)) \right] R_y(\tau)$

APPENDIX IV

Let X_1, X_2, X_3, X_4 uncorrelated random variables $X_i \wedge N(0,\sigma^2)$ and the nonlinear transformation

$$X_1 = Y\cos\phi\ \sin\theta\ \cos\omega \qquad y \geq 0$$

$$X_2 = Y\sin\phi\ \sin\theta\ \cos\omega \qquad \phi\ (-\pi,\pi)$$

$$X_3 = Y\cos\theta\ \cos\omega \qquad \theta\ (0,\pi)$$

$$X_4 = Y\sin\omega \qquad \omega\ (-\pi/_2, \pi/_2)$$

We are going to find the density functions of ϕ,θ,ω

$$f(y,\phi,\theta,\omega) = |J(y,\phi,\theta,\omega)|\ f(X_1,X_2,X_3,X_4)$$

where $J(y,\phi,\theta,\omega)$ is the Jacobian of the above transformation.

After some computations we take

$$J(y,\phi,\theta,\omega) = y^3\cos^2\omega\sin\theta \text{ and } X_1^2+X_2^2+X_3^2+X_4^2 = y^2$$

So finally

$$f_{y\theta\phi\omega}(y,\theta,\phi,\omega) = y^3\cos^2\omega\sin\theta\ \frac{1}{(2\pi)^2\sigma^4}\ e^{-\frac{y^2}{2\sigma^2}}$$

$$\frac{1}{2\pi}\cdot\left[\frac{y^3}{\sigma^4}\ \rho^{\frac{-y^2}{2\sigma^2}}\right]\left[\frac{1}{2}\sin\theta\right]\left[\frac{2}{\pi}\cos^2\omega\right]$$

So ϕ is uniform in $(-\pi,\pi)$

..is density $f_\theta(\theta) = 1/2\ \sin\theta$

ω has density $f_\omega(\omega) = \frac{2}{\pi}\cos^2\omega$

and y^2 has density of a chi-square with 4 degrees of freedom.

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