JOINT OPTIMIZATION OF INFORMATION PATTERN AND CONTROL IN SOME LINEAR QUADRATIC GAUSSIAN PROBLEMS

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Thesis submitted to the Faculty of the Graduate School
of the University of Maryland in partial fulfillment
of the requirements for the degree of
Master of Science
1978

CURRICULUM VITAE

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ABSTRACT

Title of Thesis: Joint Optimization of Information Pattern and Control in some Linear Quadratic Guassian Problems

Sheldon Wolk, Master of Science, 1978

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A quadratic cost criterion is optimized for a discrete-time stochastic control system in which each controller uses a control law which is a linear combination of observations, as determined by the information pattern. The optimal information pattern as well as the optimal control laws are obtained for controllers with no memory. An example displaying the interplay between communication costs, control costs, and the optimum information pattern is developed.

Consideration is then given to a system in which the controllers have finite memory and in which a delay is imposed on the transmission of information. This is seen to be an extension of the theory developed for the simpler case without memory.

TABLE OF CONTENTS

ACKNOWLEDGMENTS

I would like to take this opportunity to thank Dr. Baras, whose guidance made this paper possible.

CHAPTER 1

1.1 Introduction

The problem considered in this thesis is one of optimizing a quadractic cost criterion for a discrete linear system in which there are q observers that can make noisy measurements on the system, and of controllers to which they may or may not communicate. It has been shown [7] that due to the decentralized nature of the above problem that the optimal control laws may in general be non-linear. Nevertheless, we will constrain the inputs to be a linear combination of the observations made by some subset of the set of observers.

A sub-optimal problem very similar to the one described above was examined by Chong and Athans [4]. However, in their case no communication was allowed between controllers. Here, we will optimize over both the allowable control laws and the allowable information patterns.

The problem will first be reduced to a set of deterministic equations, to which the maximum principle will be applied. This will lead to a two point boundary value problem in which the state and costate equations are coupled. A method of solution for this problem is presented in Chapter 3, along with an example for a system with two states, two controllers, and two observers.

Finally, it is shown that the conditions for the optimality are necessary, and that there is indeed a solution to the above two point boundary value problem.

1.2 Linear O-memory Control Laws

Consider a discrete linear time invariant system with q controllers, q observers, the following dynamics:

$$x(i+1) = Ax(i) + w(i) + \sum_{j=1}^{q} B_j u_j(i) \qquad (1.2.1)$$

and the following observations model:

$$y_k(i) = C_k x(i) + v_k(i), \; k = 1,2,3,\ldots,q \qquad (1.2.2)$$

where:

$x(i)$ is the nxl state vectors

$y_k(i)$ is a vector of the observations of the kth controller at time i and is $\ell_k \times 1$

$u_j(i)$ is a $p_j$xl vector of the jth controller at time i

$w(i)$ is the noise associated with the dynamics

$v_k(i)$ is the noise associated with the observers.

It is assumed that the dynamics are known at all control stations. Thus each controller can use A, B_j, and C_k $1 \le j \le q$, $1 \le k \le q$ as known quantities.

Further, there are several assumptions on the random variables:

i) $v_{k_1}(i)$ and $v_{k_2}(t)$ are independent if $k_1 \neq k_2$ or $i \neq t$

ii) w is independent of v_k for all k

iii) $x(o)$ is independent of w and v_k for all k

iv) w and v_k are zero mean Gaussian white noise processes

v) The second moment of the noise $w(i)$ and of the noise $v_k(i)$ is known at all stations for all times i

We further impose a certain structure on the information pattern:

vi) If control station k_1 communicates to station k_2, then k_1 will transmit to k_2 at time i its whole observation vector y_{k_1}

vii) Two control stations either communicate for all time or never communicate

(The underlying assumption here is that the cost of communication is incurred primarily from setting up the communication link, not in using it.)

Finally,

The Control Laws are of the following form:

viii) At each time t each control station has available a subset of $y_k(t)$ $k=1,2,\ldots,q$

ix) The control has to be a linear combination of the above data basis.

The cost is quadratic in form with weighting matrices R, Q_i, and $\Delta(i)$ with $R=R^T$ and $Q_i=Q_i^T$

$$J=E\ \{\sum_{i=0}^{n-1} x^T(i)Rx(i) + u_1^T(i)Q_1u(i)\}+ \dots$$

$$+ u_q^T(i)Q_qu_q(i) + \sum_{i=1}^{N-1} tr[\alpha\Delta^T(i)] \tag{1.2.3}$$

where $\alpha_{ij} = \begin{cases} 1 \text{ if station } i \text{ sends information to station } j \\ 0 \text{ otherwise} \end{cases}$

$\Delta_{ij}(t)$ is the cost of communication between i and station j, at time t

The problem is simply to find the control u(i) and the information pattern α that will minimize the above cost for the given dynamics.

1.3 Reduction to a Two Point Boundary Value Problem

In this section the problem described in Section 1.2 will be reduced to the solution of a two point boundary value problem through the use of the maximum principle.

In order to simplify notation, a number of matrices and vectors will be defined here:

$Y(t) \triangleq (y_1^T(t)\ y_2^T(t)\ \dots\ y_q^T(t))^T$ is the $\ell \times 1$ (where $\ell = \sum_{j=1}^{q} \ell_j$) vector made up of the concatonation of all observation vectors

$U(t) \triangleq (u_1^T(t)\ u_2^T(t)\ \dots\ u_q^T(t))^T$ is the pxl (where $p=\sum_{j=1}^{q} p_j$) vector made up of the concatonation of all control vectors

and, simlarly, for $C_i, v_i,$ and B_i:

$C \triangleq (C_1^T C_2^T\ \dots\ C_q^T)^T$ and the dimension of C is ℓxn

$B \triangleq (B_1 B_2\ \dots\ B_q)$ the dimension of B is nxp

$v_o(t) \triangleq (v_1^T(t)\ v_2^T(t)\ \dots\ v_q^T(t))^T$ the dimension of v_o is ℓx1

The pxp matrix determining the weighting for the cost due to the controllers is the block diagonal matrix Q where

$$Q = \begin{pmatrix} Q_1 & o & \dots & \dots & o \\ o & Q_2 & & & o \\ o & & \ddots & & \vdots \\ \vdots & & & \ddots & o \\ o & o & \dots & o & Q_q \end{pmatrix}$$

Finally, the following matrices will define the second moment of the noise:

$W(t)_{nxn} \triangleq E\{w(t)w^T(t)\}$ is the second moment of the noise in the state equation

$V(t)_{\ell x \ell} \triangleq E\{V_o(t)V_o^T(t)\}$ is the second moment of the noise due to observation..

Note that

$$V(t) = E\begin{pmatrix} V_1(t)V_1^T(t) & o & \cdots & o \\ o & \ddots & & \vdots \\ \vdots & & \ddots & o \\ o & o \cdots & o & V_q(t)V_q^T(t) \end{pmatrix} \text{ since } E\{V_i(t)V_j^T(t)\}=0$$

by assumption (i) and (iv) section 1.2

With the above definitions, the system dynamics become:

$$x(i+1) = Ax(i)+w(i)+BU(i) \tag{1.3.1}$$

$$Y(i) = Cx(i)+V_o(i) \tag{1.3.2}$$

And the cost is

$$J = E\{\sum_{i=0}^{N-1} x^T(i)Rx(i)+U^T(i)QU(i)\}+\sum_{i=0}^{N-1} tr(\alpha\Delta^T(i)) \tag{1.3.3}$$

Now in section (1.2) the admissible control laws at time i were constrained to be a linear combination of the observations at time i of some subset of observers. This subset is to be determined by the information pattern, α. Therefore, the control law at time i is in fact

$$U(i) = g(i)Y(i) \tag{1.3.4}$$

Where g(i) is a $p \times \ell$ matrix, the jk^{th} block of which is dimension $p_j \times \ell_k$ and is described by:

$$[g(i)]^{jk} = \begin{cases} [o] \text{ if } \alpha_{kj} = 0 \\ [G(i)]^{jk} \text{ if } \alpha_{kj} \neq 0 \end{cases} \tag{1.3.5}$$

and $[G(i)]^{jk}$ is some yet to be determined matrix

Equation (1.3.5) simply enforces the constraints due to the information pattern. That is, if α is such that the k^{th} station does not communicate to the j^{th} station, then the appropriate block of g(i) is forced to zero by equation (1.3.5)

Since $[g(i)]^{jk} = 0$ for $\alpha_{kj} = 0$, g(i) is equivalently given by the $p \times \ell$ matrix the jk^{th} block of which is

$$[g(i)]^{jk} = \alpha_{kj}^{kj} [G(i)]^{jk} \tag{1.3.6a}$$

For instance, for the case in which there are exactly 2 controllers u_1, u_2 and 2 observers y_1, y_2 and they are all scalars, equation (1.3.4) becomes

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \alpha_{11} G^{11} & \alpha_{21} G^{12} \\ \alpha_{12} G^{21} & \alpha_{22} G^{22} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

where G^{11}, G^{12}, G^{21}, G^{22} are all scalars.

This case will be explored in greater detail in Chapter 3 of this thesis.

There is, however, a more desireable expression for g(i), but in order to use it, some definitions will have to be made:

Let γ_k be a symmetric pxp matrix such that the ms^{th} block of γ_k is given by

$$[\gamma_k]^{ms} \triangleq \begin{cases} I_{p_k x p_k} & \text{if } m=s=k \\ 0_{p_m x p_s} & \text{else} \end{cases}$$

where I is the identity matrix and 0 is the zero matrix

Similarly, let δ_j be a symmetric $\ell \times \ell$ matrix such that the ms^{th} block of δ_j is given by:

$$[\delta_j]^{ms} \triangleq \begin{cases} I_{\ell_j x \ell_j} & \text{if } m=s=j \\ 0_{\ell_m \times \ell_s} & \text{else} \end{cases}$$

Now, an alternative formulation for g(i), equivalent to (1.3.6a) is

$$g(i) = \sum_{k=1}^{q} \sum_{j=1}^{q} \alpha_{jk}\gamma_k G(i)\delta_j \tag{1.3.6b}$$

Notice that the constant coefficients γ_k and δ_j simply force the appropriate blocks of G(i) to zero as in equation (1.3.6a).

At this point a more useful expression for the cost than equation (1.3.3) can be found.

By substitution of equation (1.3.2) into (1.3.4),

$$U(i) = g(i)Cx(i)+g(i)V_o(i) \tag{1.3.7}$$

And substituting (1.3.7) into (1.3.3) to determine the cost,

$$J = E\{\sum_{i=0}^{N-1} x^T(i)Rx(i)+[g(i)Cx(i)+g(i)V_o(i)]^T Q[g(i)Cx(i)+g(i)V_o(i)]\}+$$

$$+ \sum_{i=0}^{N-1} tr(\alpha\Delta^T(i))$$

Regrouping terms,

$$J = E\{\sum_{i=0}^{N-1} x^T(i)[R+C^Tg^T(i)Qg(i)C]x(i)+x^T(i)[C^Tg^T(i)Qg(i)]V_o(i)+$$

$$+ V_o^T(i)[g^T(i)Qg(i)C]x(i)+V_o^T[g^T(i)Qg(i)]V_o(i)\}+\sum_{i=0}^{N-1} tr(\alpha\Delta^T(i))$$

By Lemma A.1 of the Appendix,

$$E\{x^T(i)[C^Tg^T(i)Qg(i)]V_o(i)\}=0 \ , \ i=0,1, \ \ldots \ N-1.$$

and $E\{V_o^T(i)[g^T(i)Qg(i)C]x(i)\}=0$, $i= 0, \ \ldots \ N-1.$

Thus,

$$J = E\{\sum_{i=0}^{N-1} x^T(i)[R+C^Tg^T(i)Qg(i)C]x(i)+V_o^T(i)[g^T(i)Qg(i)]V_o(i)\}+$$

$$+ \sum_{i=0}^{N-1} tr(\alpha\Delta^T(i)) \tag{1.3.8}$$

Note, however that $x^T(i)[R+C^Tg^T(i)Qg(i)C]\ x(i)$ is a scalar.

$$x^T(i)[R+C^Tg^T(i)Qg(i)C]x(i) = tr\ x^T(i)[R+C^Tg^T(i)Qg(i)C]x(i)$$

$$=tr\{[R+C^Tg^T(i)Qg(i)C]x(i)x^T(i)\} \tag{1.3.9}$$

By a similar argument,

$$V_o^T(i)[g^T(i)Qg(i)]V_o(i) = tr[g^T(i)Qg(i)]V_o(i)V_o^T(i) \tag{1.3.10}$$

Substituting (1.3.9) and (1.3.10) into (1.3.8),

$$J = E\{\sum_{i=0}^{N-1} tr[R+C^Tg^T(i)Qg(i)C]x(i)x^T(i)\}+$$

$$+E\{\sum_{i=0}^{N-1} tr[g^T(i)Qg(i)]V_o(i)V_o^T(i)\}+\sum_{i=o}^{N-1} tr(\alpha\Delta^T(i)) \tag{1.3.11}$$

By commutativity of the trace with expectation

$$J = \sum_{i=0}^{N=1} \mathrm{tr}E\{[R+C^T g^T(i)Qg(i)C]x(i)x(i)^T\}+$$

$$+\mathrm{tr}E\{g^T(i)\,Qg(i)V_o^T(i)V_o^q\}+\mathrm{tr}\alpha\Delta^T(i). \tag{1.3.12}$$

By lemma A.2 of the Appendix,

$$J = \sum_{i=0}^{N-1} \mathrm{tr}\{[R+C^T g^T(i)Qg(i)C]P(i)\}+\mu_x^T(i)[R+C^T g^T(i)Qg(i)C]\mu_x(i)+$$

$$+\ \mathrm{tr}\{[g^T(i)Qg(i)]V(i)\}+\mu_v^T(i)[g^T(i)\ [g^T(i)[Qg(i)]\mu_v(i)+\mathrm{tr}\alpha\Delta^T(i) \tag{1.3.13}$$

However, by hypothesis, $\mu_v(i)=0$

Thus,

$$J = \sum_{i=0}^{N-1} \mathrm{tr}[R+C^T g^T(i)Qg(i)C]P(i)+\mu_x^T(i)[R+C^T g^T(i)Qg(i)C]\mu_x(i)+$$

$$+\mathrm{tr}[g^T(i)Qg(i)]V(i)+\mathrm{tr}\alpha\Delta^T(i) \tag{1.3.14}$$

This is the desired expression for the cost that must be optimized over all possible α, and $g(i)$

Note that (1.3.14) expresses the cost as a function of $\mu(i)$, $P(i)$, $g(i)$, α, and known quantities. So, now a recursive expression for the variables $P(i)$ and $\mu(i)$ will be derived.

Consider the dynamics of (1.3.1)

$$x(i+1) = Ax(i)+w(i)+BU(i)$$

Substituting equation (1.3.7) into (1.3.1),

$$x(i+1) = Ax(i)+w(i)+B[g(i)Cx(i)+g(i)V_o(i)]$$

Rearranging terms,

$$x(i+1) = [A+Bg(i)C]x(i)+w(i)+Bg(i)V_o(i) \quad (1.3.15)$$

To simplify notation, we define

$$M(i) \triangleq A+Bg(i)C \quad (1.3.16)$$

Thus,

$$x(i+1) = M(i)x(i)+w(i)+Bg(i)V_o(i) \quad (1.3.17)$$

Taking the expected value of both sides,

$$E\{x(i+1)\} \triangleq \mu_x(i+1) = M(i)\mu_x(i)+E\{w(i)\}+Bg(i)E\{V_o(i)\}$$

However, as the expected value of $w(t)$ and $V_o(t)$ is zero for all t,

$$\mu_x(i+1) = M(i)\mu_x(i) \quad (1.3.18)$$

Now, to develop a similar sort of recurrsion for $P(t)$, subtract $\mu_x(i+1)$ from both sides of equation (1.3.17)

$$x(i+1)-\mu_x(i+1) = M(i)x(i)-\mu_x(i+1)+w(i)+Bg(i)V_o(i) \quad (1.3.19)$$

And, substituting for $\mu_x(i+1)$ in the right hand side of equation (1.3.19) using equation (1.3.18),

$$x(i+1)-\mu_x(i+1) = M(i)[x(i)-\mu_x(i)] + w(i)+Bg(i)V_o(i) \tag{1.3.20}$$

By definition,

$$P_x(i+1) = E\{[x(i+1)-\mu_x(i+1)][x(i+1)-\mu_x(i+1)]^T$$

Thus,

$$P_x(i+1) = E\{[M(i)(x(i)-\mu_x(i))+w(i)+Bg(i)V_o(i)]$$

$$[M(i)(x(i)-\mu_x(i))+w(i)+Bg(i)V_o(i)]^T\} \tag{1.3.21}$$

Expanding this expression,

$$P_x(i+1) = E\{M(i)(x(i)-\mu_x(i) \quad x(i)-\mu_x(i))^T M^T(i)+w(i)w^T(i)+$$

$$+Bg(i)V_o(i)V_o^T(i)g^T(i)B^T\} \tag{1.3.22}$$

This is true by the independence of x(i), w(i), and $V_o(i)$, and the assumptions that $V_o(i)$ and w(i) have zero means,

And so,

$$P_x(i+1) = M(i)P_x(i)M^T(i)+W(i)+Bg(i)V(i)g^T(i)B^T \tag{1.3.23}$$

This is the desired recursion for P(i).

Equations (1.3.18), (1.3.23) and (1.3.14) provide a convenient reformulation of original problem:

$$\text{Minimize} \sum_{i=0}^{N-1} \text{tr}[R+C^T g^T(i)Qg(i)C]P(i) + \mu^T(i)[R+C^T g^T(i)Qg(i)C]\mu(i) +$$

$$+\text{tr}[g^T(i)Qg(i)]V(i) + \text{tr}\,\alpha\Delta^T(i) \qquad (1.3.23a)$$

subject to:

$$\mu(i+1) = M(i)\mu(i)$$

and $$P(i+1) = M(i)P(i)M^T(i) + W(i) + Bg(i)V(i)g^T(i)B^T$$

The remainder of this section is centered around this optimization problem and its solution (basically) the application of the minimum principle. In order to do so, however, it will be expedient to appropriately define the setting for the problem.

To this end, we define elements of a vector space $\mathcal{T}$ over the field of real numbers R by

$$T_j = \begin{pmatrix} \mu_j \\ P_j \end{pmatrix}$$

where μ_j is an n x 1 vector and P_j is an n x n matrix

Define now vector addition by

$$T_1+T_2 = \begin{pmatrix} \mu_1 \\ P \end{pmatrix} + \begin{pmatrix} \mu_2 \\ P_2 \end{pmatrix} \triangleq \begin{pmatrix} \mu_1 + \mu_2 \\ P_1 + P_2 \end{pmatrix}$$

where $\mu_1 + \mu_2$ is the usual element by element sum of two n x 1 vectors and $P_1 + P_2$ is the usual element by element sum of two n x n matrices. The above definition of vector addition has the following properties:

i) If $T_1 \varepsilon \mathcal{T}$ and $T_2 \varepsilon \mathcal{T}$ then $T_1+T_2 \varepsilon \mathcal{T}$ (closure)

ii) $T_1+T_2 = T_2 + T_1 \quad T_1, T_2 \varepsilon \mathcal{T}$ (commutivity)

iii) $(T_1+T_2)+T_3 = T_1+(T_2+T_3) \quad T_1, T_2, T_3 \varepsilon \mathcal{T}$ (associativity

iv) there exists a vector $\underline{0}$ such that

$T_1+ \ = T_1 \quad T_1 \varepsilon \mathcal{T}$ (identity)

v) for every vector $T_1 \varepsilon \mathcal{T}$ there exists a unique vector

$-T_1 \varepsilon \mathcal{T}$ such that $T_1+(-T_1) = o$ (additive inverse)

Now define scalar multiplication as follows:

$$aT_1 = a\begin{pmatrix}\mu_1 \\ P_1\end{pmatrix} \triangleq \begin{pmatrix}a\mu_1 \\ aP_1\end{pmatrix}$$

for $a \varepsilon R$ and $T_1 \varepsilon \mathcal{T}$

where $a\mu_1$ is the nxl vector formed by multiplying each element of μ_1 by the scalar a

and aP_1 is the nxn matrix formed by multiplying each element by P_1 by the scal r a

This definition of scalar multiplication has the following properties:

vi) for every $a \varepsilon R$ and every $T_1 \varepsilon \mathcal{T}$, $aT_1 \varepsilon \mathcal{T}$ (closure)

vii) $a(bT_1) = (ab)T_1$ for $a, b \varepsilon R$, $T_1 \varepsilon \mathcal{T}$ (associativity)

viii) $1T_1 = T_1$ for $T_1 \varepsilon \mathcal{T}$ (identity)

ix) $a(T_1+T_2) = aT_1+aT_2$ $a\varepsilon R$ $T_1,T_2 \varepsilon \mathcal{T}$ (distributivity)

x) $(a+b)T_1 = aT_1+bT_1$ $a,b,\varepsilon R$ $T_1 \varepsilon \mathcal{T}$ (distributivity)

From the above properties [(i) through (x)] it is clear that $\mathcal{T}$ is indeed a vector space over the reals.

Now, define the inner product of two vectors T_1 and T_2 denoted $<T_1,T_2>$as:

$$<T_1,T_2> \equiv < \begin{pmatrix} \mu_1 \\ P_1 \end{pmatrix}, \begin{pmatrix} \mu_2 \\ P_2 \end{pmatrix} > \stackrel{\Delta}{=} \mu_1^T \mu_2 + \mathrm{tr} P_1^T P_2$$

The above definitions may now be applied to equations (1.3.18), (1.3.23) and (1.3.14).

We define first the following functions:

$$f(i,T,g) = T(i+1)-T(i)$$

$$\text{where } T(t) = \begin{pmatrix} \mu(t) \\ P(t) \end{pmatrix}$$

For the remainder of this section we drop the subindex x of μ_x

Rewriting then eqns. (1.3.18) and (1.3.23),

$$f(i,T,g) = \begin{pmatrix} \mu(i+1)-\mu(i) \\ P(i+1)-P(i) \end{pmatrix} = \begin{pmatrix} (M(i) - I_{nxn})\mu(i) \\ M(i)P(i)M^T(i)+W(i) \end{pmatrix} + \tag{1.3.24}$$

$$+ \begin{pmatrix} 0 \\ -P(i)+Bg(i)V(i)g^T(i)B^T \end{pmatrix} \tag{1.3.25}$$

and for the cost, from eqn. (1.3.14)

$$f_o(i,T,g(i)) \triangleq -tr[R+C^Tg^T(i)Qg(i)C]P(i) +$$

$$-\mu^T(i)[R+C^Tg^T(i)Qg(i)C]\mu(i) +$$

$$-tr[g^T(i)Qg(i)]V(i)-tr\alpha\Delta^T(i) \tag{1.3.26}$$

(Notice that $\sum_{i=0}^{N-1} f_o = -\mathcal{J}$. Thus, maximizing f_o is equivalent to minimizing $\mathcal{J}$, and the maximum principle will be used)

By definition of the inner product <• , •> it is clear that the adjoint variables ("costates," Langrange multipliers) for this problem are also elements of . So let

$$\begin{pmatrix} \lambda(i) \\ \Lambda(i) \end{pmatrix} \varepsilon\ \mathcal{T}$$

be the adjoint variables in the formulation of the maximum principle. For the moment we proceed to establish the necessary conditions leaving the rigorous justification of all the steps taken for the next section of this chapter.

We define then the Hamiltonian (following [1])

$$H \triangleq f_o(i,T,g)+\left\langle \begin{matrix}\lambda(i+1)\\ \Lambda(i+1)\end{matrix}, f(i.T,g)\right\rangle \tag{1.3.27}$$

$$H = f_o(i,T,g)+\lambda^T(i+1)\ (M(i)-I)\mu(i)+$$

$$+tr\Lambda^T(i+1)[M(i)P(i)M^T(i)-P(i)+W(i)+$$

$$+Bg(i)V(i)g^T(i)B^T]$$

Note that H is a function of $\mu(i)$, $P(i)$, $\lambda(i+1)$, $\Lambda(i+1)$

and $G^{jk}(i)$ for all integers j, k such that

$\alpha_{kj} = 1 \quad o<j \leq q \;,\quad 0\leq k\leq q$

This refers to the formulation for g(i) given by

equation (1.3.6a). Specifically, H is <u>not</u> a function

of $G^{jk}(i)$ for j,k such that $\alpha_{kj} = 0$.

So, assuming that an optimal solution exists and that the constraint qualifications are satisfied, the necessary conditions for an optimum solutions are [1]:

$$\frac{\partial H}{\partial \lambda(i+1)} = \mu(i+1)-\mu(i), \frac{\partial H}{\partial \Lambda(i+1)} = P(i+1)-P(i)$$

produces the dynamics (1.3.28)

$$\frac{\partial}{\partial \mu(i+1)} = \lambda(i)-\lambda(i+1), \frac{\partial H}{\partial P(i)} = \Lambda(i)-\Lambda(i+1)$$

produces the adjoint eqns. (1.3.29)

$$\frac{\partial}{\partial G^{jk}} = 0 \quad \text{for all } j,k \text{ such that } \alpha_{kj}=1 \tag{1.3.30}$$

From equations (1.3.28) and lemma A.4 of the Appendix,

$$\frac{\partial H}{\partial \lambda(i+1)} = (M(i)-I)\mu(i) = \mu(i+1)-\mu(i) \tag{1.3.31}$$

and

$$\frac{\partial H}{\partial \Lambda(i+1)} = [M(i)P(i)M^T(i) \quad (i)+W(i)+Bg(i)V(i)g^T(i)B^T]$$

$$= P(i+1)-P(i) \tag{1.3.32}$$

These are, of course, simply the dynamic equations (1.3.24) and (1.3.25).

Now to find $\dfrac{\partial H}{\partial \mu(i)}$.

From eqn. (1.3.26) since there is only one term that is a function of $\mu(i)$,

$$\frac{\partial f_o(i,\cdot,\cdot)}{\partial \mu(i)} = \frac{\partial}{\partial \mu(i)} \left(^{-}tr\mu^T(i)\,[R+C^Tg^T(i)Qg(i)C]\mu(i)\right) \qquad (1.3.33)$$

By lemma A.3 of the appendix, and (1.3.33),

$$\frac{\partial f_o}{\partial \mu(i)} = {}^{-}[R+C^Tg^T(i)Qg(i)C]\mu(i) - [R+C^Tg^T(i)Qg(i)C]^T\mu(i) \qquad (1.3.34)$$

But Both R and Q are symmetric hy hypothesis.

Thus,

$$\frac{\partial f_o}{\partial \mu(i)} = {}^{-}2(R+C^Tg^T(i)Qg(i)C)\mu(i) \qquad (1.3.35)$$

And,

$$\frac{\partial}{\partial \mu(i)} \lambda^T(i+1)(M(i)-I)\mu(i)$$

$$= \frac{\partial}{\partial \mu(i)} (tr\,\mu\,(i)\lambda^T(i+1)(M(i)-I)) \qquad (1.3.36)$$

By lemma A.4 of the appendix, and eqn. (1.3.36),

$$\frac{\partial}{\partial \mu(i)} \lambda^T(i+1)(M(i)-I)\mu(i) = (M(i)-I)^T\lambda(i+1) \qquad (1.3.37)$$

Combining equations (1.3.35) and (1.3.37),

$$\frac{\partial H}{\partial \mu(i)} = -2(R+C^T g^T(i)Qg(i)C)\mu(i)+(M(i)-I)^T\lambda(i+1) \tag{1.3.38}$$

Similarly, for $\dfrac{\partial H}{\partial P(i)}$,

From eqn. (1.3.26)

$$\frac{\partial f_o(i,.,.)}{\partial P(i)} = \frac{\partial}{\partial P(i)}(-tr[R+C^T g^T(i)Qg(i)C]P(i)) \tag{1.3.39}$$

By lemma A.4 of the appendix, equation (1.3.39), and the symmetry of R and Q,

$$\frac{\partial f_o}{\partial P(i)} = -[R+C^T g^T(i)Qg(i)C] \tag{1.3.40}$$

Now,

$$\frac{\partial}{\partial P(i)} tr\Lambda^T(i+1)[M(i)P(i)M^T(i)-P(i)+W(i)+ +Bg(i)V(i)g^T(i)B^T] = \frac{\partial}{\partial P(i)} tr\Lambda^T(i+1)[M(i)P(i)M^T(i)-P(i)] \tag{1.3.41}$$

Due to properties of the trace operator

$$\frac{\partial}{\partial P(i)} \operatorname{tr}\Lambda^T(i+1)[M(i)P(i)M^T(i)-P(i)] =$$

$$= \frac{\partial}{\partial P(i)} \operatorname{tr}P(i)M^T(i)\Lambda^T(i+1)M(i)-\operatorname{tr}P(i)\Lambda^T(i+1) \qquad (1.3.42)$$

By lemma A4

$$\frac{\partial}{\partial P(i)} \operatorname{tr}\Lambda^T(i+1)[M(i)P(i)M^T(i)-P(i)] =$$

$$= M^T(i)\Lambda(i+1)M(i)-\Lambda(i+1) \qquad (1.3.43a)$$

so, combining equations (1.3.40) and (1.3.43a)

$$\frac{\partial H}{\partial P(i)} = -[R+C^Tg^T(i)Qg(i)C]+M^T(i)\Lambda(i+1)M(i)-\Lambda(i+1) \qquad (1.3.43b)$$

Finally, coming equations (1.3.29), (1.3.38) and (1.3.43b) to produce the adjoint equations,

$$\begin{pmatrix} \lambda(i)-\lambda(i+1) \\ \Lambda(i)-\Lambda(i+1) \end{pmatrix} = \begin{pmatrix} -2(R+C^Tg^T(i)Qg(i)C)\mu(i) \\ -(R+C^Tg^T(i)Qg(i)C) \end{pmatrix} +$$

$$+ \begin{pmatrix} (M(i) - I)^T\lambda(i+1) \\ M^T(i)\Lambda(i+1)M(i)-\Lambda(i+1) \end{pmatrix} \qquad \begin{matrix} (1.3.44) \\ (1.3.45) \end{matrix}$$

Now, all that remains is to maximize the Hamiltonian by satisfying equation (1.3.30). The Hamiltonian is a polynomial function, and thus everywhere differentiable.

Thus, equation (1.3.30) is equivalent to requiring that:

$$\sum_{k=1}^{q} \sum_{j=1}^{q} \alpha_{jk}\gamma_k \frac{\partial H}{\partial G(i)} \quad \delta j = 0 \tag{1.3.46}$$

Where here the partial differentiation is done with respect to the entire matrix G(i) and coefficients of $\frac{\partial H}{G(i)}$ simply force blocks of $\frac{\partial H}{\partial G}$ corresponding to $[G(i)]^{jk}=0$ (i.e., no communication) to zero.

The important point here is that H is in fact a function of $[G(i)]^{jk}$ j,k such that $\alpha_{kj}=1$ (see note after eqn. 1.3.27). However, if H is interpreted as a function of every element in the matrix G(i), everywhere that a term involving any $[G(i)]^{jk}$ such that $\alpha_{jk}=0$ appears the coefficient of that term will be zero, and hence the derivative of that term will be zero. In short,

$$\frac{\partial H(\mu(i),P(i),\lambda(i+1),\Lambda(i+1),\mathcal{G}(i))}{\partial[G(i)]^{jk}} = \frac{\partial H(\mu(i),P(i),\lambda(i+1),\underline{\Lambda}(i+1)g(i))}{\partial[G(i)]^{jk}}$$

where $\mathcal{G}(i) = \{[G(i)]_{jk} | \alpha_{kj}=1\}$

and so, equation (1.3.30) is equivalent to

$$\sum_{k=1}^{q} \sum_{j=1}^{q} \alpha_{jk}\gamma_k \frac{\partial}{\partial G(i)} \{H(\mu(i), P(i), \lambda(i+1), \Lambda(i+1), g(i)\}\delta_j = 0$$

Some formulas will now be derived so that the differentiation in in equation (1.3.46) can be performed in an explicit way.

We have

$$tr[g^T(i)Sg(i)T] =$$

$$= tr \sum_{k=1}^{q} \sum_{j=1}^{q} [\alpha_{jk}\gamma_k G(i)\delta_j]^T S \sum_{m=1}^{q} \sum_{s=1}^{q} \alpha_{sm}\gamma_m G(i)\delta_s\ T \tag{1.3.47}$$

$$tr[g^T(i)Sg(i)T] =$$

$$= \ tr \sum_{k=1}^{q} \sum_{j=1}^{q} \sum_{m=1}^{q} \sum_{s=1}^{q} \alpha_{jk}\alpha_{sm}\delta_j^T G^T(i)\gamma_k^T S\gamma_m G(i)\delta_s T \tag{1.3.48}$$

Let $h(G(i)) \triangleq g(i)$ by equation (1.3.6b)

Then,

$$\frac{\partial}{\partial G(i)} \ tr\ [h^T(G(i))Sh(G(i))T] =$$

$$= \sum_{k=1}^{q} \sum_{j=1}^{q} \sum_{m=1}^{q} \sum_{s=1}^{q} \alpha_{jk}\alpha_{sm} \frac{\partial}{\partial G(i)} \ tr[\delta_j^T G^T(i)\gamma_k^T S\gamma_m G(i)\delta_s T] \tag{1.3.49}$$

By properties of the trace,

$$\frac{\partial}{\partial G(i)} \operatorname{tr}[h^T(G(i))Sh(G(i))T] =$$

$$= \sum_{k=1}^{q} \sum_{j=1}^{q} \sum_{m=1}^{q} \sum_{s=1}^{q} \alpha_{jk}\alpha_{sm} \frac{\partial}{\partial G(i)} \operatorname{tr}\ [G^T(i)\gamma_k^T S\gamma_m G(i)\delta_s T\delta_j^T \qquad (1.3.50)$$

$$\frac{\partial}{\partial G(i)} \operatorname{tr}\ [h^T(G(i))Sh(G(i))T] = \sum_{k=1}^{q} \sum_{j=1}^{q} \sum_{m=1}^{q} \sum_{s=1}^{q} \alpha_{jk}\ \alpha_{sm}$$

$$\{\ [\gamma_k^T S\gamma_m]G(i)[\delta_s T\delta_j^T] + [\gamma_m^T\ S^T\gamma_k]G(i)[\delta_j T^T\delta_s^T]\} \qquad (1.3.51)$$

But γ's and δ's are symmetric matrices, so

$$\frac{\partial}{\partial G(i)} \operatorname{tr}\ [h^T(G(i))Sh(G(i))T] = \sum_{k=1} \sum_{j=1} \sum_{m=1} \sum_{s=1} \alpha_{jk}\ \alpha_{sm}$$

$$\{\gamma_k S[\gamma_m G(i)\delta_s]T\delta_j^T + \gamma_m S^T[\gamma_k G(i)\delta_j]T^T\delta_s^T\ \} \qquad (1.3.52)$$

Rearranging terms,

$$\frac{\partial}{\partial G(i)} \operatorname{tr}\ [h^T(G(i))sh(G(i)T] = \sum_{k=1}^{q} \sum_{j=1}^{q} \sum_{m=1}^{q} \sum_{s=1}^{q} \alpha_{jk}\gamma_k S$$

$$\alpha_{sm}[\gamma_m G(i)\delta_s]\ T\delta_j^T + \alpha_{sm}\gamma_m S^T\{\alpha_{jk}[\gamma_k G(i)\delta_j]\}T^T\delta_s^T$$

or,

$$\frac{\partial}{\partial G(i)} \text{tr } [h^T(G(i))Sh(G(i))T] =$$

$$= \sum_{k=1}^{q} \sum_{j=1}^{q} \alpha_{jk}\gamma_k S \sum_{m=1}^{q} \sum_{s=1}^{q} \alpha_{sm}[\gamma_m G(i)\delta_s] \; T\delta_j^T \; +$$

$$+ \sum_{m=1}^{q} \sum_{s=1}^{q} \alpha_{sm}\gamma_m S^T \sum_{k=1}^{q} \sum_{j=1}^{q} \alpha_{jk}[\gamma_k G(i)\delta_j] \; T^T\delta_s^T \qquad (1.3.53)$$

Now, substituting into equation (1.3.53) with (1.3.6b)

$$\frac{\partial}{\partial G(i)} \text{tr}[h^T(G(i))Sh(G(i))T] = \sum_{k=1}^{q} \sum_{j=1}^{q} \alpha_{jk}\gamma_k Sg(i)T\delta_j^T \; +$$

$$+ \sum_{m=1}^{q} \sum_{s=1}^{q} \alpha_{sm}\gamma_m S^T g(i)T^T\delta_s^T \qquad (1.3.54)$$

And, finally:

$$\frac{\partial}{\partial G(i)} \text{tr } [h^T(G(i))Sh(G(i))T] =$$

$$= \sum_{k=1}^{q} \sum_{j=1}^{q} \alpha_{jk}\gamma_k Sg(i)T\delta_j^T + \alpha_{jk}\gamma_k S^T g(i)T^T\delta_j^T \qquad (1.3.55)$$

This is one of the necessary formulas to compute the maximum of the Hamiltonian.

It will also be necessary to compute the value of

$$\frac{\partial}{\partial G(i)} \operatorname{tr}[h(G(i))]^T S = \frac{\partial}{\partial G}\{\operatorname{tr}\sum_{k=1}^{q}\sum_{j=1}^{q}\alpha_{jk}[\gamma_k G(i)\delta_j]^T S\} \tag{1.3.56}$$

Interchanging the order of operations,

$$\frac{\partial}{\partial G(i)} \operatorname{tr}[h(G(i))]^T S = \sum_{k=1}^{q}\sum_{j=1}^{q}\alpha_{jk}\frac{\partial}{\partial G(i)} \operatorname{tr}[\delta_j G^T(i)\gamma_k]S \tag{1.3.57}$$

$$\frac{\partial}{\partial G(i)} \operatorname{tr}[h(G(i))]^T S = \sum_{k=1}^{q}\sum_{j=1}^{q}\alpha_{jk}\frac{\partial}{\partial G(i)} \operatorname{tr}[G^T(i)\gamma_k S\delta_j] \tag{1.3.58}$$

By lemma A.4 of the Appendix, and (1.3.58),

$$\frac{\partial}{\partial G(i)} \operatorname{tr}[h(G(i))]^T S = \sum_{k=1}^{q}\sum_{j=1}^{q}\alpha_{jk}\gamma_k S\delta_j \tag{1.3.59}$$

Similarly, using the second part of lemma A.4,

$$\frac{\partial}{\partial G(i)} \operatorname{tr}[h(G(i))]S = \sum_{k=1}^{q}\sum_{j=1}^{q}\alpha_{jk}\gamma_k S^T\delta_j \tag{1.3.60}$$

Equations (1.3.55), (1.3.59), and (1.3.60) are the formulas necessary to compute the derivation of the Hamiltonian, which we now proceed to do.

Consider first

$$\frac{\partial f_o}{\partial G(i)} = \frac{\partial}{\partial G(i)} \{-tr[R+C^T h^T(G(i))Qh(G(i))C]P(i) +$$

$$- tr\mu^T(i)[R+C^T h^T(G(i))Qh(G(i))C]\mu(i) +$$

$$- tr[h^T(G(i))Qh(G(i))V(i)]\} \tag{1.3.61}$$

Keeping only terms involving G(i)

$$\frac{\partial f_o}{\partial G(i)} = \frac{\partial}{\partial G(i)} \{-tr[h^T(G(i))Qh(G(i))CP(i)C^T] +$$

$$+ -tr[h^T(G(i))Qh(G(i))C\mu(i)\mu^T(i)C^T] +$$

$$- tr[h^T(G(i))Qh(G(i))V(i)]\} \tag{1.3.62}$$

From equation (1.3.55) and the symmetry of Q, P(i) and V(i)

$$\frac{\partial f_o}{\partial G(i)} = -2 \sum_{k=1}^{q} \sum_{j=1}^{q} \alpha_{jk}\gamma_k Qg(i)CP(i)C^T\delta_j +$$

$$+ \alpha_{jk}\gamma_k Qg(i)C\mu(i)\mu^T(i)C^T\delta_j +$$

$$+ \alpha_{jk}\gamma_k Qg(i)V(k)\delta_j \tag{1.3.63}$$

Rearranging terms,

$$\frac{\partial f_o}{\partial G(i)} = -2 \sum_{k=1}^{q} \sum_{j=1}^{q} \alpha_{jk}\gamma_k Qg(i)[CP(i)C^T +$$

$$+ C\mu(i)\mu^T(i)C^T+V(i)]\delta_j$$

Differentiating the rest of the Hamiltonian with respect to G(i),

From equations (1.3.24) and (1.3.25),

$$<\left| \begin{matrix} \lambda(i+1) \\ \Lambda(i+1), \end{matrix} \right. f> = \lambda^T(i+1)(M(i)-I)\mu(i) +$$

$$+tr\Lambda^T(i+1)[M(i)P(k)M^T(i) +$$

$$- P(i)+W(i)+Bg(i)V(i)g^T(i)B^T] \qquad (1.3.64)$$

Thus, eliminating terms not functions of G(i),

$$\frac{\partial}{\partial G(i)} < \begin{matrix} \lambda(i+1) \\ \Lambda(i+1), \end{matrix} f> = \frac{\partial}{\partial G(i)}\{\lambda^T(i+1)Bh(G(i))C\mu(i) +$$

$$+ tr(\Lambda^T(i+1)[(A+Bh(G(i))C)\ P(i)(A+Bh(G(i))C)^T+$$

$$+ Bh(G(i))V(i)h^T(G(i)B^T]\} \qquad (1.3.65)$$

Rearranging terms,

$$\frac{\partial}{\partial G(i)} <\begin{pmatrix}\lambda(i+1)\\ \Lambda(i+1)\end{pmatrix}, f > = \frac{\partial}{\partial G(i)} \{ tr[h(G(i))C\mu(i)\lambda^T(i+1)B] +$$

$$+ tr\ h^T(G(i)B^T\Lambda^T(i+1)AP(i)C^T +$$

$$+ tr\ h^T(G(i))B^T\Lambda^T(i+1)Bh(G(i))CP(i)C^T +$$

$$+ tr\ h\ (G(i))CP(i)A^T\Lambda^T(i+1)B +$$

$$+ tr\ h^T(G(i))B^T\Lambda^T(i+1)Bh(G(i))V(i) \qquad (1.3.66$$

By equations (1.3.55), (1.3.59) and (1.3.60),

$$\frac{\partial}{\partial G(i)} < \begin{pmatrix}\lambda(i+1)\\ \Lambda(i+1)\end{pmatrix}, f > =$$

$$= \sum_{k=1}^{q} \sum_{j=1}^{q} \alpha_{jk}\gamma_k [B^T\lambda(i+1)\mu^T(i)C^T + B^T\Lambda^T(i+1)AP(i)C^T$$

$$+ B^T\lambda(i+1)AP(i)C^T]\delta_j +$$

$$+ \sum_{k=1}^{q} \sum_{j=1}^{q} \alpha_{jk}\gamma_k [B^T\Lambda^T(i+1)Bg(i)CP(i)C^T +$$

$$+ B^T\Lambda(i+1)Bg(i)CP(i)C^T +$$

$$+ B^T\Lambda^T(i+1)Bg(i)V(i) + B^T\Lambda(i+1)Bg(i)V(i)]\delta_j \qquad (1.3.67)$$

Simplifying,

$$\frac{\partial}{\partial G(i)} < \begin{pmatrix}\lambda(i+1)\\ \Lambda(i+1)\end{pmatrix}, f > = \sum_{k=1}^{q} \sum_{j=1}^{q} \alpha_{jk}\gamma_k \{B^T\lambda(i+1)\mu^T(i)C^T +$$

$$+ B^T[\Lambda(i+1)+\Lambda^T(i+1)]AP(i)C^T +$$

$$+ B^T[\Lambda(i+1)+\Lambda^T(i+1)]Bg(i)[CP(i)C^T+V(i)]\}\ \delta_j \qquad (1.3.68)$$

So, combining equations (1.3.63) and (1.3.68),

$$\frac{\partial H}{\partial G(i)} = \sum_{k=1}^{q} \sum_{j=1}^{q} \alpha_{jk}\gamma_k \{-2Qg(i)[CP(i)C^T+C\mu(i)\mu^T(i)C^T+V(i)] +$$

$$+ B^T\lambda(i+1)\mu^T(i)C^T + B^T[\Lambda(i+1)+\Lambda^T(i+1)]AP(i)C^T +$$

$$+ B^T[\Lambda(i+1)+\Lambda^T(i+1)]\ Bg(i)[CP(i)C^T+V(i)]\}\ \delta_j \qquad (1.3.69)$$

to maximize the Hamiltonian, the requirement is not that

$\frac{\partial H}{\partial G(i)} = 0$, but it is actually given by

equation (1.3.46):

$$\sum_{k=1}^{q} \sum_{j=1}^{q} \alpha_{jk}\gamma_k \frac{\partial H}{\partial G(i)}\ \delta_j = 0$$

But, since $\gamma_{k_1} \gamma_{k_2} = \begin{cases} 0 & k_1 \neq k_2 \\ \gamma_{k_1} & k_1 = k_2 \end{cases}$

and $\delta_{j_1} \delta_{j_2} = \begin{cases} 0 & j_1 \neq j_2 \\ \delta_{j_1} & j_1 = j_2 \end{cases}$

and $(\alpha_{jk})^2 = \begin{cases} 1 \text{ if } \alpha_{jk} = 1 \\ 0 \text{ if } \alpha_{jk} = 0 \end{cases}$

Substituting (1.3.69) into (1.3.46),

$$\sum_{k_1=1}^{q} \sum_{j_1=1}^{q} \alpha_{j_1 k_1} \gamma_{k_1} \sum_{k_2=1}^{q} \sum_{j_2=1}^{q} \alpha_{j_2 k_2} \gamma_{k_2}$$

$$\{O(y(i), P(i), \mu(i), \Lambda(i+1)\} \; \delta_{j_2} \delta_{j_1} = 0$$

And so, equation (1.3.46) actually reduces to equation (1.3.69)

For clarity let us summarize the results so far: If an optimal control exists and appropriate conditions are satisfied, necessary conditions for optimality are provided by

Dynamic Equations:

$$\begin{pmatrix} \mu(i+1) \\ P(i+1) \end{pmatrix} - \begin{pmatrix} \mu(i) \\ P(i) \end{pmatrix} = \begin{pmatrix} M(i) - I)\mu(i) \\ M(i)P(i)M^T(i) + W(i) - P(i) + Bg(i)V(i)g^T(i)B^T \end{pmatrix} \quad \begin{matrix} (1.3.24) \\ (1.3.25) \end{matrix}$$

Adjoint Equations

$$\begin{pmatrix} \lambda(i) \\ \Lambda(i) \end{pmatrix} - \begin{pmatrix} \lambda(i+1) \\ \Lambda(i+1) \end{pmatrix} = \begin{pmatrix} -2(R+C^T g^T(i)Qg(i)C)\mu(i) \\ -(R+C^T g^T(i)Qg(i)C) \end{pmatrix} +$$

$$+ \begin{pmatrix} (M(i) - I)^T \lambda(i+1) \\ M^T(i)\Lambda(i+1)M(i) - \Lambda(i+1) \end{pmatrix} \quad \begin{matrix} (1.3.44) \\ (1.3.45) \end{matrix}$$

$$\begin{pmatrix} \mu(o) \\ P(o) \end{pmatrix} \text{ is given and } \begin{pmatrix} \lambda(N) \\ \Lambda(N) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

which are coupled via the condition to maximize the Hamiltonian,

$$\sum_{k=1}^{q} \sum_{j=1}^{q} \alpha_{jk}\gamma_k\{-2Qg(i)[CP(i)C^T + C\mu(i)\mu^T(i)C^T + V(i)] +$$

$$+ B^T\lambda(i+1)\mu^T(i)C^T + B^T[\Lambda(i+1)+\Lambda^T(i+1)]\ AP(i)C^T +$$

$$+ B^T[\Lambda(i+1)+\Lambda^T(i+1]Bg(i)[CP(i)C^T+V(i)]\}\delta_j \qquad (1.3.69)$$

From equation (1.3.45) the symmetry of R, Q and $\Lambda(N)$, it is clear that $\Lambda(i+1)$ is symmetric $i \leq N=1$

Thus,

$$\Lambda(i+1) = \Lambda^T(i+1) \qquad (1.3.70)$$

Consider now equation (1.3.45)

$$\Lambda(i) = -(R+C^Tg^T(i)Qg(i)C) + M^T(i)\Lambda(i+1)M(i)$$

Right multiplying both sides by $2\mu(i)$,

$$2\Lambda(i)\mu(i) = -2(R+C^Tg^T(i)Qg(i)C)\ \mu(i) + 2M^T(i)\Lambda(i+1)M(i)\mu(i) \qquad (1.3.71)$$

By equation (1.3.18),

$$2\Lambda(i)\mu(i) = -2(R+C^Tg^T(i)Qg(i)C)\mu(i)+2M^T(i)\Lambda(i+1)\mu(i+1) \qquad (1.3.72)$$

But, this is the same recursion in $2\Lambda(i)\mu(i)$ as equation (1.3.44) is in $\lambda(i)$; and, both have the same terminal conditions. Thus,

$$2\Lambda(i)\mu(i) = \lambda(i) \tag{1.3.73}$$

Substituting equations (1.3.70) and (1.3.73) into equation (1.3.69),

$$\sum_{k=1}^{q} \sum_{j=1}^{q} \alpha_{jk}\gamma_k\{-2Qg(i)[CP(i)C^T + C\mu(i)\mu^T(i)C^T + V(i)] + $$

$$+ 2B^T\Lambda(i+1)\mu(i+1)\mu^T(i)C^T +$$

$$+2B^T\Lambda(i+1)AP(i)C^T + 2B^T\Lambda(i+1)Bg(i)[CP(i)C^T+V(i)]\}\delta_j \tag{1.3.74}$$

And using equation (1.3.18)

$$2B^T\Lambda(i+1)\mu(i+1)\mu^T(i)C^T = 2B^T\Lambda(i+1)M(i)\mu(i)\mu^T(i)C^T \tag{1.3.75}$$

Defining

$$\Pi(i) = \mu(i)\mu^T(i) + P(i) \tag{1.3.76}$$

Substituting equations (1.3.76) and (1.3.75) into (1.3.74),

$$\sum_{k=1}^{q} \sum_{j=1}^{q} \alpha_{jk}\gamma_k\{-Qg(i)[C\Pi(i)C^T+V(i)] +$$

$$+ B^T\Lambda(i+1)A\Pi(i)C^T + B^T\Lambda(i+1)Bg(i)[C\Pi(i)C^T+V(i)]\}\ \delta_j=0 \tag{1.3.77}$$

Combining terms,

$$0 = \sum_{k=1}^{q} \sum_{j=1}^{q} \alpha_{jk}\gamma_k \{[B^T\Lambda(i+1)B-Q]g(i)[C\Pi(i)C^T+V(i)] +$$

$$+ B^T\Lambda(i+1)A\Pi(i)C^T]\delta_j \tag{1.3.78}$$

From the dynamics equations (1.3.24) and (1.3.25),

$$P(i+1)+\mu(i+1)\mu^T(i+1) = \Pi(i+1) = M(i)\Pi(i)M^T(i) +$$

$$+ W(i)+Bg(i)V(i)g^T(i)B^T \tag{1.3.79}$$

The necessary conditions for optimality have at this point been reduced to the solution of the two point boundary problem of equations (1.3.45), (1.3.79) and (1.3.78), repeated here for clarity.

$$\Lambda(i) = -(R+C^Tg^T(i)Qg(i)C) + M^T(i)\Lambda(i+1)M(i) \tag{1.3.80}$$

$$\Pi(i+1) = M(i)\Pi(i)M^T(i) + W(i) + Bg(i)V(i)g^T(i)B^T \tag{1.3.81}$$

$$\sum_{k=1}^{q} \sum_{j=1}^{q} \alpha_{jk}\gamma_k\{[B^T\Lambda(i+1)B-Q]g(i)[C\Pi(i)C^T+V(i)] +$$

$$+ B^T\Lambda(i+1)A\Pi(i)C^T\}\delta_j = 0 \tag{1.3.82}$$

With boundary conditions

$$\Pi(o) = P(o) + \mu(o)\mu^T(o) \tag{1.3.83}$$

$$\Lambda(N) = 0 \tag{1.3.84}$$

where $M(i) = A + Bg(i)C$ and $g(i)= \sum_{k=1}^{q} \sum_{j-1}^{q} \alpha_{jk}\gamma_k G(i)\delta_j$ (1.3.85)

This simplified form of the necessary condition for optimality, can be obtained in a more direct way if we introduce $\Pi(i)$ in (1.3.23a) from the beginning. This is due to the observation that the cost depends only on the quadratic combination of the two groups of $\mu(i)$, $P(i)$ which we called $\Pi(i)$ in (1.3.76). Thus the problem we have to solve is

$$\max J = \sum_{i=0}^{N-1} (-\mathrm{tr}\ [R+C^T g^T(i)Qg(i)C]\Pi(i) +$$

$$\mathrm{tr}\ [g^T(i)Qg(i)V(i)] - \mathrm{tr}\alpha\Delta^T(i)) \qquad (1.3.86)$$

over all admissible α, $g(\cdot)$,

subject to

$$\Pi(i+1) = M(i)\Pi(i)M(i)^T + W(i) + Bg(i)V(i)g^T(i)B^T \qquad (1.3.87)$$

$$\Pi(o) = P(o) + \mu(o)\mu^T(o)$$

where $M(i)$, $g(i)$ are as in (1.3.85). It is now clear that if an optimal solution exists and certain conditions are satisfied , the optimizing solution must satisfy (1.3.80)-(1.3.85).

We now proceed to provide the rigorous justification and the final results of this chapter in the next section.

1.4 Rigorous Results for O-memory Problem

We first establish existence of an optimal solution to the problem posed in Section 1.2. An optimal solution is specified by two components.

i) The information (communication) pattern α^*_{kj}

ii) The linear gains that relate o-memory linear control laws to observations $[G^*(i)]^{jk}$, $i=0,1,\ldots N-1$ which of course satisfy the information pattern constraints.

$$[G^*(i)]^{jk} = 0, \quad \text{if } \alpha^*_{kj} = 0$$

We note here that with very little additional work optimization with respect to the size of each controller's observation and control vector can be also performed, working an identical development as in this and the previous section.

It is clear that for each possible information pattern, the non zero blocks of the G(i) matrix over which we optimize are fixed. To establish then existence of a jointly optimal solution (i.e., with respect to control and information pattern) we first establish existence of an optimal solution for every fixed information pattern . Let

Γ = set of all possible information patterns

= set of all $q \times q$ matrices with elements 0 or 1 off the diagonal and 1 on the diagonal (1.4.1)

Clearly Γ is a finite set with cardinality $2^{q(q-1)}$

Given $\alpha\varepsilon\Gamma$ we want to show that the optimization problem (1.3.86) (1.3.87) where α is now fixed has a solution. Let I_α be the set of pairs of indices (j,k) $j_k\varepsilon\{1,\ldots,q\}$ such that $\alpha_{kj}=1$. Then the control laws admissible to this information pattern are as in (1.3.5). So we really have to solve for the optimal $p_j x \ell_k$ matrices $[G(i)]^{jk}$, i=0...N-1 $(j,k)\ \varepsilon I_\alpha$, where in our usual notation the super index ^{jk} indicates the (appropriate size) jk^{th} block of G(i) (which is $p \times \ell$).

From (1.3.85) and (1.3.87) it follows that $\Pi(i)$, i=0, 1, ... N-1 is a continuous function of the $[G(\ell)]^{jk}$, $(j,k)\varepsilon I_\alpha$, $\ell=0,1,\ldots i-1$, in the sense that each element of $\Pi(i)$ is a continuous function of the elements of $(G(\ell))^{jk}$, $(j,k)\ \varepsilon I_\alpha$, $\ell=0,1,\ldots i-1$. Consequently, for each admissible set of $[G(i)]^{jk}$, i=0, ..., N-1, $(j,k)\varepsilon I_\alpha$, in view of 1.3.86, J is a continuous function of the elements of those matrices (i.e., the $[G(i)]^{jk}$'s). We indicate this by writing:

$$J = F(\{[G(i)]^{jk}, i=0,\ldots,N-1, (j,k)\ \varepsilon I_\alpha\}) \tag{1.4.2}$$

where F is a continuous function

$$F : R^{n_\alpha} \rightarrow R$$

and

$$n_\alpha = \sum_{i=0}^{N-1} \sum_{\substack{j,k \\ (j,k)\varepsilon I_\alpha}} P_j \ell_k \tag{1.4.3}$$

Since $R \geq 0$, $Q_i \geq 0$ and all elements of $\Delta(i) \geq 0$, it follows from (1.3.86) that for any admissible sequence $g(i)$, $i=0, \ldots N-1$, $J \leq 0$. The sequence $g(i) = 0$ is admissible for any information matrix and gives the value

$$\begin{aligned} J_o &= F(\{[0]^{jk}, i=0, \ldots N-1, (j,k)\varepsilon I_\alpha\}) \\ &= \sum_{i=0}^{N-1} -trR\Pi(i)-tr\alpha\Delta^T(i) \end{aligned} \tag{1.4.4}$$

with

$$\begin{aligned} \Pi(i+1) &= A\Pi(i)A^T+W(i) \\ \Pi(o) &= P(o)+\mu(o)\mu(o)^T \end{aligned} \tag{1.4.5}$$

so $J<0$, and J_o is finite.

As a result for given we have

$$\mathcal{J}_o \leq \sup_{\substack{\text{over}\\ \text{all}\\ \text{adm}\\ g(\ )}} = \sup_{\substack{\text{over}\\ \text{all}\\ \text{adm.}\\ [G(\cdot)]^{jk}\\ [j,k)\varepsilon I_\alpha}} F(\cdot) \leq 0 \tag{1.4.6}$$

We have used above the result of lemma A.5 in the Appendix.

Let now

$$\Phi_\alpha = \text{ subset of } R^{n_\alpha} \text{ such that}$$

$$\mathcal{J}_o \leq F(\{[G(i)]^{jk},\ i=0,\ \ldots\ N-1,\ (j,k)\varepsilon I_\alpha\}) \leq \tag{1.4.7}$$

Φ_α is closed and bounded in view of the quadratic form of (1.3.86) (1.3.87). But now since $F(\cdot)$ is continuous on Φ_α which is closed and bounded in R^{n_α} it is well known that there exist a point in Φ_α where $F(\cdot)$ attains its maximum. That is there exists an $\omega^*_\alpha \varepsilon \Phi_\alpha$

$$\mathcal{J}_o \leq F(\omega^*_\alpha) = \max_{\varepsilon\Phi_\alpha} F(\omega) = \sup_{\omega\varepsilon\Phi_\alpha} F(\omega) \leq 0 \tag{1.4.8}$$

Then (1.4.6) and (1.4.8) establish that given α, there exist $[G^*(i)]^{jk}$, $i=0,\ \ldots\ N-1$, $(j,k)\varepsilon I_\alpha$ such that

$$F(\{[G^*(i)]^{jk},\ k=0,\ \ldots\ N=1,\ (j,k)\varepsilon I_\alpha\}) \geq$$

$$\geq F(\{\text{any other admissible } [G(i)]^{jk}\}) \tag{1.4.9}$$

We have then

Proposition 1.4.1: There is an information pattern α^*, and compatible sequence of linear gains $[G^*(i)]^{jk}$, $(j,k)\varepsilon I_\alpha{}^*$, i=0, ... N-1, which are jointly optimal for the problem posed in (1.2.3).

Proof: The arguments leading to (1.4.9) showed that for each $\alpha\varepsilon\Gamma$ there exist compatible G_α such that

$$J(\alpha G^*{}_\alpha) \geq (\alpha, \text{ any other } G \text{ compatible with } \alpha) \tag{1.4.10}$$

Let α^*, $G^*_{\alpha*}$ be defined via

$$J(\alpha^*, G^*_{\alpha_*}) = \alpha\varepsilon\Gamma \quad \max \quad (\alpha G^*_\alpha) \tag{1.4.11}$$

Since Γ is finite this establishes the result.

We have thus rigorously established existence of an optimal solution. We now proceed to justify the use of the maximum principle, which leads to the two point boundary value problem (1.3.80)-(1.3.85).

Equations (1.3.80) through (1.3.84) are in fact necessary conditions for optimality if and only if the constraint qualification is satisfied. This reduces to showing that the gradients of the constraint equation (1.3.81) are linearly independent [1].

So, we define

$$f(i+1) \triangleq \pi(i+1)-M(i)\pi(i)M^T(i)-W(i)-Bg(i)V(i)g^T(i)B^T = 0 \qquad (1.4.12)$$

and as usual $(f(i+1))_{jk}$ will denote the jk^{th} element of $f(i+1)$

It must be shown then that there exist no scalars $\lambda_{jk}(i)$

$j = 1,2,\ldots, n \quad k=1,2,\ldots, n$ such that

$$\frac{\partial}{\partial \pi(N-1)} \sum_{i=0}^{N-2} \lambda_{jk}(i+1)(f(i+1))_{jk} = \underline{0} \qquad (1.4.13)$$

$$\frac{\partial}{\partial \pi(N-2)} \sum_{i=0}^{N-2} \lambda_{jk}(i+1)(f(i+1))_{jk} = \underline{0} \qquad (1.4.14)$$

$$\vdots$$

$$\frac{\partial}{\partial \pi(1)} \sum_{i=0}^{N-2} \lambda_{jk}(i+1)(f(i+1))_{jk} = \underline{0} \qquad (1.4.15)$$

$$\frac{\partial}{\partial g(N-1)} \sum_{i=0}^{N-2} \lambda_{jk}(i+1)(f(i+1))_{jk} = \underline{0} \qquad (1.4.16)$$

$$\vdots$$

$$\frac{\partial}{\partial g(o)} \sum_{i=0}^{N-2} \lambda_{jk}(i+1)(f(i+1))_{jk} = \underline{0} \qquad (1.4.17)$$

except the trivial case $\lambda_{jk}(i+1) = 0$

for all integers j, k, i such that $1 \leq j \leq n$, $1 \leq k \leq n$,

$0 \leq i \leq N-2$

But equation (1.4.13) reduces to

$$\frac{\partial}{\partial \pi(N-1)} \quad \lambda_{jk}(N-1)(\underline{f}(N-1))_{jk} = 0 \qquad (1.4.18)$$

It is clear that the only solution to this equation is

$$\lambda_{jk}(N-1) = 0 \quad j = 1,2, \ldots, n \quad k = 1,2, \ldots, n \qquad (1.4.19)$$

Substituting equation (1.4.19) into (1.4.14) and simplifying, as above, we have

$$\frac{\partial}{\partial \pi(N-2)} \quad \lambda_{jk}(N-2)(f(N-2))_{jk} = \underline{0} \qquad (1.4.20)$$

This produces the solution

$$\lambda_{jk}\ (N-2) = 0 \qquad j = 1,2, \ldots, n \quad k = 1,2, \ldots, n \quad (1.4.21)$$

Proceeding in a similar fashion, it may be seen that the only solution to equations (1.4.13) to (1.4.17) is

$$\lambda_{jk}(i+1) = 0 \qquad j = 1,2, \ldots, n$$
$$k = 1,2, \ldots, n$$
$$i = 0,1, \ldots, N-2$$

And, so it has been shown that equations (1.3.80) through (1.3.84) constitute necessary conditions for optimality.

CHAPTER 2

2.1 Introduction

Again we consider the problem of optimizing the cost of a stochastic system over both the set of all allowable linear control laws and the set of all information patterns. However, unlike Chapter 1, here we allow the control laws to be a function of past as well as present observations. This seemingly inocuous generalization of the previous problem greatly increases the difficulty of computing a solution. The reason for this is that now cross moments of the noise and state variables appear, where they were absent before. This results in an increase in the number of states, and a corresponding increase in the number of constraint equations.

The resulting two point boundary value problem may in principle be solved; however, the computation involved is indeed quite complex.

In the classical case (that is, with one controller) the separation principle exists and a Kalman filter obviates the above complexity. This suggests the possible alternative, not examined here, of introducing a filter linearly into the control law. This would again be a sub-optimal scheme for a decentralized problem, but may simplify it.

2.2 Linear Finite Memory Control Laws

The problem considered in this chapter differs from that in Chapter 1 only in as much as each controller is now a linear combination, the previous k observations of some subset of observers, instead of being a linear combination of a subset of the present observations. Otherwise, all assumptions, statistics and dynamics are identical. They are, however, repeated here for convenience.

Dynamics

$$x(i+1) = Ax(i) + W(i) + \sum_{j=1}^{q} B_j u_j(i) \tag{2.2.1}$$

Observation model

$$y_j(i) = C_j x(i) + v_j(i) \qquad j=1,\ 2,\ .\ .\ .,\ q \tag{2.2.2}$$

where:

$x(i)$ is the $n \times 1$ state vector

$y_j(i)$ is an $\ell_j \times 1$ vector of the observations of the j^{th} controller at time i

$u_j(i)$ is a $p_j \times 1$ vector of the j^{th} controller at time i

$w(i)$ is the noise associated with the dynamics

$v_j(i)$ is the noise associated with the observations.

The dynamics are known at all stations.

Assumptions on the Random Variables:

i) $v_{j_1}(i)$ and $v_{j_2}(t)$ are independent if $j_1 \neq j_2$ or $i \neq t$

ii) w is independent of $v_j(t)$ for all j

iii) x(o) is independent of w and v_j for all j

iv) w and v_j are zero mean white noise processes

v) The second moment of the noise w(i) and of the noise $v_j(i)$ is known at all stations for all time i.

Assumptions on the Information Pattern:

vi) If control station j_1 communicates to station j_2, the j_1 will transmit at time i its whole observation vector y_{j_1}

vii) Two control stations either communicate for all time or never communicate

Assumptions on Control Laws:

viii) At each time t each control station has available a subset of $y_j(\tau)$ $j = 1, 2, \ldots, q$

where

$$0 \leq \tau \leq t \quad \text{if } t < k$$

and

$$t - k \leq \tau \leq t \quad \text{if } t \geq k$$

ix) The control is a linear combination of the above data basis.

The cost is quadratic in form with weighting matrices R, Q, and $\Delta(i)$ where $R = R^T$ and $Q = Q^T$

$$J = E\{\sum_{i=0}^{n-1} x^T(i)Rx(i) + u_1^T(i)Q_1u_.(i) + \ldots + u_q^T(i)Q_qu_q(i)\} +$$

$$+ \sum_{i=0}^{n-1} \alpha\Delta^T(i)$$

where

$$\alpha_{ij} = \begin{cases} 1 \text{ if station i sends information to station j} \\ 0 \text{ otherwise} \end{cases}$$

Δ_{ij} is the cost of communication between station i and station j

The problem is to find the control u(i) and the information pattern α that will minimize the above cost for the given dynamics.

2.3 Reduction to a Two Point Boundary Value Problem

In this section the equations necessary to solve the problem described in Section 2.2 will be derived in a manner very similar to that of Chapter 1. And, much as in Chapter 1, some definitions will first be made.

$$B \triangleq (B_1 B_2 \ \ldots, \ B_q) \qquad \text{where } B \text{ is } n \times p$$

$$U(i) \triangleq (u_1^T(i) \ u_2^T(i) \ \ldots, \ u_q^T(i))^T \quad U(i) \text{ is } p \times n$$

$$Y(i) \triangleq (y_1^T(i) \ y_2^T(i) \ \ldots, \ y_q^T(i))^T \quad Y(i) \text{ is } \ell \times 1$$

$$C \triangleq (C_1^T \ \ C_2^T \ \ldots, \ C_q^T)^T \qquad C \text{ is } \ell \times n$$

$$v(i) \triangleq (v_1^T(i) \ v_2^T(i) \ \ldots, \ v_q^T(i))^T \quad v(i) \text{ is } \ell \times 1$$

and

$$p = \sum_{j=1}^{q} p_i \quad , \quad \ell = \sum_{j=1}^{q} \ell_j$$

This produces the following dynamics and observation model:

$$x(i+1) = Ax(i) + w(i) + BU(i) \tag{2.3.1}$$

$$Y(i) = Cx(i) + v(i) \tag{2.3.2}$$

However, the case being considered is one of k-step finite memory. Thus, the control U at time i is a linear combination of the observations from time i back to time i-k of some subset of observers, as determined by α. That is,

$$U(i) = \begin{cases} g_i(i)Y(i) + g_{i-1}(i)Y(i-1) + \ldots \\ \qquad + g_{i-k}(i)Y(i-k) & \text{if } i \geq k \\ g_i(i)Y(i) + g_{i-1}(i)Y(i-1) + \ldots \\ \qquad + g_o(i)Y(o) & \text{if } i < k \end{cases} \tag{2.3.3}$$

(Note that the subscript on the matrix $g_j(i)$ refers to the time of the observation and the argument refers to the time of the application of the control.

Substituting equation (2.3.2) into equation (2.3.3)

$$U(i) = \sum_{j=[0,\ i-k]}^{i} g_j(i)[Cx(j) + v(j)] \tag{2.3.4}$$

where

$$[0,\ i-k] \triangleq \begin{cases} 0 & \text{if } 0 \geq i-k \\ i-k & \text{if } 0 < i-k \end{cases} \tag{2.3.5}$$

Consider now the cost

$$J = E \sum_{i=0}^{n-1} x^T(i)Rx(i) + U^T(i)QU(i) + \sum_{i=0}^{N-1} \mathrm{tr}\alpha\Delta^T(i) \tag{2.3.6}$$

where

$$Q \triangleq \begin{pmatrix} Q_2 & & & 0 \\ & Q_2 & & \\ & & \ddots & \\ 0 & & & Q_q \end{pmatrix} \text{ and } Q \text{ is } p \times p$$

Substituting equation (2.3.4) into equation (2.3.5),

$$J = E\{\sum_{i=0}^{N-1} x^T(i)Rx(i) + [\sum_{j=[0,i-k]}^{i} g_j(i)(Cx(j)+v(j))]^T$$

$$Q[\sum_{j=[0,i-k]}^{i} g_j(i)(Cx(j)+v(j))]\} + \mathrm{tr} \sum_{i=0}^{N-1} \alpha\Delta^T(i) \tag{2.3.7}$$

Expanding this expression,

$$J = E \sum_{i=0}^{N-1} x^T(i)Rx(i) + [\sum_{j=[0,i-k]}^{i} x^T(j)C^T g_j^T(i)]$$

$$Q[\sum_{j=[0,i-k]}^{i} g_j(i)Cx(j)] +$$

$$+ [\sum_{j=[0,i-k]}^{i} x^T(j)C^T g_j^T(i)]Q\sum_{j=[0,i-k]}^{i} g_j(i)v(j)] +$$

$$+ [\sum_{j=[0,i-k]}^{i} v^T(j)g_j^T(i)]Q[\sum_{j=[0,i-k]}^{i} g_j(i)Cx(j)] +$$

$$+ [\sum_{j=[0,i-k]}^{i} v^T(j)g_j^T(i)]Q\sum_{j=[0,i-k]}^{i} g_j(i)v(j) +$$

$$+ \sum_{i=0}^{N-1} tr\alpha\Delta^T(i) \qquad (2.3.8)$$

The next-to-last term of equation (2.3.8) has a very straightforward evaluation:

$$E = \{\sum_{j=[0,i-k]}^{i} v^T(j)g_j^T(i)]Q[\sum_{j=[0,i-k]}^{i} g_j(i)v(j)]\}$$

$$= E\{\sum_{j=[0,i-k]}^{i} \sum_{s=[0,i-k]}^{i} v^T(j)g_j^T(i)Qg_s(i)v(s)\} \qquad (2.3.9)$$

But, this is just a scalar, and due to assumption (iv),

$$E\{[\sum_{j=[0,i-k]}^{i} v^T(j)g_j^T(i)]Q[\sum_{j=[0,i-k]}^{i} g_j(i)v(j)]\}$$

$$= \sum_{j=[0,i-k]}^{i} tr\ g_j^T(i)Qg_j(i)V(j) \qquad (2.3.10)$$

Where

$$V(j) \triangleq E\ [v(j)v^T(j)] \qquad (2.3.11)$$

The other terms in equation (2.3.8) will not evaluate as easily as the one above. So, much as in the case of zero memory, recursions in new state variables will be found. To this end we define:

$$P(j,m) \triangleq E[\ x(j)x^T(m)]$$

$$S(j,m) \triangleq E[v(j)x^T(m)]$$

$$T(j,m) \triangleq E[w(j)x^T(m)]$$

$$W(j) \triangleq E[w(j)w^T(j)]$$

From equations (2.3.1) and (2.3.4)

$$x(i+1) = Ax(i) + w(i) + B \sum_{j=[0,i-k]}^{i} g_j(i)[Cx(j)+v(j)] \qquad (2.3.12)$$

So,

$$S^T(m,i+1) = AS^T(m,i) + B \sum_{j=[0,i-k]}^{i} g_j(i)\{CS^T(m,j) +$$

$$+E[v(j)v^T(m)]\} \qquad (2.3.13)$$

Further, from the assumption of independence

$$S^T(m,0) = 0 \quad \text{for all } m \qquad (2.3.14)$$

Similarly, from equation (2.3.12),

$$T^T(m,i+1) = AT^T(m,i) + E[w(i)w^T(m)] + +B\sum_{j=[0,i-k]}^{i} g_j(i)CT^T(m,j) \qquad (2.3.15)$$

Where

$$T^T(m,o) = 0 \qquad (2.3.16)$$

Now all that remains to redefine the problem is to find a recursion in the variable $P(\cdot,\cdot)$

From equation (2.3.12),

$$P(i+1,m) = E[x(i+1)x^T(m)] \qquad (2.3.17)$$

$$P(i+1,m) = E\ [Ax(i)+w(i) + B\sum_{j=[0,i-k]}^{i} g_j(i)(Cx(j)+v(j))]x^T(m) \qquad (2.3.18)$$

$$P(i+1,m) = AP(i,m) + T(i,m) + + B\sum_{j=[0,i-k]}^{i} g_j(i)[CP(j,m)+S(j,m)] \qquad (2.3.19)$$

where

$$P(0,0) = P_o \text{ (a known quantity)} \qquad (2.3.20)$$

Similarly,

$$P(m,i+1) = P(m,i)A^T+T^T(i,m) + + \sum_{j=[0,i-k]}^{i} (S^T(j,m) + P^T(j,m)C^T)g_j^T(i)B^T \qquad (2.3.21)$$

Finally, from equation (2.3.10, (2.3.8) and the definitions, the cost is

$$J = \sum_{i=0}^{N-1} \text{tr } RP(i,i) + [\text{tr}\sum_{j=[0,i-k]}^{i} \sum_{h=[0,i-k]}^{i} C^T g_j^T(i) Q g_h(i) CP(h,j)] +$$

$$+ [\text{tr}\sum_{j=[0,i-k]}^{i} \sum_{h=[0,i-k]}^{i} C^T g_j^T(i) Q g_h(i) S(h,j)] +$$

$$+ [\text{tr}\sum_{j=[0,i-k]}^{i} \sum_{h=[0,i-k]}^{i} g_j^T(i) Q g_h(i) CS^T(j,h)] +$$

$$+ [\sum_{j=[0,i-k]}^{i} \text{tr } g_j^T(i) Q g_j(i) V(j)] + \text{tr}\alpha\Delta^T(i) \qquad (2.3.22)$$

And so, the problem becomes one of minimizing equation (2.3.22) subject to equations (2.3.13) to (2.3.16), (2.3.19) and (2.3.20). Notice that equation (2.3.21) is unnecessary since it is just the transpose of (2.3.20).

It is interesting to note here that if k=0 (i.e., if the control at time i is a function of observations at time i only), then equation (2.3.22) reduces to equation (1.3.23a)

Further, since the cost is not functionally dependent on S(.,.) when k=0, the constraint equations (2.3.13) to (2.3.16) are not relevant. And finally, since for k=0 the cost is a function of $P(\tau,\tau)$, equations (2.3.19) and (2.3.21) reduce to the constraints of equation (1.3.23a).

Now, in order to define the Lagrangian, we first define

$$f_o \triangleq \text{the summand of } J \text{ in equation (2.3.22)} \tag{2.3.23}$$

$$f_1 \triangleq S^T(m,\ i+1)-AS^T(m,i) +$$

$$- B\sum_{j=[0,i-k]}^{i} g_j(i)[CS^T(m,j)+E\{v(j)v^T(m)\}] = 0 \tag{2.3.24}$$

$$f_2 \triangleq T^T(m,i+1)-AT^T(m,i)-E[w(i)w^T(m)] +$$

$$- B\sum_{j=[0,i-k]}^{i} g_j(i)CT^T(m,j) = 0 \tag{2.3.25}$$

$$f_3 \triangleq P(i+1,m) - AP(i,m)-T(i,m) +$$

$$- B\sum_{j=[0,i-k]}^{i} g_j(i)[CP(j,m)+S(j,m)] = 0 \tag{2.3.26}$$

Thus, we form the Lagrangian

$$L = \sum_{i=0}^{N-1} f_o + \sum_{i=0}^{N-1}\sum_{m=0}^{N-1} \operatorname{tr} \Lambda^T_1(m,i+1)f_1 +$$

$$+ \sum_{i=0}^{N-1}\sum_{m=0}^{N-1} \operatorname{tr} \Lambda^T_2(m,i+1)f_2 + \sum_{i=0}^{N-1}\sum_{m=0}^{N-1} \operatorname{tr} \Lambda^T_3(m,i+1)f_3 \tag{2.3.27}$$

The requirements for optimality are:

$$\frac{\partial L}{\partial P(s,t)} = 0 \tag{2.3.28}$$

$$\frac{\partial L}{\partial S(s,t)} = 0 \tag{2.3.29}$$

$$\frac{\partial L}{\partial T(s,t)} = 0 \tag{2.3.30}$$

$$\frac{\partial L}{\partial g_s(t)} = 0 \tag{2.3.31}$$

$$\frac{\partial L}{\partial \Lambda_j(s,t)} = 0 \qquad j=1,2,3 \tag{2.3.32}$$

[This is effectively the same set of requirements as those of equations (1.3.28) to (1.3.30)]

Equation (2.3.32) produces the dynamics (i.e. equations (2.3.24) to (2.3.26)

Next, consider equation (2.3.28). Since f_1 and f_2 are not functions of $P(\cdot,\cdot)$,

$$\frac{\partial L}{\partial P(s,t)} = \frac{\partial}{\partial P(s,t)} \sum_{i=0}^{N-1} f_o + \sum_{i=0}^{N-1} \sum_{m=0}^{N-1} \mathrm{tr}\Lambda_3^T (m,i+1)\, f_3 \tag{2.3.33}$$

Now, it is clear that

$$\frac{\partial J}{\partial P(s,t)} = \sum_i [C^T g_t^T(i) Q g_s(i) C]^T \quad \begin{array}{l} \text{for all i such that} \\ [0,i-k] \le s \le i \quad s \ge 0 \\ [0,i-k] \le t \le i, \ t \ge 0 \end{array} \tag{2.3.34a}$$

when $s \neq t$

$$\frac{\partial J}{\partial P(s,t)} = \sum_i [C^T g_s^T(i) Q g_s(i) C]^T + R \quad \text{for all } i \text{ such that}$$

$$\text{When } s = t \qquad [0,i-k] \leq s \leq i \qquad (2.3.34b)$$

$$\text{Let } (s,t) \triangleq \begin{cases} s & \text{if } s \leq t \\ t & \text{if } t \leq s \end{cases} \qquad (2.3.35)$$

Then,

$$\frac{\partial J}{\partial P(s,t)} = \begin{cases} \sum\limits_{i=[s,t]}^{(s,t)+k} C^T g_t^T(i) Q g_s(i) C & \text{when } t \neq s \\ (R + \sum\limits_{i=s}^{s+k} C g_s^T(i) Q g_s(i) C) & \text{when } t = s \end{cases} \qquad (2.3.36)$$

(where it is understood that $\sum\limits_{i=a}^{b} f(i) = 0$ when $a > b$)

And, from equation (2.3.26),

$$\frac{\partial}{\partial P(s,t)} \sum_{i=0}^{N-1} \sum_{m=0}^{N-1} \text{tr } \Lambda_3^T(m,i+1)\, f_3$$

$$= \Lambda_3(t,s) - [\Lambda_3^T(t,s+1)A]^T - \sum_{i=s}^{s+k} C^T g_s^T(i) B^T \Lambda_3(t,i+1) \qquad (2.3.37)$$

Combining equations (2.3.36), (2.3.37) and (2.3.28)

$$\Lambda_3(t,s) = -[\sum_{i=[s,t]}^{[s,t]+k} C^T g_t^T(i) Q g_s(i) C] + A^T \Lambda_3(t,s+1) +$$

$$+ \sum_{i=s}^{s+k} C^T g_s^T(i) B^T \Lambda_3(t,i+1) \tag{2.3.38a}$$

when $s \neq t$

$$\Lambda_3(t,s) = -[R + \sum_{i=s}^{s+k} C^T g_s^T(i) Q g_s(i) C] + A^T \Lambda_3(t,s+1) +$$

$$+ \sum_{i=s}^{s+k} C^T g_s^T(i) B^T \Lambda_3(t,i+1)$$

when $s = t$ (2.3.38b)

Next, we consider equation (2.3.29)

$$\frac{\partial J}{\partial S(s,t)} = \sum_{j=[0,i-k]}^{i} \sum_{h=[0,i-k]}^{i} \frac{\partial}{\partial S(s,t)} \{ \mathrm{tr} C^T g_j^T(i) Q g_h(i) S(h,j) +$$

$$+ \mathrm{tr}\, g_j^T(i) Q g_h(i) C S^T(j,h) \} \tag{2.3.39}$$

So,

$$\frac{\partial J}{\partial S(s,t)} = 2 \sum_{i=[s,t]}^{(s,t)+k} g_s^T(i) Q g_t(i) C \tag{2.3.40}$$

And from equations (2.3.24) and (2.3.26),

$$\frac{\partial}{\partial S(s,t)} \sum_{i=0}^{N-1} \sum_{m=0}^{N-1} \text{tr } \Lambda_1^T(m,i+1) f_1 +$$

$$+ \text{ tr } \Lambda_3^T(m,i+1) f_3 = \Lambda_1^T(s,t) - \Lambda_1^T(s,t+1)A +$$

$$- \sum_{i=t}^{t+k} [\Lambda_1^T(s,i+1) B g_t(i) C] - [\sum_{i=s}^{s+k} g_s^T(i) B^T \Lambda_3(t,i+1)] \qquad (2.3.41)$$

Combining equations (2.3.40),(2.3.41), and (2.3.29),

$$\Lambda_1^T(s,t) = [^{-}2 \sum_{i=(s,t)}^{(s,t)+k} g_s^T(i) Q g_t(i) C] + \Lambda_1^T(s,t+1)A +$$

$$+ [\sum_{i=t}^{t+k} \Lambda_1^T(s,i+1) B g_t(i) C] + [\sum_{i=s}^{s+k} g_s^T(i) B^T \Lambda_3(t,i+1) \qquad (2.3.42)$$

The above procedure must be applied to equation (2.3.30)

$$\frac{\partial L}{\partial T(s,t)} = \frac{\partial}{\partial T(s,t)} \sum_{i=0}^{N-1} \sum_{m=0}^{N-1} \{\text{tr } \Lambda_2^T(m, i+1) f_2\} +$$

$$+\{\text{tr } \Lambda_3^T(m,i+1) f_3\} \qquad (2.3.43)$$

$$\frac{\partial L}{\partial T(s,t)} = \Lambda_2^T(s,t) - \Lambda_2^T(s,t+1)A - [\sum_{i=t}^{t+k} \Lambda_2^T(s,i+1) B g_t(i) C] +$$

$$- \Lambda_3(t,s+1) \qquad (2.3.44)$$

$$\Lambda_2^T(s,t) = \Lambda_2^T(s,t+1)A + \Lambda_3(t,s+1) +$$

$$+ \sum_{i=t}^{t+k} \Lambda_2^T(s,i+1)Bg_t(i)C \qquad (2.3.45)$$

Finally, equation (2.3.31) must be used.

$$\frac{\partial J}{\partial g_s(t)} = 2[\sum_{j=[0,t-k]}^{t} Qg_j(t)CP(j,s)C^T + Qg_j(t)S(j,s)C^T +$$

$$+Qg_j(t)CS^T(s,j)] + Qg_s(t)V(s)$$

when $[0,t-k] \leq s \leq t$

$$\frac{\partial J}{\partial g_s(t)} = 0 \qquad \text{otherwise} \qquad (2.3.46)$$

And,

$$\frac{\partial J}{\partial g_s(t)} \sum_{i=0}^{N-1} \sum_{m=0}^{N-1} [\mathrm{tr}\ \Lambda_1^T(m,i+1)f_1] + [\mathrm{tr}\ \Lambda_2^T(m,i+1)f_2] +$$

$$+ \mathrm{tr}\ \Lambda_3^T(m,i+1)f_3] = -\sum_{m=0}^{N-1} B^T \Lambda_1(m,t+1)[S(m,s)C^T +$$

$$+ E\{v(m)v^T(s)\}] - \sum_{m=0}^{N-1} B^T\Lambda_2(m,t+1)T(m,s)C^T +$$

$$- \sum_{m=0}^{N-1} B^T\Lambda_3(m,t+1)\ [P(m,s)C^T+S^T(s,m)]$$

when $[0,t-k] \leq s \leq t$

$$= 0 \qquad \text{Otherwise} \qquad (2.3.47)$$

Combining equations (2.3.46) and (2.3.47),

$$0 = 2[Qg_s(t)V(s) + \sum_{j=[0,t-k]}^{t} Qg_j(t)CP(j,s)C^T +$$

$$+ Qg_j(t)S(j,s)C^T + Qg_j(t)CS^T(s,j)] +$$

$$- \sum_{m=0}^{N-1} [B^T \Lambda_1(m,t+1)S(m,s)C^T + B^T \Lambda_1(m,t+1)E\{v(m)v^T(s)\} +$$

$$+ B^T \Lambda_2(m,t+1)T(m,s)C^T + B^T \Lambda_3(m,t+1)P(m,s)C^T +$$

$$+ B^T \Lambda_3(m,t+1)S^T(s,m)]$$

$$\text{when } [0,t-k] \leq s \leq t \qquad (2.3.48)$$

So, we now have the following equations as necessary conditions for optimality:

$$S^T(m,i+1) - AS^T(m,i) - B \sum_{j=[0,i-k]}^{i} g_j(i)[CS^T(m,j) +$$

$$+ E\{v(j)v^T(m)\}] = 0 \qquad (2.3.49)$$

$$T^T(m,i+1) - AT^T(m,i) - E\{w(i)w^T(m)\} +$$

$$- B \sum_{j=[0,i-k]}^{i} g_j(i)CT^T(m,j) = 0 \qquad (2.3.50)$$

$$P(i+1,m) - AP(i,m) - T(i,m) +$$

$$- B \sum_{j=[0,i-k]}^{i} g_j(i)[CP(j,m) + S(j,m)] = 0 \qquad (2.3.51)$$

$$\Lambda_3(t,s) = - \sum_{i=[s,t]}^{(s,t)+k} C^T g_t^T(i) Q g_s(i) C] + A^T \Lambda_3(t,s+1) +$$

$$+ \sum_{i=s}^{s+k} C^T g_s^T(i) B^T \Lambda_3(t,i+1) \quad \text{when } s \neq t \qquad (2.3.52a)$$

$$\Lambda_3(t,s) = -[R + \sum_{i=s}^{s+k} C^T g_s^T(i) Q g_s(i) C] + A^T \Lambda_3(t,s+1) +$$

$$+ \sum_{i=s}^{s+k} C^T g_s^T(i) B^T \Lambda_3(t,i+1) \quad \text{when } s = t \qquad (2.3.52b)$$

$$\Lambda_1^T(s,t) = [-2 \sum_{i=[s,t]}^{(s,t)+k} g_s^T(i) Q g_t(i) C] + \Lambda_1^T(s,t+1) A +$$

$$+ [\sum_{i=t}^{t+k} \Lambda_1^T(s,i+1) B g_t(i) C] + [\sum_{i=s}^{s+k} g_s^T(i) B^T \Lambda_3(t,i+1) \qquad (2.3.53)$$

$$\Lambda_2^T(s,t) = \Lambda_2^T(s,t+1) A + \Lambda_3(t,s+1) + \sum_{i=t}^{t+k} \Lambda_2^T(s,i+1) B g_t(i) C \qquad (2.3.54)$$

$$\sum_{h=1}^{q} \sum_{r=1}^{q} \alpha_{rh}\, \gamma_h \{2[Qg_s(t)V(s) + \sum_{j=[0,t-k]}^{t} Qg_j(t)CP(j,s)C^T +$$

$$+ Qg_j(t)S(j,s)C^T + Qg_j(t)CS^T(s,j)] - \sum_{m=0}^{N-1} [B^T\Lambda_1(m,t+1)S(m,s)C^T +$$

$$+ B^T\Lambda_1(m,t+1)E\{v(m)v^T(s)\} + B^T\Lambda_2(m,t+1)T(m,s)C^T +$$

$$+ B^T\Lambda_3(m,t+1)P(m,s)C^T + B^T\Lambda_3(m,t+1)S^T(s,m)]\}\ \delta_h = 0$$

$$\text{when } [0,t-k] \leq s \leq t \qquad (2.3.55)$$

(Where the matrices α, γ, δ have been introduced into the last equation exactly as in Chapter One.)

And the initial conditions

$$S^T(m,0) = 0 \qquad (2.3.56)$$

$$T^T(m,0) = 0 \qquad (2.3.57)$$

$$P(0,0) = P_o \qquad (2.3.58)$$

$$\Lambda_j(m,N) = 0 \quad j=1,2,3 \qquad (2.3.59)$$

An important point to note about equations (2.3.52) to (2.3.55) is that the upper limit of summation may not exceed N-1 (since the state vector x is not specified at the terminal time N. As a result of this observation, applying the initial conditions to the adjoint equations will produce solutions for $\Lambda_j(s,t)$, j=1,2,3; for all s,t once $({}_r ch)$ is known. And similarly, for equations (2.3.49) to (2.3.51), once $g_r(h)$ is known, they may be integrated forwards in time.

Perhaps one of the more straightforward methods, then of solving the two point boundary value problem of equations (2.3.49) to (2.3.59) is use of the method of steepest descent [6]:

1. Pick an admissible set $g_j^o(s)$ $0 \leq j,s \leq N-1$
2. Integrate equations (2.3.49) to (2.3.51) forwards in time
3. Integrate equations (2.3.52) to (2.3.54) backwards in time
4. Consider equation (2.3.55) as an expression for the gradient. So, define $\nabla g_s(t)$ = left side of equation (2.3.55)
5. If $||\nabla g_s(t)||$ is small enough, stop.
6. Otherwise let $g_s^h(t) = \beta \nabla g_g(t) + g_s^o(t)$ and go to step 2.

On solving the problem for every allowable information pattern, the optimum pattern may be found. The above method will be discussed more thoroughly in Chapter 3. It was introduced here simply to indicate that a solution can indeed be found based on the above equation.

Finally, it should be noticed that if a delayed sharing pattern is to be introduced into the above equations, it may be done quite simply by changing the limits of summation. Specifically, the upper limits in equations (2.3.23) to (2.3.26) become τ and the lower limits become $[0, \tau-k]$ when the input at time t is a function of k observations from time τ backwards and when $\tau < t$.

CHAPTER 3

3.1 Example of a Zero Memory Problem

In this chapter we will further examine the two point boundary value problem of equations (1.3.80) to (1.3.85). As a closed form solution to these equations is not feasible, gradient methods, discussed later, are probably the most expedient means of solving them. But first the relatively simple case where:

i) the terminal time N=2

ii) the dimension of the state vector n=2

iii) the number of controllers q=2

iv) each controller is a scaler (P_j=1, j=1,2)

is considered

From equations (1.3.80) and (1.3.84),

$$\Lambda(2) = \underline{0} \tag{3.1.1}$$

$$\Lambda(1) = -(R+C^T g^T(1)Qg(1)C) \tag{3.1.2}$$

Similarly, from equations (1.3.81) and (1.3.83),

$$\pi(0) = \pi(0) \text{ known} \tag{3.1.3}$$

$$\pi(1) = M(0)\pi(0)M^T(0)+W(0)+Bg(0)V(0)g^T(0)B^T \tag{3.1.4}$$

And from equation (1.3.82),

$$\sum_{k=1}^{q} \sum_{j=1}^{q} \alpha_{jk}\gamma_k\{-Qg(1)[C\pi(1)C^T+V(1)]\}\delta_j = \underline{0} \tag{3.1.5}$$

$$\sum_{k=1}^{q} \sum_{j=1}^{q} \alpha_{jk}\gamma_k \{B^T\Lambda(1)B-Q]g(0)[C\pi(0)C^T+V(0)]+$$

$$+ B^T\Lambda(1)A\pi(0)C^T\} \delta_j = \underline{0} \tag{3.1.6}$$

Now, substituting equations (3.1.1) to (3.1.4) into equations (3.1.5) and (3.1.6)

$$\sum_{k=1}^{q} \sum_{j=1}^{q} \alpha_{kj}\gamma_k\{-Qg(1)[C(M(0)\pi(0)M^T(0) +$$

$$+ W(0)+Bg(0)V(0)g^T(0)B^T)C^T+V(1)]\} \delta_j = \underline{0} \tag{3.1.7}$$

$$\sum_{k=1}^{q} \sum_{j=1}^{q} \alpha_{kj}\gamma_k \{[-B^T(R+C^Tg^T(1)Qg(1)C)B-Q]g(0)[C\pi(0)C^T+V(0)] +$$

$$- B^T(R+C^Tg^T(1)Qg(1)C)A\pi(0)C^T\} \delta_j = \underline{0} \tag{3.1.8}$$

These equations can be solved by the following method:

1. Equation (3.1.7) always produces a solution g(1) = 0
2. Inserting this into equation (3.1.8) produces four linear equations in four unknowns which may easily be solved by well-known methods.

Consider then, the following problem

$$x(i+1) = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} x(i) + w(i) + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} U(i) \qquad (3.1.9)$$

$$Y(i) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} x(i) + V_o(i) \qquad (3.1.10)$$

where

$$P(0) = \frac{1}{2} I$$

$$\mu(0) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$W(0) = W(1) = V(0) = V(1) = I$$

$$Q = \begin{pmatrix} 0 & 0 \\ 0 & \sigma \end{pmatrix}$$

$$R = \begin{pmatrix} 1 & 0 \\ 0 & \varepsilon \end{pmatrix}$$

and I is the identity matrix

Substituting the above values into equation (3.1.7) we obtain

$$\alpha_{12}\{g_{21}(1)[4-2g_{22}(0)+2g_{21}(0)+2g_{22}^2(0)+2g_{21}^2(0)] +$$

$$+ g_{22}(1)[-1-g_{21}(0)- g_{11}(0)-2g_{11}(0)g_{21}(0)+g_{12}(0) +$$

$$- 2g_{12}(0)g_{22}(0)]\} = 0 \qquad (3.1.11)$$

$$\alpha_{22}\{g_{21}(1)[-1-g_{21}(0)-g_{11}(0)-2g_{11}(0)g_{21}(0)+g_{21}(0)-2g_{12}(0)g_{22}(0)] +$$

$$+ g_{22}(1)[3+2g_{11}(0)+2g_{11}^2(0)+2g_{12}^2(0)]\}= 0 \qquad (3.1.12)$$

where, as usual, $g_{jk}(i) = \alpha_{kj} G_{jk}$ from equation (1.3.6a)

Note that the above equations leave $g_{12}(1)$ and $g_{11}(1)$ unspecified. This is due to the choice of Q. As Q has a nonzero 2,2 element only, it costs nothing to use arbitrary inputs $g_{11}(1)$ and $g_{12}(1)$ unless they cause a change in state (see equation (1.3.86)). However, from the dynamic equations (1.3.87), it is clear that g(1) does not affect $\pi(0)$ or $\pi(1)$, and since there is no terminal weighting in the expression for the cost (i.e., no term like $X^T(N)RX(N)$), the inputs $g_{11}(1)$ and $g_{12}(1)$ may indeed be arbitrary.
To simplify the ensuing equations, we choose

$$g_{11}(1) = g_{12}(1) = 0 \qquad (3.1.13)$$

(Notice that this choice is valid for all admissible information patterns.)
There are two solutions to equations (3.1.11) and (3.1.12)

1) $g_{22}(1) = 0$, $g_{21}(1) = 0$, g(0) to be determined (3.1.14)

2) $g_{22}(1)$, $g_{21}(1)$ arbitrary.

$$[1+g_{21}(0)-g_{11}(0)\ \ g_{11}(0)g_{21}(0)-g_{12}(0)+2g_{12}(0)g_{22}(0)]^2 =$$

$$= 3+2g_{11}(0)+2g_{11}^2(0)+2g_{12}^2(0)][4-2g_{22}(0)+2g_{21}(0)+2g_{22(0)}^2+2g_{21}^2(0)]$$

(3.1.15)

Clearly for the conditions (i) to (iv) listed at the beginning of this section, there will in general be more than one solution to equation (3.1.7). The implication here is that the optimum solution may not be unique (assuming, of course, that both solutions to equation (3.1.7) are allowable. That is, for example, that equation (3.1.5) does not overspecify the problem). Since $g(1)=0$ is a reasonable choice for either (3.1.14) or (3.1.15), this is the solution we will pursue.

Thus, substituting equations (3.1.14) and (3.1.13) into equation (3.1.8),

$$\sum_{k=1}^{q} \sum_{j=1}^{q} \alpha_{kj}\gamma_k\{[B^TRB+Q]g(0)[C\pi(0)C^T+V(0)] +$$

$$- B^TRA\pi(0)C^T\}\ \delta_j = \underline{0} \qquad (3.1.16)$$

This produces the following equations:

$$\alpha_{11}[2g_{11}(0)+1] = 0 \tag{3.1.17}$$

$$\alpha_{21}[2g_{12}(0)] = 0 \tag{3.1.18}$$

$$\alpha_{12}[2(\sigma+\varepsilon)g_{21}(0)+\varepsilon] = 0 \tag{3.1.19}$$

$$\alpha_{22}[2(\sigma+\varepsilon)g_{22}(0)-\varepsilon] = 0 \tag{3.1.20}$$

We will assume that each controller communicates with its own observer at all times. That is,

$$\alpha_{11} = \alpha_{22} = 1 \tag{3.1.21}$$

This may or may not in general produce an optimum information pattern, however, it is not an unreasonable assumption from the standpoint of modeling a real system.

The solution to equations (3.1.17) to (3.1.21) is

$$g_{11}(0) = -\frac{1}{2} \quad \text{for all information patterns} \tag{3.1.22}$$

$$g_{22}(0) = \frac{\varepsilon}{2(\sigma+\varepsilon)} \quad \text{for all information patterns} \tag{3.1.23}$$

And, from equation (3.1.18),

$$g_{12}(0) = 0 \qquad \text{for all information patterns} \tag{3.1.24}$$

with $g_{21}(0)$ yet to be determined,

For ease of notation, we will define the following information patterns,

$$\alpha^1 \triangleq \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\alpha^2 \triangleq \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$\alpha^3 \triangleq \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

$$\alpha^4 \triangleq \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

Then, since $g_{12}(0) = 0$ the information patterns α^1 and α^3 will produce the same input $g(0)$ and the same dynamics $\pi(1)$. Similarly, α^2 and α^4 will produce the same input $g(0)$ and dynamics $\pi(1)$. So considering each case separately,,

$$\alpha = \alpha^1$$

$$g_{11}(0) = -\frac{1}{2}$$

$$g_{21}(0) = g_{12}(0) = 0$$

$$g_{22} = \frac{\varepsilon}{2(\sigma+\varepsilon)}$$

$$\pi(1) = \begin{matrix} \frac{7}{2} & \frac{7}{2} - \frac{3}{2} \\ \frac{-3}{2} & 3 - \frac{\varepsilon}{\sigma+\varepsilon} + \frac{3\varepsilon^2}{4(\sigma+\varepsilon)^2} \end{matrix}$$

$$J^1 = \frac{\overset{5}{\cancel{-9}}}{2} - \overset{3}{\cancel{4}}\varepsilon + \frac{\varepsilon^2}{\sigma+\varepsilon} - \frac{\cancel{3}\varepsilon^3}{2\cancel{4}(\sigma+\varepsilon)^2} - \frac{\sigma\varepsilon^2}{2(\sigma+\varepsilon)} +$$

$$- \Delta_{11}(0) - \Delta_{22}(0) - \Delta_{11}(1) - \Delta_{22}(1) \qquad (3.1.25)$$

$$\alpha = \alpha^3$$

$$J^3 = -\frac{9}{2} - 4\varepsilon + \frac{\varepsilon^2}{\sigma+\varepsilon} - \frac{3\varepsilon^3}{4(\sigma+\varepsilon)^2} - \frac{\sigma\varepsilon^2}{2(\sigma+\varepsilon)^2} +$$

$$- \Delta_{11}(0) - \Delta_{21}(0) - \Delta_{22}(0) - \Delta_{11}(1) +$$

$$- \Delta_{21}(1) + \Delta_{22}(1) \qquad (3.1.26)$$

$$\alpha = \alpha^2$$

$$g_{11}(0) = -\frac{1}{2}$$

$$g_{12}(0) = 0$$

$$g_{21}(0) = \frac{-\varepsilon}{2(\sigma+\varepsilon)}$$

$$g_{22}(0) = \frac{\varepsilon}{2(\sigma+\varepsilon)}$$

$$\pi(1) = \begin{matrix} \frac{3}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 3 - \frac{2\sigma}{\sigma+\varepsilon} + \frac{\varepsilon^2}{(\sigma+\varepsilon)^2} \end{matrix}$$

$$J^2 = \frac{-5}{2} - 3\varepsilon + \frac{\varepsilon^2}{\sigma+\varepsilon} - \frac{3}{4}\frac{\varepsilon^3}{(\sigma+\varepsilon)^2} - \frac{\varepsilon^2}{(\sigma+\varepsilon)} +$$

$$- \Delta_{11}(0) - \Delta_{21}(0) - \Delta_{22}(0) - \Delta_{11}(1) - \Delta_{21}(1) - \Delta_{22}(1)$$

(3.1.27)

$$\alpha = \alpha^4$$

$$J^4 = -\frac{5}{2} - 4\varepsilon + \frac{2\sigma\varepsilon}{\sigma+\varepsilon} - \frac{\varepsilon^3}{(\sigma+\varepsilon)^2} - \frac{\varepsilon^3\sigma}{(\sigma+\varepsilon)^2} -$$

$$- \Delta_{11}(0) - \Delta_{12}(0) - \Delta_{21}(0) - \Delta_{22}(0) +$$

$$- \Delta_{11}(1) - \Delta_{12}(1) + \Delta_{21}(1) - \Delta_{22}(1) \qquad (3.1.28)$$

where the cost has been calculated using

$$J = - \operatorname{tr} R[\pi(1)+\pi(0)] - 2\operatorname{tr} g^T(0)Qg(0) +$$

$$+ \operatorname{tr} \alpha\Delta^T(1) - \alpha\Delta^T(0) \qquad (3.1.29)$$

which is simply equation (3.1.86) after the appropriate substitutions for the above problem.

Notice then that α is never a better choice than α^1 since $\Delta_{jk}(\cdot) \geq 0$. Similarly, α^4 is never a better choice than α^2. Thus the optimum information pattern will be either α^1 or α^2.

Consider the case in which $\varepsilon = 0$. Then,

$$J^1 = -\frac{5}{2} - \Delta_{11}(0) - \Delta_{22}(0) - \Delta_{11}(1) - \Delta_{22}(1) \qquad (3.1.30)$$

$$J^2 = -\frac{5}{2} - \Delta_{11}(0) - \Delta_{21}(0) - \Delta_{22}(0) - \Delta_{11}(1) +$$

$$- \Delta_{21}(1) - \Delta_{22}(1) \qquad (3.1.31)$$

That is, the optimum cost is less with a more complicated communications structure, as long as the cost of that communication is not too great. Specifically, for $\varepsilon = 0$.

The optimum cost is J^2 if

$$\Delta_{21}(0) + \Delta_{21}(1) \leq 2 \tag{3.1.32}$$

When $\sigma = 0$, corresponding to $Q = \underline{0}$, we have

$$J^1 = -\frac{5}{2} - \frac{5}{2}\varepsilon - \Delta_{11}(0) - \Delta_{22}(0) - \Delta_{11}(1) - \Delta_{22}(1) \tag{3.1.33}$$

$$J^2 = -\frac{5}{2} - \frac{5\varepsilon}{4} - \Delta_{11}(0) - \Delta_{21}(0) - \Delta_{22}(0) +$$

$$- \Delta_{11}(1) - \Delta_{21}(1) - \Delta_{22}(1) \tag{3.1.34}$$

with a result similar to that above.

3.2 Gradient Algorithm for the Solution of the Two Point Boundary Value Problem

It is clear from the above example that for $N > 2$ the above method of solution will quickly become quite difficult, as solving for $\Lambda(\cdot)$ and $\pi(\cdot)$ in terms of $g(\cdot)$ can be a very demanding task.

So, turning now to more general methods of solution, it should first be pointed out that it is not the intent of this chapter to present the most efficient algorithm to solve equations (1.3.80) to (1.3.85) but rather to demonstrate that methods do indeed exist, and to suggest some of the alternatives. In Chapter 2, the method of steepest descent was presented.

We repeat it here for equations (1.3.80) to (1.3.85):

1) Guess an admissble $g^0(0)$
2) Using the boundary conditions (1.3.83) to (1.3.84) integrate equations (1.3.80) to (1.3.81) backwards and forwards in time, respectively.
3) Compute ∇H from equation (1.3.82)
4) If $||\nabla H||$ is small enough, stop
5) otherwise, let $g^{j+1}(0) = g^j + \beta^j(\nabla H)$
6) to to step 2.

The major advantage of this method is its simplicity. It does, however, have two drawbacks that can be quite serious. The first lies in choosing an optimum step size β. There are many schemes for this. One possibility is to continue to increase β until the Hamiltonian fails to decrease in value and let that be the choice of β. However, a more efficient method can be found inasmuch as the two-point boundary value problem is quadratic in nature. It basically involves differentiating equation (1.3.82) with respect to β (see Appendix B).

The second difficulty alluded to earlier is the convergence rate of the alogrithm. This will vary from problem to problem and will be a function of the contour of H. In the classical quadratic case, the Hamiltonian produces ellipsoids. If these are highly eccentric, convergence will be quite slow (actually, a function of the eigenvalues of the weighting matrix) [5].

So, the procedure for solving equations (1.3.80) to (1.3.86) is:

1) Select an admissible information pattern α
2) Set k=0
3) Choose values for the elements of $g^k(i)$, $g^k(1)$, . . . $g^k(N)$ that are admissible for the selected information pattern

4) Using the values chosen in step (3) [step (9)] integrate the dynamic and adjoint equations to determine $\pi^k(1)$, $\pi^k(2)$, ..., $\pi^k(N)$ and $\Lambda^k(N-1)$, $\Lambda^k(N-2)$, ..., $\Lambda^k(0)$.

5) Using equation (B.2), compute the gradient of the Hamiltonian

6) If $||(\nabla H)_k|| < \varepsilon 1$ go to step (11).

7) Using equation (B.8) compute the new stepsize

8) If $\beta < \varepsilon 2$ go to step (11)

9) Set $g^{k+1}(i) = g^k(i) + \beta_k (\nabla H)_k$ $\quad i=0,1, \ldots, N-1$

10) Go to step (4)

11) Calculate the value of the cost from equation (1.3.86)

12) Store the above results

13) If there are no other admissible information patterns to to step (15)

14) Choose a new information pattern and go to step (2)

15) Choose as optimum the set $g(0)$, $g(1)$, ..., $g(N-1)$, α that produces the smallest cost.

16) Done

To achieve faster rates of convergence, second order gradient algorithms or conjugate gradient algorithms may be employed.

Again, as the problem is basically quadratic in nature, these techniques should be quite effective. The reader is referred to [5] or almost any text on numerical optimization techniques.

APPENDICES

APPENDIX A

Definition

Let g be a p x ℓ matrix and let f map g into some scalar f(g)

Then define

$$\frac{\partial}{\partial g} f(g) \triangleq F(g)$$

where F(g) is a p x ℓ matrix such that $(F(g))_{ij} = \frac{\partial}{\partial g_{ij}} f(g)$

Lemma A.1

This Lemma follows a derivation from [2].

Let x(t) and v(t) be independent vector random variables with

$E[v(t)] = 0$ and s an unknown parameter matrix.

Then,

$$E[x^T(t)Sv(t)] = 0$$

Proof:

$$E[x^T(t)Sv(t)] = E\{[x(t)-\mu_x(t)]^T S[v(t)-\mu_v(t)] + x^T(t)S\mu_v(t) + \\ + \mu_x^T(t)Sv(t) - \mu_x^T(t)S\mu_v(t)\}$$

and since

$$E[x^T(t)S\mu_v(t)] = E[\mu_x^T Sv(t)]$$

$$E[x^T(t)Sv(t)] = E\{[x(t)-\mu_x(t)]^T S[v(t)-\mu_v(t)] + \mu_x^T(t)S\mu_v(t)\}$$

But, $x^T(t)Sv(t)$ is a scalar. So,

$$E[x^T(t)Sv(t)] = E[tr x^T(t)Sv(t)]$$
$$= E\{tr[x(t)-\mu_x(t)]^T S[v(t)-\mu_v(t)]\}$$
$$+ \mu_x^T(t)S\mu_v(t)$$

From properties of the trace,

$$E[x^T(t)Sv(t)] = E\{tr\ S[v(t)-\mu_v(t)][x(t)-\mu_x(t)]^T +$$
$$+ \mu_x^T(t)S\mu_v(t)\} =$$
$$= tr(S)E\{[v(t)-\mu_v(t)][x(t)-\mu_x(t)]^T\} + \mu_x^T(t)S\mu_v(t)$$

By hypothesis,

$\mu_v(t) = 0$, and

$v(t)$ and $x(t)$ are independent.

Thus,

$$E[x^T(t)Sv(t)] = tr(S)\ E\{[v(t)-\mu_v(t)]\}\ E\{[x(t)-\mu_x(t)]^T\} + 0$$

and since

$$E[v(t) - \mu_v(t)] = 0$$

$$E[x^T(t)Sv(t)] = 0$$

Lemma A.2

This Lemma follows a derivation in [2]

Let x(t) be a vector random variable

Then

$$E[x^T(t)Sx(t)] = \text{tr } SP_x(t)+\mu_x^T(t)S\mu_x(t)$$

where

$$\mu_x(t) = E[x(t)]$$

and

$$P_x(t) = E\{[x(t)-\mu_x(t)][x(t)-\mu_x(t)]^T\}$$

and

S an unknown parameter matrix

Proof:

$$E[x^T(t)Sx(t)] = E\{[x(t)-\mu_x(t)]^T S[x(t)-\mu_x(t)] + \\ + \mu_x^T(t)S\mu_x(t)\}$$

Since

$x^T(t)Sx(t)$ is a scalar,

$$E[x^T(t)Sx(t)] = E\{tr[x^T(t)Sx(t)]\}$$

$$E[x^T(t)Sx(t)] = E\{tr[x(t)-\mu_x(t)]^T S[x(t)-\mu_x(t)] + \mu_x^T(t)S\mu_x(t)\}$$

And, from properties of the trace

$$E[x^T(t)Sx(t)] = E\{trS[x(t)-\mu_x(t)][x(t)-\mu_x(t)]^T\} +$$

$$+ E\{\mu_x^T(t)S\mu_x(t)\}$$

Thus,

$$E[x^T(t)Sx(t)] = tr\ E\{S[x(t)-\mu_x(t)][x(t)-\mu_x(t)]^T\} + \mu_x^T(t)S\mu_x(t)$$

$$= tr\ SP_x(t) + \mu_x^T(t)S\mu_x(t)$$

Lemma A.3

Let g be a p x ℓ matrix, S be a p x p constant matrix and T be an ℓ x ℓ constant matrix.

Then,

$$\frac{\partial}{\partial g}\{tr[g^TSgT]\} = SgT + S^TgT^T$$

Proof

$$(g^TSgT)_{jk} = \sum_{n=1}^{p}(g^T)_{jn}\sum_{m=1}^{p}S_{nm}\sum_{h=1}^{\ell}g_{mh}T_{hk}$$

$$= \sum_{n=1}^{p}\sum_{m=1}^{p}\sum_{h=1}^{\ell}(g^T)_{jn}S_{nm}g_{mh}T_{hk}$$

$$tr(g^TSgT) = \sum_{k=1}^{\ell}\sum_{n=1}^{p}\sum_{m=1}^{p}\sum_{h=1}^{\ell}g_{nk}S_{nm}g_{mh}T_{hk}$$

So,

$$\frac{\partial}{\partial g_{ij}} \{tr(g^T SgT)\} = \sum_{m=1}^{p} \sum_{h=1}^{\ell} S_{im} g_{mh} T_{hj} + \sum_{k=1}^{\ell} \sum_{n=1}^{p} g_{nk} S_{ni} T_{jk}$$

$$= \sum_{h=1}^{\ell} T_{hj} \sum_{m=1}^{p} S_{im}\, g_{mh} + \sum_{k=1}^{\ell} T_{jk} \sum_{n=1}^{p} g_{nk} S_{ni}$$

$$= \sum_{h=1}^{\ell} [\sum_{m=1}^{p} S_{im} g_{mh}]\, T_{hj} + \sum_{k=1}^{\ell} [\sum_{n=1}^{p} (S^T)_{in} g_{nk}]\, (T^T)_{kj}$$

$$= (SgT)_{ij} + (S^T g T^T)_{ij}$$

Thus, from the definition of the derivative given above

$$\frac{\partial}{\partial g} \{tr[g^T SgT]\} = SgT + S^T g T^T$$

Lemma A.4

Let g be a p x ℓ martrix and S be a constant ℓ x p matrix
Then,

$$\frac{\partial}{\partial g} \{tr[gS]\} = S^T$$

and

$$\frac{\partial}{\partial g} \{tr[g^T S]\} = S$$

Proof

$$tr(gS) = \sum_{i=1}^{p} \sum_{k=1}^{\ell} g_{ik}\, S_{ki}$$

$$\frac{\partial}{\partial g_{ij}} \{ tr(gS)\} = S_{ji}$$

$$\frac{\partial}{\partial g} \{tr(gS)\} = S^T$$

Now, note that

$$\frac{\partial}{\partial g}\{tr[g^TS]\} = \frac{\partial}{\partial g}\{tr[(g^TS)^T]\} = \frac{\partial}{\partial g}\{tr(S^Tg)\}$$

$$= \frac{\partial}{\partial g}\{tr(gS^T)\}$$

So, using the first part of this lemma,

$$\frac{\partial}{\partial g}\{tr\ [g^TS]\} = S$$

Lemma A.5

Let $A \geq 0$, $B \geq 0$ both real symmetric matrices. Then

$$tr[AB] \geq 0.$$

Proof [4]

Find P orthogonal such that

$P^TAP = D = \text{diag}\ [\lambda_i]$ from [3]

Then $tr\ [AB] = tr\ [P^TABP]$

$$= tr\ [P^TAP\ P^TBP]$$

$$= tr\ [\text{diag}[\lambda_i] \cdot P^TBP]$$

$$= \sum_i \lambda_i (P^TBP)_{ii}$$

Now, since $A \geq 0$, $\lambda_i \geq 0$ and $(P^T BP)_{ii} \geq 0$ since $(P^T BP)_{ii} =$

$$= e_i^T P^T BPe_i = (Pe_i)^T BPe_i \geq 0$$

by $B \geq 0$ (here $e_i = [0 \ . \ . \ .010 \ . \ . \ . \ 0]^T$
↑ith position

So, $\text{tr} [AB] \geq \min_i \lambda_i \sum_i (P^T BP)_{ii} = (\min_i \lambda_i) \text{Tr}[B] \geq 0.$

APPENDIX B

Here we derive the optimum stepsize for a first order gradient algorithm to solve equations (1.3.80) to (1.3.86).

From equations (1.3.86) and (1.3.87) we define

$$H = tr[R+C^T g^T(i)Qg(i)C]\pi(i) - tr[g^T(i)Qg(i)V(i)] - tr\ \alpha\Delta^T(i) +$$

$$+ tr\ \Lambda^T(i+1)\ [M(i)\pi(i)M^T(i)+W(i)+Bg(i)V(i)g^T(i)B^T-\pi(i+1)] \tag{B.1}$$

And from equation (1.3.82), it may be seen that the gradient of H is

$$\nabla H = 2\sum_{k=1}^{q}\sum_{j=1}^{q}\alpha_{jk}\gamma_k\{[B^T\Lambda(i+1)B-Q]g(i)[C\pi(i)C^T+V(i)] +$$

$$+ B^T\Lambda(i+1)A\pi(i)C^T]\}\delta_j = 0 \tag{B.2}$$

Now define

$$g_{m+1} = g_m + \beta_m(\nabla H)_m \tag{B.3}$$

as the m + 1 iteration of g where β_m is the (scalar) stepsize.

In order to select an optimum stepsize, we will now substitute equations (B.2) and (B.3) into equations (B.1) and differentiate with respect to β_m. (For notational simplicity, the argument i will be left out of the next few equations. It is understood, however, to still be there.)

Thus,

$$H(g_m+\beta_m(\nabla H)_m) = - \operatorname{tr}[R+C^T(g_m+\beta_m(\nabla H)_m)^T Q(g_m+\beta_m(\nabla H)_m)C]\pi_m +$$

$$- \operatorname{tr}\ [(g_m+\beta_m(\nabla H)_m)^T Q(g_m+\beta_m(\nabla H)_m V(i)] +$$

$$- \operatorname{tr}\ \alpha\Delta^T(i) +$$

$$+ \operatorname{tr}\ \Lambda_m^T(i+1)\{[A+B(g_m+\beta_m(\nabla H)_m)C]\pi_m[C^T(g_m^T+\beta_m(\nabla H)_m^T)B^T A^T]$$

$$+ W(i) + B[g_m+\beta_m(\nabla H)_m]V(i)[g_m^T+\beta_m(\nabla H)_m^T]B^T - \pi_m(i+1) \qquad \text{(B.4)}$$

Differentiating, now, with respect to β_m,

$$0 = - \operatorname{tr}\ [C^T g_m^T Q(\nabla H)_m C+C^T(\nabla H)_m^T\ Qg_m C+2\beta_m C^T(\nabla H)_m C]\ \pi_m +$$

$$- \operatorname{tr}\ [g_m^T Q(\nabla H)_m+(\nabla H)_m^T Qg_m+2\beta_m(\nabla H)_m Q(\nabla H)_m]V(i) +$$

$$+ \operatorname{tr}\ \Lambda_m^T(i+1)\{(A+Bg_m C)\pi_m C^T(\nabla H)_m^T B^T+B(\nabla H)_m C\pi_m(C^T g_m^T B^T+A^T) +$$

$$+ 2\beta_m B(\nabla H)_m C\ {}_m[C^T(\nabla H)_m^T B^T] + Bg_m V(i)(\nabla H)_m^T B^T +$$

$$+ B(\nabla H)_m V(i) g_m^T B^T + 2\beta_m B(\nabla H)_m V(i)(\nabla H)_m^T B^T\} \qquad \text{(B.5)}$$

Combining terms involving β_m,

$$2\beta_m \{\mathrm{tr}\ C^T(\nabla H)_m^T Q(\nabla H)_m C\pi_m + \mathrm{tr}\ (\nabla H)_m^T Q(\nabla H)_m V(i) +$$

$$- \mathrm{tr}\ \Lambda_m^T(i+1)[B(\nabla H)_m C\pi_m C^T(\nabla H)_m^T B^T - B(\nabla H)_m V(i)(\nabla H)_m^T B^T]\}$$

$$= -\mathrm{tr}\ [C^T g_m^T Q(\nabla H)_m C + C^T(\nabla H)_m^T Q g_m C]\pi_m(i) +$$

$$- \mathrm{tr}\ [g_m^T W(\nabla H)_m + (\nabla H)_m^T Q g_m]V(i) +$$

$$+ \mathrm{tr}\ \Lambda_m^T(i+1)\{(A+Bg_m C)\pi_m(i)C^T(\nabla H)_m^T B^T +$$

$$+ B(\nabla H)_m C\pi_m(C^T g_m^T B^T + A^T) + Bg_m V(i)(\nabla H)_m^T B^T + B(\nabla H)_m V(i) g_m^T B^T\} \qquad (B.6)$$

And combining terms in equation (B.6)

$$2\beta_m \{ - \mathrm{tr}(\nabla H)_m^T[B^T\Lambda_m(i+1)B-Q](\nabla H)_m[C\pi_m(i)C^T+V(i)]\}$$

$$= -\mathrm{tr}\ [g_m^T(i)Q(\nabla H)_m + (\nabla H)_m^T Q g_m(i)][C\pi_m(i)C^T+V(i)] +$$

$$+ \mathrm{tr}\ B^T\Lambda_m(i+1)Bg_m[C\pi_m(i)C^T+V(i)](\nabla H)_m^T +$$

$$+ \mathrm{tr}\ B^T\Lambda_m(i+1)B(\nabla H)_m[C\pi_m(i)C^T+V(i)]g_m^T(i) +$$

$$+ \mathrm{tr}\ B^T\Lambda_m(i+1)A\pi_m(i)C^T(\nabla H)_m^T + \mathrm{tr}\ \Lambda_m^T(i+1)B(\nabla H)_m C\pi_m(i)A^T \qquad (B.7)$$

$$2\beta_m \; \{ - \text{tr} \; \{ (\nabla H)_m^T \; [B^T \Lambda_m (i+1) B - Q] (\nabla H)_m [C\pi_m (i) C^T + V(i)\} \}$$

$$= \text{tr} \; [B^T \Lambda_m (i+1) B - Q] g_m (i) [C\pi_m (i) C^T + V(i)] (\nabla H)_m^T \; +$$

$$+ \text{tr} \; [B^T \Lambda_m (i+1) B - Q] (\nabla H)_m [C\pi_m (i) C^T + V(i)] g_m^T (i) \; +$$

$$+ 2 \; \text{tr} \; B^T \Lambda_m (i+1) A\pi_m (i) C^T (\nabla H)_m^T \tag{B.8}$$

when equation (B.8) is solved explicitly for β_m, the optimal stepsize may be determined.