ENCE 353 Solutions to Midterm 2

Question 1 (15 points) Consider the cantilevered beam structure shown in Figure 1.

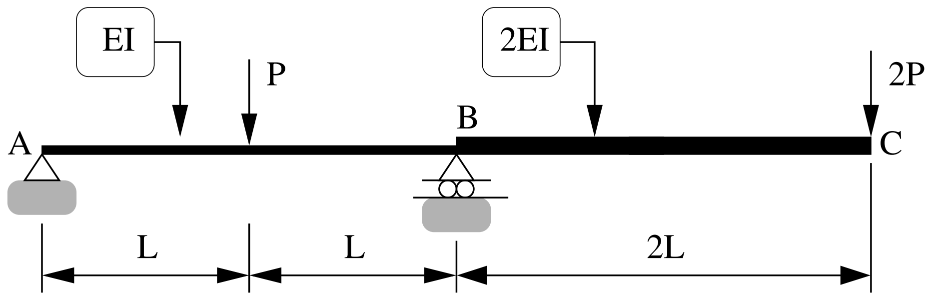


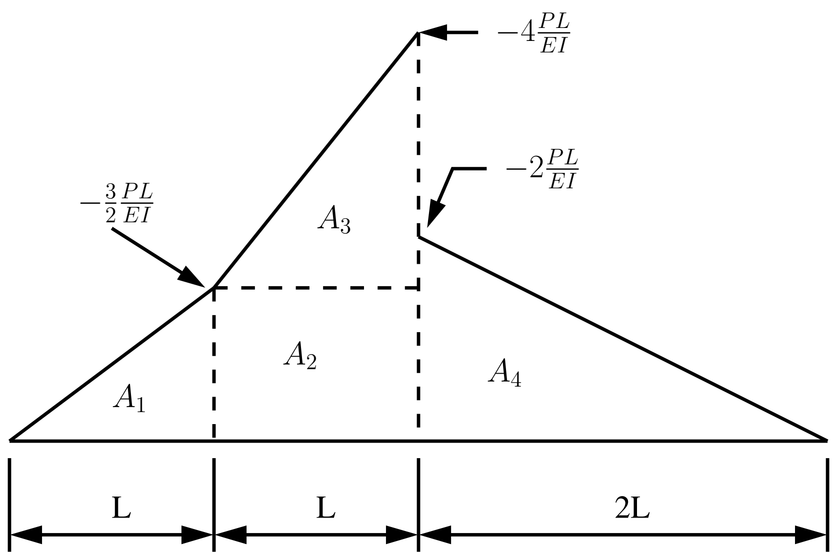
Figure 1. Front elevation view of a cantilevered beam structure.

Notice that segments A-B and B-C have cross-sectional properties EI and 2EI, respectively.

Part [1a] (3 pts) Compute and draw the M(x)/EI diagram for the complete beam A-B-C.

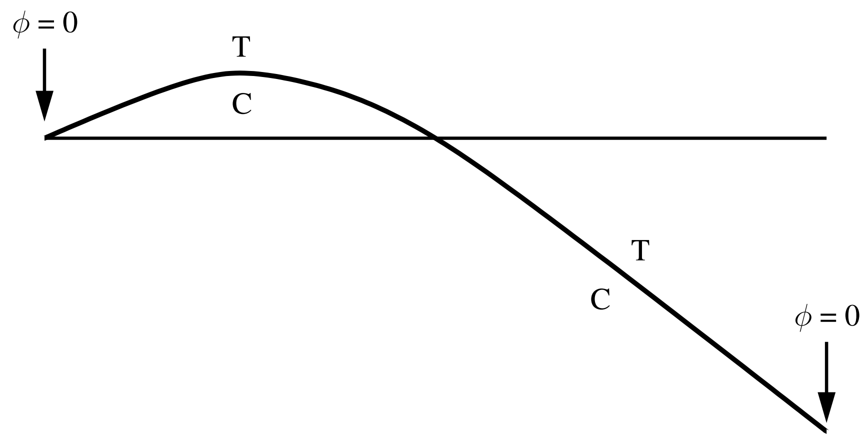
Solution: From statics:

$$V_b = \frac{9}{2}P \quad \text{and} \quad V_a = -\frac{3}{2}P. \tag{1}$$



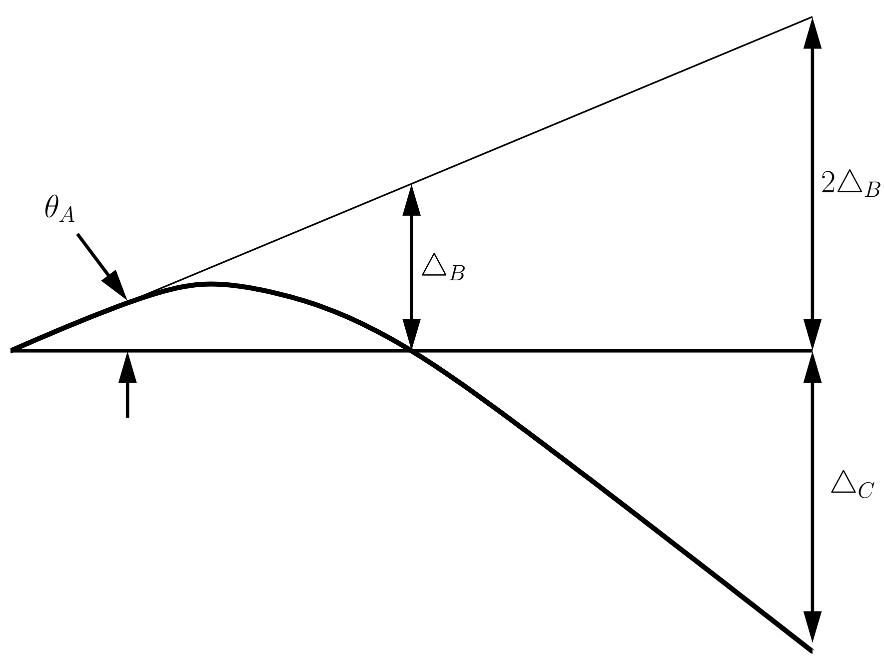
Part [1b] (4 pts) Draw and label a diagram of the deflected shape. Clearly indicate on your diagram regions (or points) of the beam having zero curvature.

Solution:



Part [1c] (4 pts) Draw and label a diagram showing how the rotation at A is related to the beam deflections at points B and C.

Solution:



Part [1d] (4 pts) Use the method of moment-area to compute the vertical deflection of the beam at point C.

Solution: From the curvature diagram:

$$A_1 = \frac{3}{2}\frac{PL}{EI}\frac{L}{2} = \frac{3}{4}\frac{PL^2}{EI}. \tag{2}$$

Similarly,

$$A_2 = \frac{3}{2}\frac{PL^2}{EI}, \quad A_3 = \frac{5}{4}\frac{PL^2}{EI}, \quad \text{and} \quad A_4 = 2\frac{PL^2}{EI}. \tag{3}$$

For the $\triangle_B$ calculation:

$$\bar{x}_1 = \frac{4}{3}L, \quad \bar{x}_2 = \frac{L}{2} \quad \text{and} \quad \bar{x}_3 = \frac{L}{3}. \tag{4}$$

Hence,

$$\triangle_B = A_1\bar{x}_1 + A_2\bar{x}_2 + A_3\bar{x}_3 = \frac{26}{12}\frac{PL^3}{EI}. \tag{5}$$

For the $2\triangle_B + \triangle_C$ calculation:

$$\bar{x}_1 = \frac{10}{3}L, \quad \bar{x}_2 = \frac{5}{2}L, \quad \bar{x}_3 = \frac{7}{3}L, \quad \text{and} \quad \bar{x}_4 = \frac{4}{3}L. \tag{6}$$

Hence,

$$2\triangle_B + \triangle_C = A_1\bar{x}_1 + A_2\bar{x}_2 + A_3\bar{x}_3 + A_4\bar{x}_4 = \frac{142}{12}\frac{PL^3}{EI}. \tag{7}$$

Combining equations 5 and 7:

$$\triangle_C = \left[\frac{142 - 52}{12}\right]\frac{PL^3}{EI} = \frac{15}{2}\frac{PL^3}{EI}. \tag{8}$$

Question 2 (15 points) Figure 2 is a front elevation view of a simple three-pinned arch. Vertical and horizontal loads 2P are applied at nodes B and D.

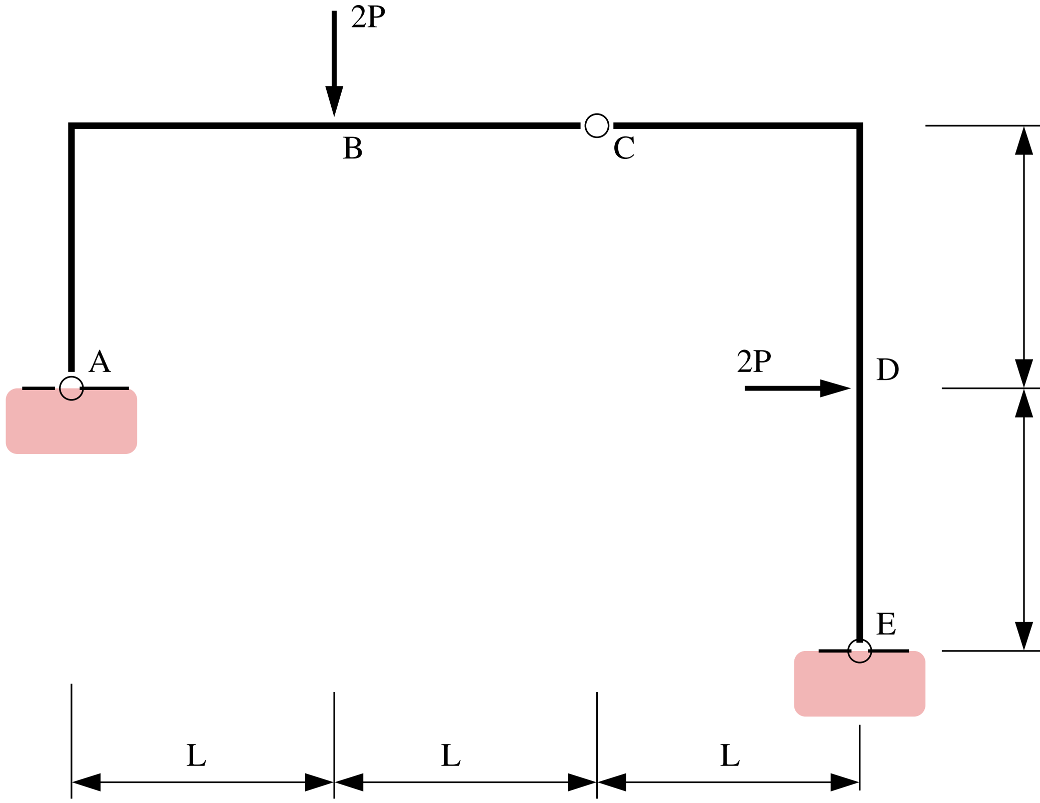


Figure 2. Front elevation view of a simple three-pinned arch.

Part [2a] (5 pts) Compute the vertical and horizontal components of reaction force at supports A and E as a function of P.

Solution: Sum forces in the vertical and horizontal directions, then take moments about C for right- and left-hand substructures:

$$\sum H = 0, \quad \longrightarrow \quad -H_A + H_E = 2P. \tag{9}$$

$$\sum V = 0, \quad \longrightarrow \quad V_A + V_E = 2P. \tag{10}$$

$$\sum M_C = 0 \quad \text{(for RHS)}, \quad \longrightarrow \quad 2PL + V_E L = H_E 2PL. \tag{11}$$

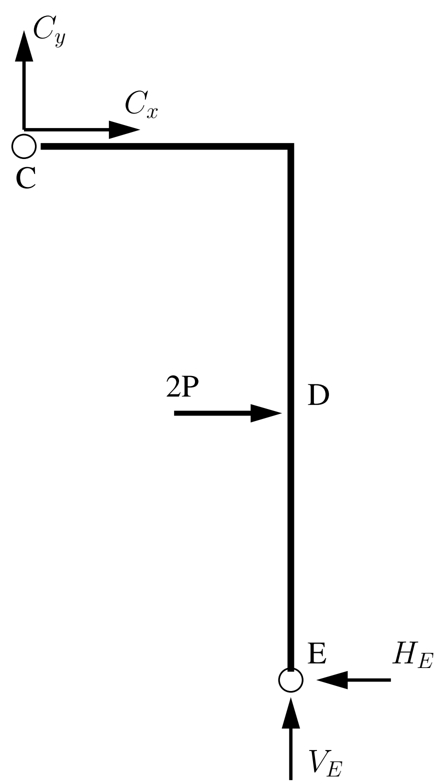
$$\sum M_C = 0 \quad \text{(for LHS)}, \quad \longrightarrow \quad 2PL + H_A L = V_A 2L \quad \longrightarrow \quad 2P + H_A = 2V_A. \tag{12}$$

Solving equations 9 – 12:

$$V_A = \frac{4}{5}P, \quad V_E = \frac{6}{5}P, \quad H_A = -\frac{2}{5}P \quad \text{and} \quad H_E = \frac{8}{5}P. \tag{13}$$

Part [2b] (5 pts) Compute and axial and shear forces transferred across the hinge at C. You can annotate Figure 2 if you think it will help to explain your solution.

Solution: Let's examine equilibrium of the right-hand substructure:



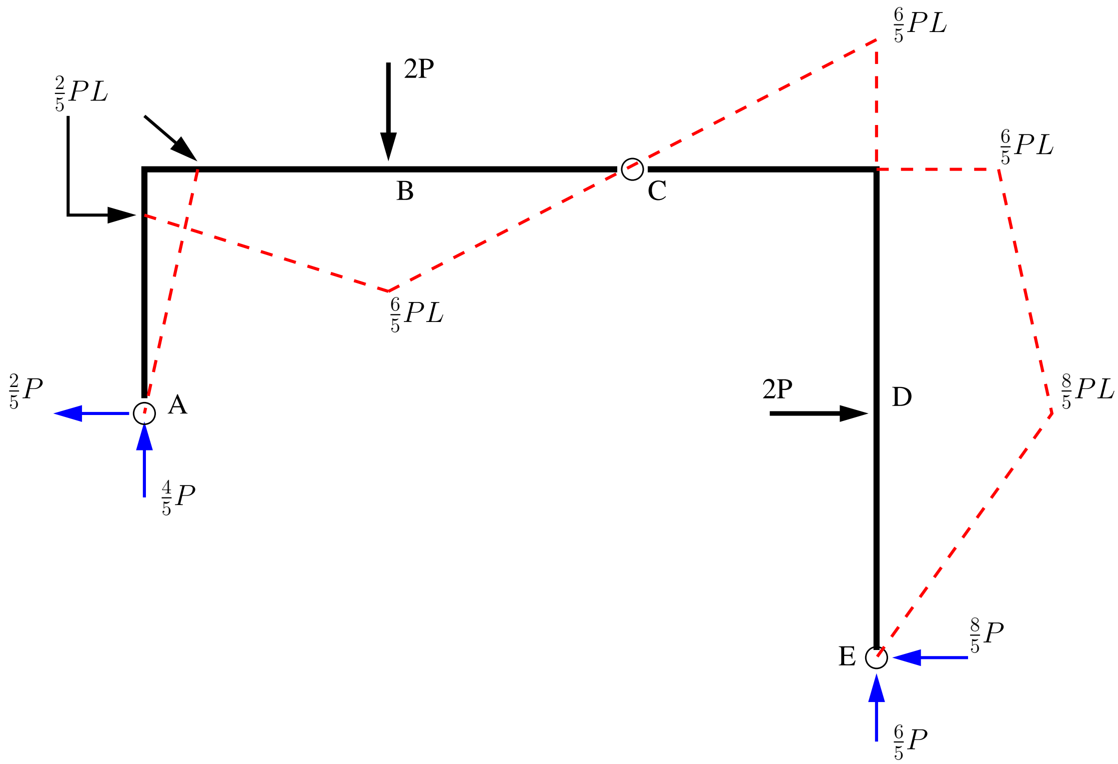
Summing forces in the x- and y- directions:

$$\sum F_y = 0, \quad C_y + V_E = 0, \longrightarrow C_y = -\frac{6}{5}P \quad \text{(shear force).} \tag{14}$$

$$\sum F_x = 0, \quad C_x + 2P = H_E, \longrightarrow C_x = -\frac{2}{5}P \quad \text{(axial force).} \tag{15}$$

Part [2c] (5 pts) Draw and label the bending moment diagram.

Solution: BMD is as follows:



Question 3 (10 points). Figure 3 is a front elevation view of a cable structure and bridge deck that carries a snow loading.

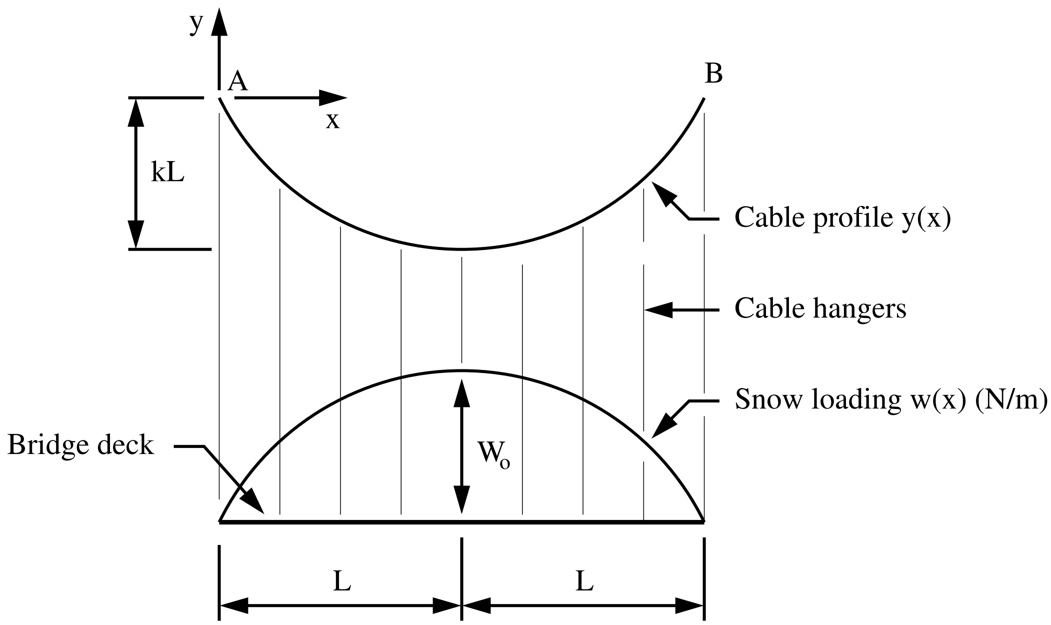


Figure 3. Cable structure carrying a snow loading.

The bridge deck and cable system have total span 2L. The cable sag is kL, where $0 < k < 1$. Roughly speaking, the distribution of snow loading follows a sinusoidal shape:

$$w(x) = W_o \sin(\frac{\pi x}{2L}) \quad (N/m). \tag{16}$$

Part [3a] (4 pts) Starting from first principles of engineering (i.e., the differential equation for cable behavior) show that the horizontal component of cable force is:

$$H = \frac{W_o}{kL}\left(\frac{2L}{\pi}\right)^2 \tag{17}$$

Solution: Behavior of the cable system is given by solutions to the differential equation:

$$\frac{d^2y}{dx^2} = \frac{W_o}{H}\sin(\frac{\pi x}{2L}) \tag{18}$$

Integrating equation 19 twice:

$$
\begin{aligned}
\frac{dy}{dx} &= \frac{-W_o}{H}\frac{2L}{\pi}\cos(\frac{\pi x}{2L}) + A, \\
y(x) &= \frac{-W_o}{H}\left(\frac{2L}{\pi}\right)^2 \sin(\frac{\pi x}{2L}) + Ax + B.
\end{aligned} \tag{19}
$$

Notice that the coordinate system is positioned at the top- left-hand support. The displacement boundary conditions are as follows:

$$
y(0) = y(2L) = 0, \quad \longrightarrow \quad B = 0. \tag{20}
$$

Also, from symmetry:

$$
\frac{dy}{dx}|_{x=L} = \frac{-W_o}{H}\frac{2L}{\pi}\cos(\frac{\pi}{2}) + A = 0, \quad \longrightarrow \quad A = 0. \tag{21}
$$

Finally, at x = L:

$$
y(L) = -kL = \frac{-W_o}{H}\left(\frac{2L}{\pi}\right)^2 \sin(\frac{\pi}{2}) \tag{22}
$$

Rearranging terms:

$$
H = \frac{W_o}{kL}\left(\frac{2L}{\pi}\right)^2 \tag{23}
$$

Part [3b] (3 pts) Determine the vertical and horizontal components of cable force at the supports A and B.

Solution: We begin by noting that:

$$
\text{Total snow loading} = \int_0^{2L} w(x) = \int_0^{2L} W_o \sin(\frac{\pi x}{2L}) dx = \frac{4W_oL}{\pi}. \tag{24}
$$

At the support reactions:

$$
\frac{dy}{dx}|_{x=0} = \frac{V}{-H} = \frac{W_o}{-H}\frac{2L}{\pi}\cos(0). \tag{25}
$$

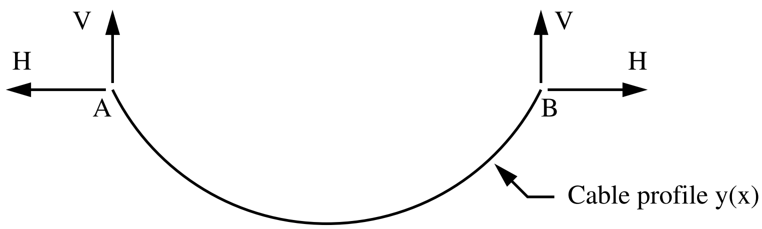


Figure 4. Horizontal and vertical support reactions.

Plugging equation 23 into 25 and rearranging terms:

$$H = \frac{W_o}{kL}\left(\frac{2L}{\pi}\right)^2 \quad \text{and} \quad V = \frac{2W_oL}{\pi}. \tag{26}$$

Notice that the total snow loading = 2V, the cable system is in equilibrium.

Part [3c] (3 pts) Show that the maximum force in the cable, T_{max}, is:

$$T_{max} = \frac{2W_oL}{\pi}\left[1 + \left(\frac{2}{k\pi}\right)^2\right]^{1/2} \tag{27}$$

Solution: The maximum cable force occurs at the supports:

$$T_{max} = \left[H^2 + V^2\right]^{1/2} = W_o\left(\frac{2L}{\pi}\right)\left[1 + \left(\frac{2}{k\pi}\right)^2\right]^{1/2} \tag{28}$$