Solutions to Homework 5

Question 1: 10 points

Problem Statement. Figure 1 shows a simple three-bar truss. The bar elements have section properties EA throughout. Horizontal and vertical loads P are applied at node C.

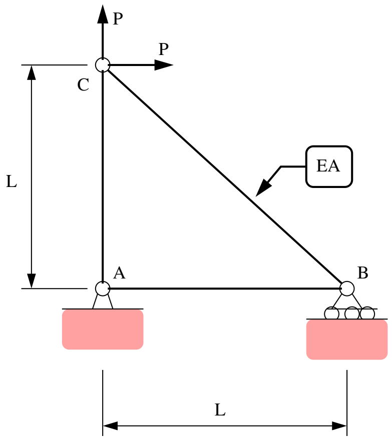
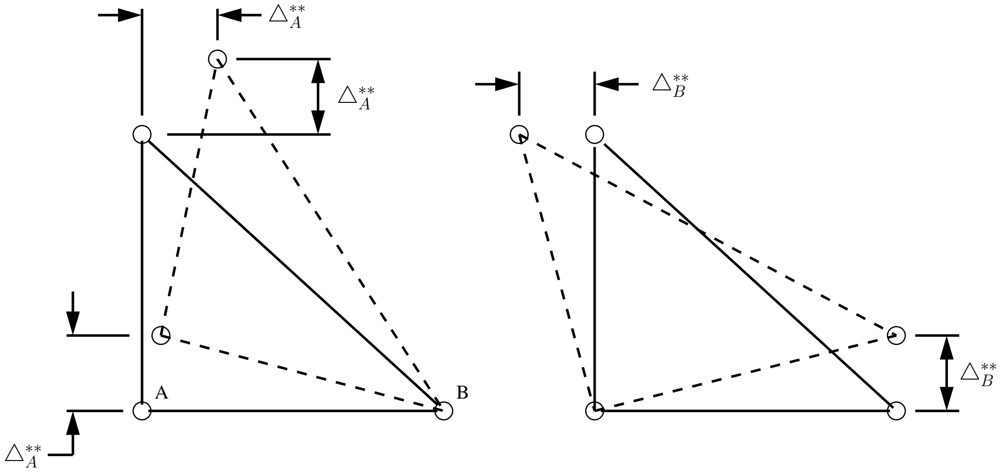


Figure 1: Simple three-bar truss.

Part [1a]. (5 pts). Use the **method of virtual displacements** to compute the vertical reaction forces at nodes A and B. Show all of your working.

Solution. Impose virtual displacements at A and B:



For reaction at A:

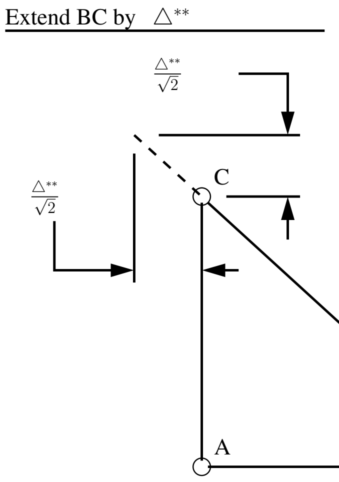
$$\sum EWD = 0, \quad \longrightarrow \quad V_A \triangle_A^{**} + P\triangle_A^{**} + P\triangle_A^{**} = 0. \quad \longrightarrow \quad V_A = -2P. \tag{1}$$

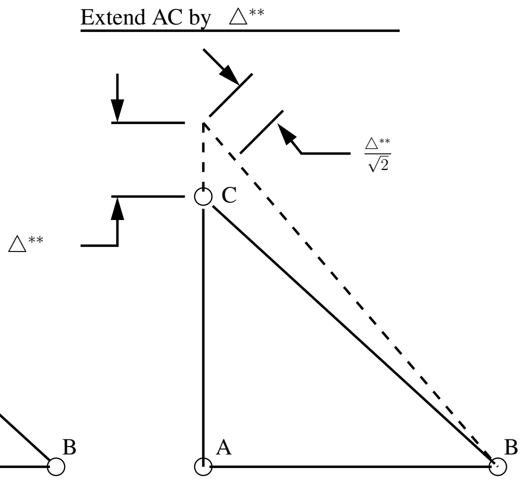
For reaction at B:

$$\sum EWD = 0, \quad \longrightarrow \quad V_B \triangle_B^{**} + P(-\triangle_B^{**}) = 0, \quad \longrightarrow \quad V_B = P. \tag{2}$$

Part [1b]. (5 pts). Use the **method of virtual displacements** to compute the member forces AC and BC. Show all of your working.

Solution. Extend members BC and AC by $\triangle^{**}$:





For member BC, energy balance $\sum$ EWD = $\sum$ IWD:

$$\left(\frac{\triangle^{**}}{\sqrt{2}}\right)(-P) + \left(\frac{\triangle^{**}}{\sqrt{2}}\right)(P) = \overline{BC}\triangle^{**} + \overline{AC}\left(\frac{\triangle^{**}}{\sqrt{2}}\right) \tag{3}$$

For member AC, energy balance $\sum$ EWD = $\sum$ IWD:

$$P\triangle^{**} = \overline{AC}\triangle^{**} + \overline{BC}\left(\frac{\triangle^{**}}{\sqrt{2}}\right) \tag{4}$$

Combining equations 3 and 4: $\overline{\text{AC}} = 2P$, and $\overline{\text{BC}} = -\sqrt{2}\text{P}$.

Question 2: 10 points

Problem Statement. The cantilevered beam structure shown in Figure 2 supports a uniformly distributed load w (N/m) between points C and D.

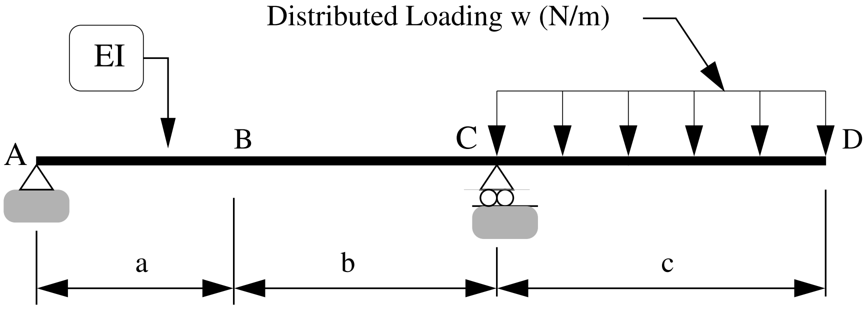
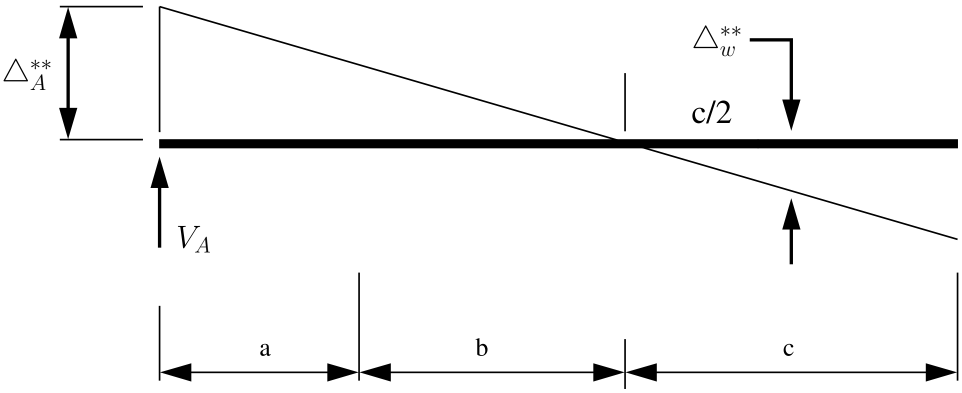


Figure 2: Front elevation view of a simple beam structure.

Part [2a]. (4 pts) Use the method of **virtual displacements** to compute formulae for the vertical reactions at A and C. Show all of your working.

Solution:

Reaction at A: Impose virtual displacement $\triangle_A^{**}$ at A;



From geometry:

$$\triangle_w^{**} = \frac{c}{2(a+b)} \triangle_A^{**} \tag{5}$$

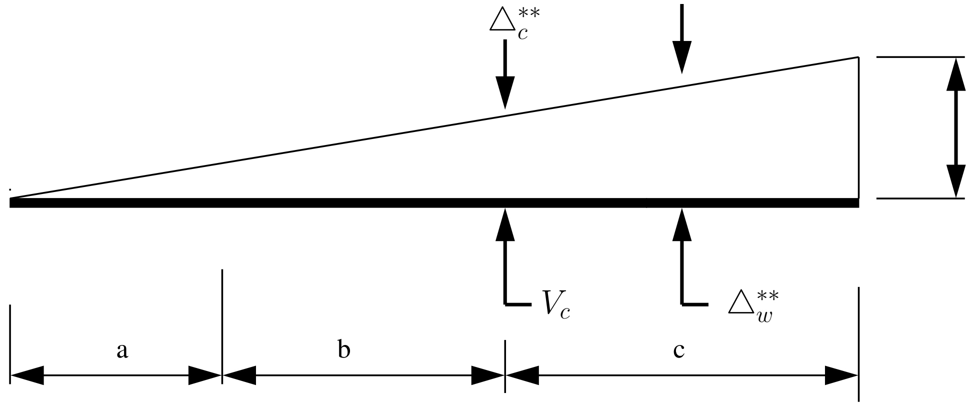
For this problem IWD = 0 and $\sum$ EWD = 0. The latter gives:

$$V_A \triangle_A^{**} + \frac{wc^2}{2(a+b)} \triangle_A^{**} = 0. \tag{6}$$

Hence,

$$V_A = \frac{-wc^2}{2(a+b)}. \tag{7}$$

Reaction at C: Impose virtual displacement $\triangle_C^{**}$ at C;



From geometry:

$$\triangle_w^{**} = \frac{a+b+c/2}{(a+b)} \triangle_C^{**} \tag{8}$$

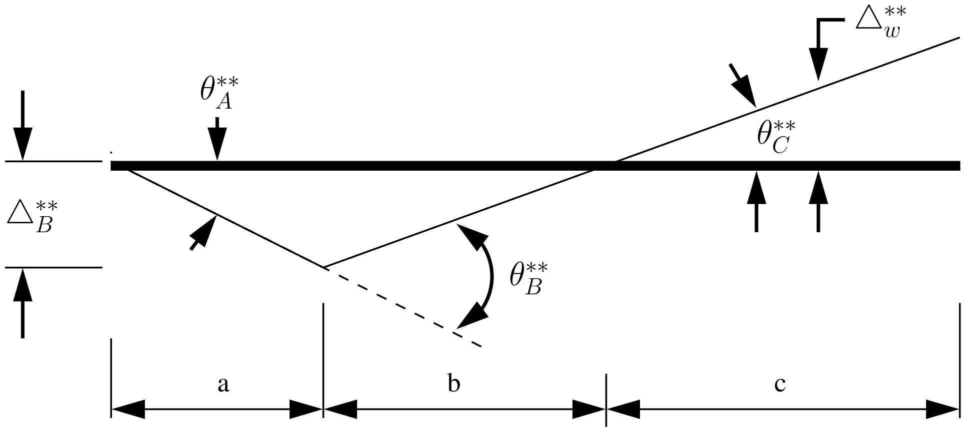
For this problem IWD = 0 and $\sum$ EWD = 0. The latter gives:

$$V_C \triangle_C^{**} - wc \triangle_w^{**} = 0. \tag{9}$$

Hence,

$$V_C = \frac{wc(a+b+c/2)}{(a+b)}. \tag{10}$$

Part [2b]. (6 pts) Use the method of **virtual displacements** to compute a formula for the bending moment at B. Show all of your working.



Solution: Impose a virtual rotation θ_B at B.

From geometry:

$$\begin{aligned} \theta_B^{**} &= 0. \\ \triangle_B^{**} &= a\theta_A^{**}. \\ \triangle_B^{**} &= b\theta_C^{**}. \\ \triangle_w^{**} &= \frac{c}{2b}\triangle_B^{**}. \end{aligned} \tag{11}$$

Energy Balance: IWD = EWD.

$$M_B^*\theta_B^{**} + w_c^*\triangle_w^{**} = 0. \tag{12}$$

Sustituting equations 11 into 12 and rearranging terms:

$$M_B\left[\frac{1}{a}+\frac{1}{b}\right]\triangle_B^{**} = -\frac{wc^2}{2b}\triangle_B^{**} \tag{13}$$

Hence,

$$M_B = \left(\frac{a}{a+b}\right)\left[\frac{-wc^2}{2}\right]. \tag{14}$$

Note: From statics: $M_B = V_A$ a.

Question 3: 10 points

Problem Statement. Consider the articulated cantilever beam structure shown in Figure 3.

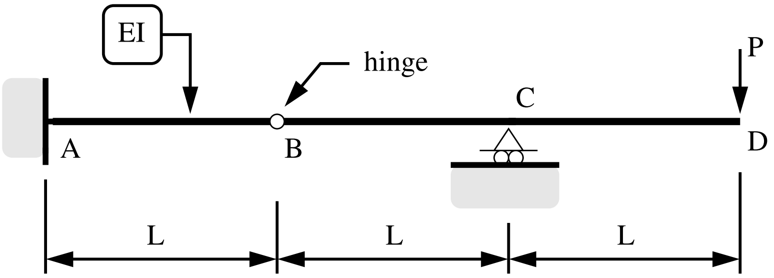
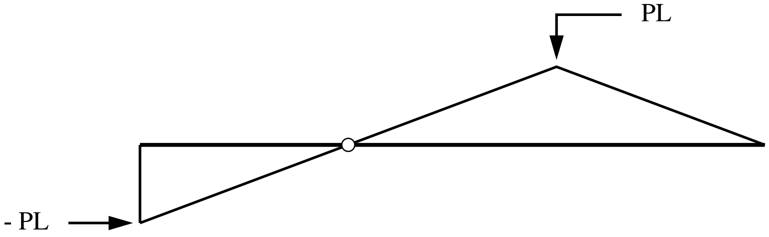


Figure 3: Elevation view of articulated cantilever beam structure.

At Point A, the cantilever is fully fixed (no movement) to a wall. Point B is a hinge. Both members have cross section properties EI. A single point load **P** (N) is applied at node D as shown in the figure.

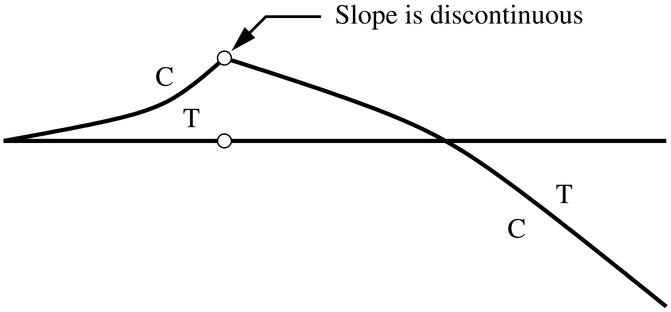
Part [3a]. (2 pts). Draw and label the bending moment diagram for this problem.

Solution: ...



Part [3b]. (2 pts). Qualitatively sketch the deflected shape. Indicate regions of tension/compression, and any points where slope of the beam is discontinuous.

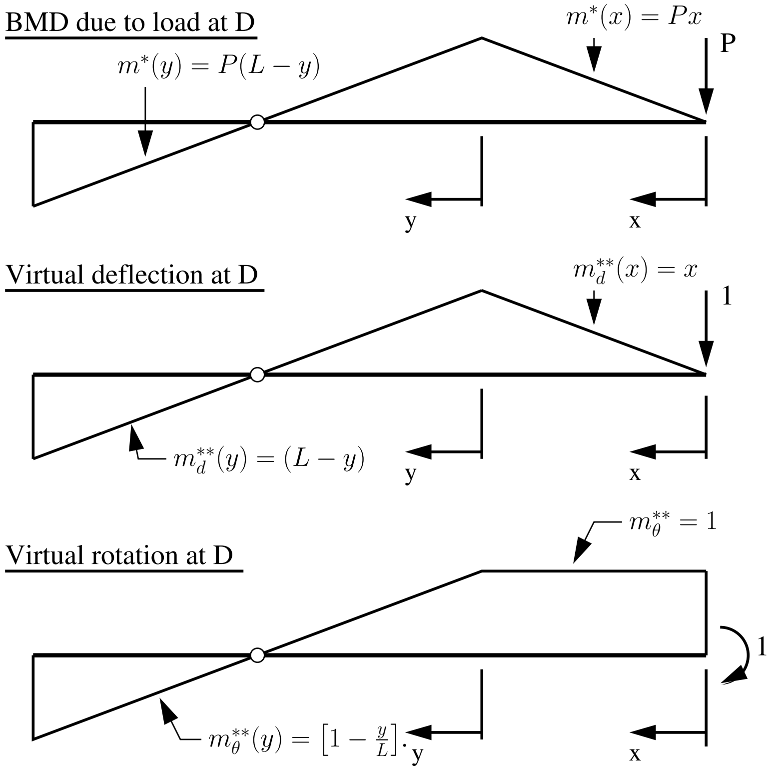
Solution: ...



Part [3c]. (6 pts). Use the method of **virtual forces** to compute the **vertical displacement** and **end rotation**

of the beam at D. Show all of your working.

Solution: BMD's due to applied load at D and unit virtual forces (point load and moment) also applied at D.



Apply virtual forces:

$$\begin{aligned}
\triangle_D &= \frac{1}{EI}\int_0^L m^*(x)m_d^{**}(x)dx + \frac{1}{EI}\int_0^{2L} m^*(y)m_d^{**}(y)dy \\
&= \frac{P}{EI}\int_0^L x^2 dx + \frac{P}{EI}\int_0^{2L}(L-y)^2 dy \\
&= \frac{PL^3}{3EI} + \frac{2}{3}\frac{PL^3}{EI} = \frac{PL^3}{EI}.
\end{aligned} \tag{15}$$

$$\begin{aligned}
\theta_D &= \frac{1}{EI}\int_0^L m^*(x)m_\theta^{**}(x)dx + \frac{1}{EI}\int_0^{2L} m^*(y)m_\theta^{**}(y)dy \\
&= \frac{P}{EI}\int_0^L x dx + \frac{P}{EIL}\int_0^{2L}(L-y)^2 dy \\
&= \frac{PL^2}{2EI} + \frac{2}{3}\frac{PL^2}{EI} = \frac{7}{6}\frac{PL^2}{EI}.
\end{aligned} \tag{16}$$

Question 4: 10 points

Problem Statement. Figure 4 is a front elevation view of a simple truss that supports vertical loads at nodes C and D.

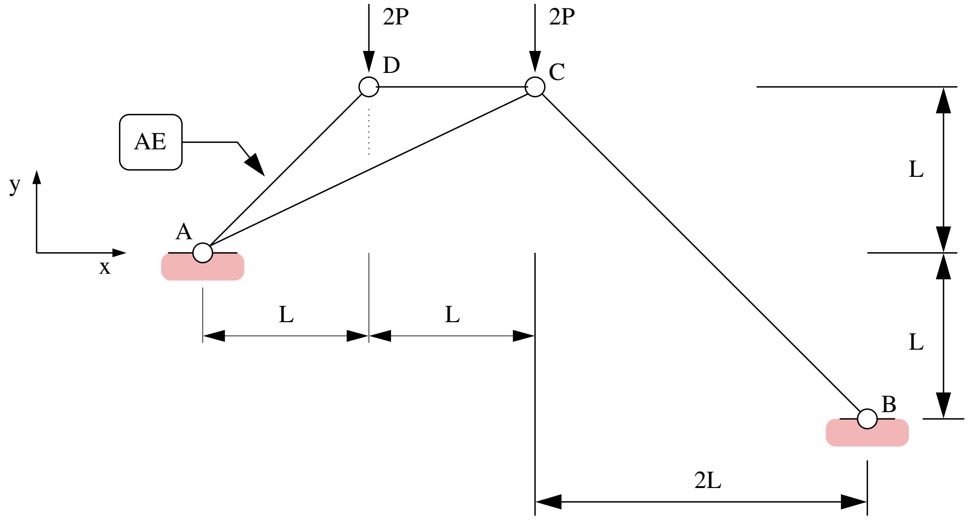


Figure 4: Front elevation view of a simple truss.

All of the truss members have cross section properties AE.

Part [4a]. (5 pts). Compute the support reactions and distribution of forces throughout the structure.

Solution: Equations of equilibrium:

$$\sum M_A = 0 \quad \longrightarrow \quad -2PL - 4PL - H_B L + 4LV_B = 0 \quad \longrightarrow \quad 4V_B = H_B + 6P \tag{17}$$

$$\sum F_x = 0 \quad \longrightarrow \quad H_A = H_B \quad \text{(not too useful)} \tag{18}$$

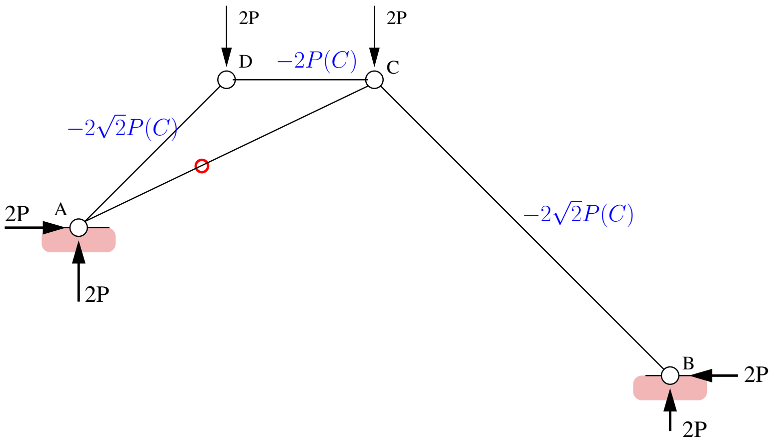
$$\sum F_y = 0 \quad \longrightarrow \quad V_A + V_B = 4P \tag{19}$$

Equilibrium at node B: $V_B = H_B$.

From equations 17 through 19:

$$V_A = V_B = 2P, \quad \longrightarrow \quad \text{and} \quad H_A = H_B = 2P. \tag{20}$$

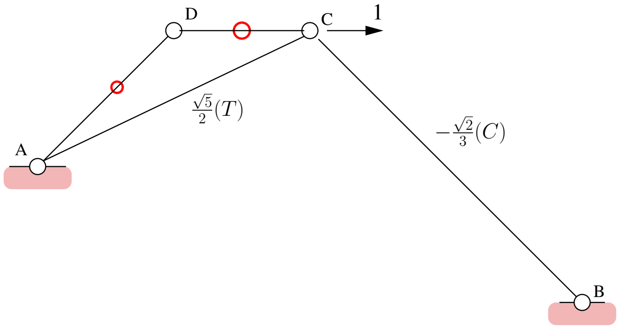
The distribution of forces is as follows:

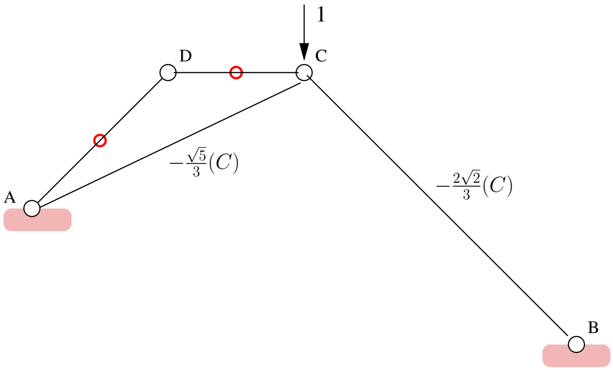


Part [4b]. (5 pts). Use the method of **virtual forces** to show that the total deflection at node C is:

$$\triangle = \frac{PL}{AE} \left[\frac{8\sqrt{10}}{3} \right]. \tag{21}$$

Solution: Apply unit loads in the x- and y-directions:





Displacements in the horizontal and vertical directions are given by:

$$\triangle_x = \sum_{i=1}^{N} \left(\frac{F_i f_h L_i}{A_i E_i} \right). \tag{22}$$

and

$$\triangle_y = \sum_{i=1}^{N} \left(\frac{F_i f_v L_i}{A_i E_i} \right). \tag{23}$$

Tabulate displacements:

Member	L/AE	F	f_h	f_v	$\sum_{i=1}^{N}\left(\frac{F_i f_h L_i}{A_i E_i}\right)$	$\sum_{i=1}^{N}\left(\frac{F_i f_v L_i}{A_i E_i}\right)$
AC	$\sqrt{5}L/AE$	0	$\sqrt{5}/2$	$-\sqrt{5}/3$	0	0
AD	$\sqrt{2}L/AE$	$-2\sqrt{2}P$	0	0	0	0
BC	$2\sqrt{2}L/AE$	$-2\sqrt{2}P$	$-\sqrt{2}/3$	$-2\sqrt{2}/3$	$8\sqrt{2}/3\frac{PL}{AE}$	$16\sqrt{2}/3\frac{PL}{AE}$
CD	L/AE	$-2P$	0	0	0	0
Summary					$8\sqrt{2}/3\frac{PL}{AE}$	$16\sqrt{2}/3\frac{PL}{AE}$

Summary:

$$\triangle_x = \frac{8\sqrt{2}}{3}\frac{PL}{AE}, \quad \text{and} \quad \triangle_y = \frac{16\sqrt{2}}{3}\frac{PL}{AE}. \tag{24}$$

Hence,

$$\text{total deflection} = \left[\triangle_x^2 + \triangle_y^2\right]^{1/2} = \frac{8\sqrt{10}}{3}\frac{PL}{AE}. \tag{25}$$

Question 5: 20 points

Problem Statement. The T-shaped beam structure shown in Figure has flexural stiffness EI throughout.

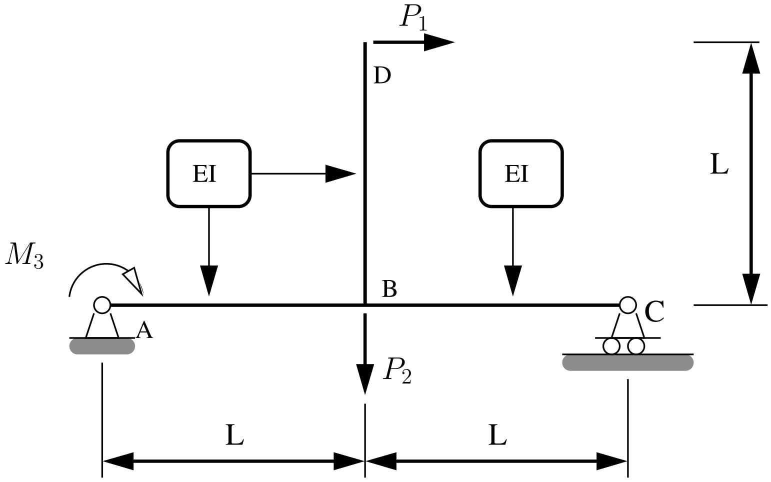
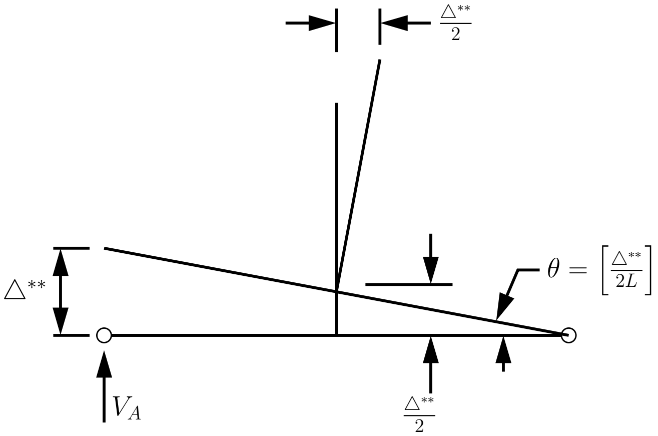


Figure 5: Front elevation view of a T-shaped beam.

Part [5a]. (5 pts). Use the **method of virtual displacements** to compute the vertical reaction force at node A.

Solution: Impose a virtual displacement $\triangle^{**}$ at A, i.e.,



Sum of external work done is zero (i.e., $\sum$ EWD = 0):

$$V_A \triangle^{**} + M_3 \theta + P_1 \frac{\triangle^{**}}{2} - P_2 \frac{\triangle^{**}}{2} = 0. \tag{26}$$

From geometry:

$$\theta = \frac{\triangle^{**}}{2L} \tag{27}$$

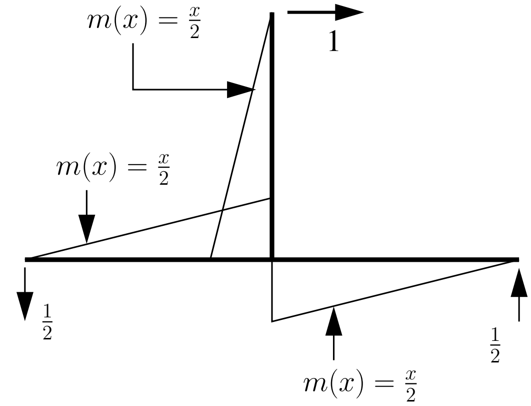
Hence,

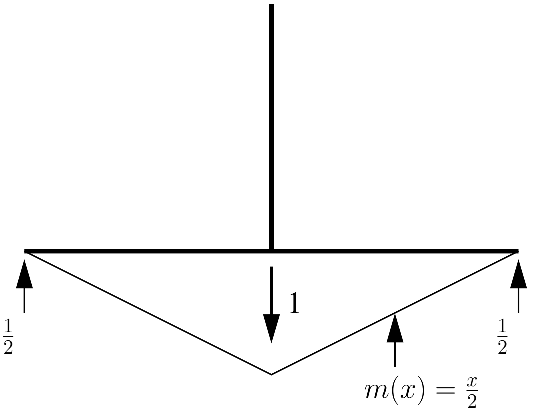
$$V_A = \frac{P_2}{2} - \frac{P_1}{2} - \frac{M_3}{2L}. \tag{28}$$

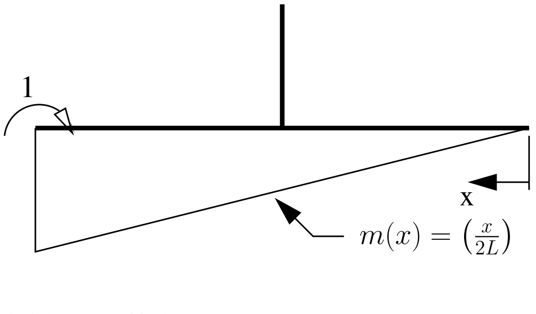
Part [5b]. (15 pts). Use the **method of virtual forces** to compute the flexibility matrix:

$$\begin{bmatrix} \triangle_{dx} \\ \triangle_{by} \\ \theta_A \end{bmatrix} = \begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \\ M_3 \end{bmatrix}. \tag{29}$$

Solution: Bending moment due to unit loads:







Flexibility coefficients:

$$\begin{aligned} f_{11} &= \frac{2}{EI}\int_0^L \left(\frac{x}{2}\right) dx + \frac{1}{EI}\int_0^L x^2 dx \\ &= \frac{L^3}{2EI}. \end{aligned} \tag{30}$$

From symmetry, f_{12} = f_{21} = 0.

$$\begin{aligned} f_{31} &= \frac{1}{EI}\int_0^L \left(\frac{-x}{2}\right)\left(1-\frac{x}{2L}\right)dx + \frac{1}{EI}\int_0^L \left(\frac{x}{2}\right)\left(\frac{x}{2L}\right)dx \\ &= \frac{-L^2}{12EI}. \end{aligned} \tag{31}$$

$$f_{22} = \frac{2}{EI}\int_0^L \left(\frac{x}{2}\right)^2 dx = \frac{L^3}{6EI}. \tag{32}$$

$$\begin{aligned} f_{32} &= \frac{1}{EI}\int_0^L \left(\frac{x}{2}\right)\left(1-\frac{x}{2L}\right)dx + \frac{1}{EI}\int_0^L \left(\frac{x}{2}\right)\left(\frac{x}{2L}\right)dx \\ &= \frac{L^2}{4EI}. \end{aligned} \tag{33}$$

$$f_{33} = \frac{1}{EI}\int_0^{2L} \left(\frac{x}{2L}\right)^2 dx = \frac{2L}{3EI}. \tag{34}$$

Flexibility Matrix:

$$f = \begin{bmatrix} L^3/2EI & 0 & -L^2/12EI \\ 0 & L^3/6EI & L^2/4EI \\ -L^2/12EI & L^2/4EI & 2L/3EI \end{bmatrix}. \tag{35}$$

Question 6: 10 points

Problem Statement. Consider the supported cantilevered beam structure shown in Figure 6.

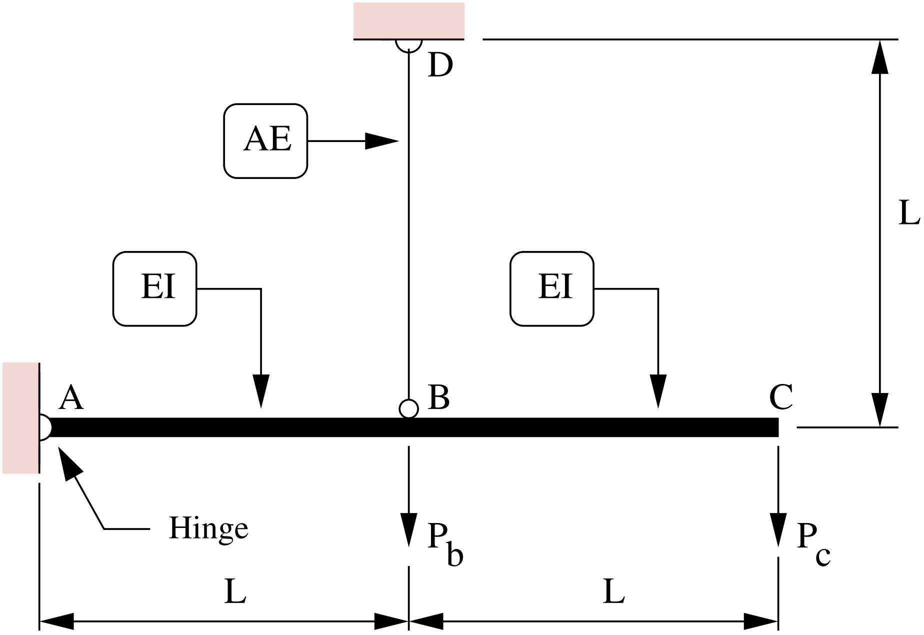
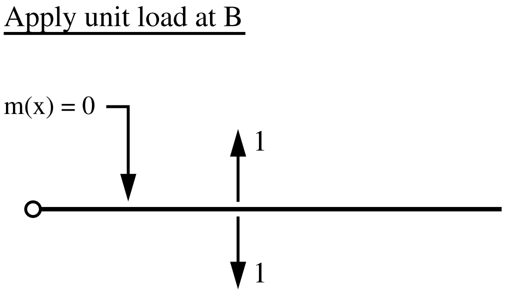


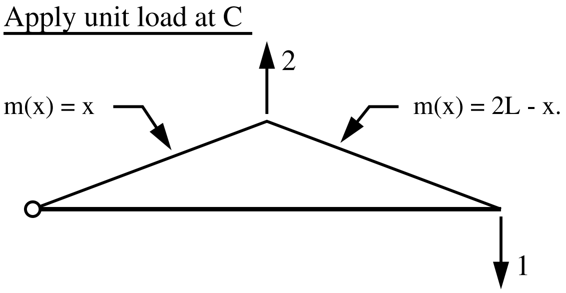
Figure 6: Front elevation view of a supported cantilevered beam structure.

Use the principle of **virtual forces** to compute the two-by-two flexibility matrix connecting displacements at points B and C to applied loads P_b and P_c, i.e.,

$$\begin{bmatrix} \triangle_b \\ \triangle_c \end{bmatrix} = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} \begin{bmatrix} P_b \\ P_c \end{bmatrix}. \tag{36}$$

Sol'n: Compute bending moment diagrams for unit forces at B and C:





Compute flexibility coefficients:

$$f_{11} = \int_0^{2L} \frac{m_1^2(x)}{EI} dx + \int_0^{L} \frac{f_1^2(x)}{AE} dx = 0 + \frac{L}{AE} = \frac{L}{AE}. \tag{37}$$

$$f_{22} == \underbrace{\int_0^{2L} \frac{m_2^2(x)}{EI} dx}_{I_1} + \underbrace{\int_0^{L} \frac{f_2^2(x)}{AE} dx}_{I_2}. \tag{38}$$

Here,

$$I_1 = \frac{1}{EI} \int_0^{2L} m_2^2(x) dx = \frac{2}{3} \frac{L^3}{EI}. \tag{39}$$

$$I_2 = \frac{1}{AE} \int_0^{L} f_2^2(x) dx = \frac{4L}{AE}. \tag{40}$$

Hence,

$$f_{22} = I_1 + I_2 = \frac{2}{3} \frac{L^3}{EI} + \frac{4L}{AE}. \tag{41}$$

Finally,

$$f_{12} = \frac{1}{EI} \int_0^{2L} m_1(x) m_2(x) dx + \frac{1}{AE} \int_0^{L} f_1(x) f_2(x) dx = \frac{2L}{AE}. \tag{42}$$

Flexibility Matrix:

$$\begin{bmatrix} \triangle_b \\ \triangle_c \end{bmatrix} = \begin{bmatrix} \frac{L}{AE} & \frac{2L}{AE} \\ \frac{2L}{AE} & \frac{2}{3}\frac{L^3}{EI} + \frac{4L}{AE} \end{bmatrix} \begin{bmatrix} P_b \\ P_c \end{bmatrix}. \tag{43}$$