

Solutions to Homework 4

Question 1: 10 points

Figure 1 is a front elevation view of a simply supported beam that carries a triangular load.

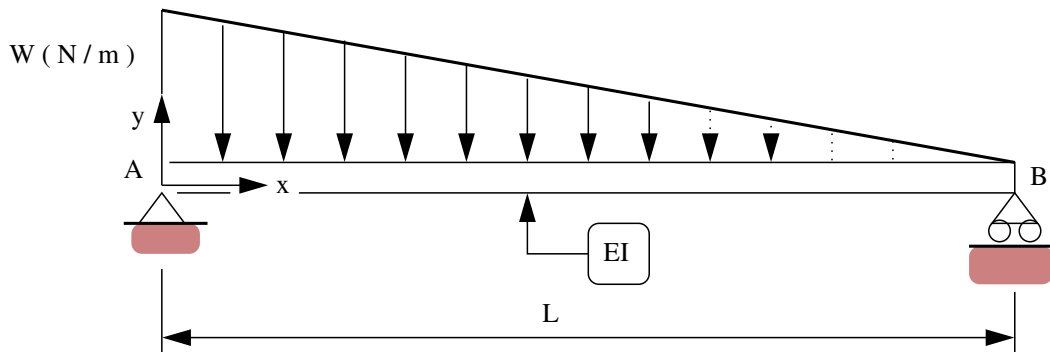


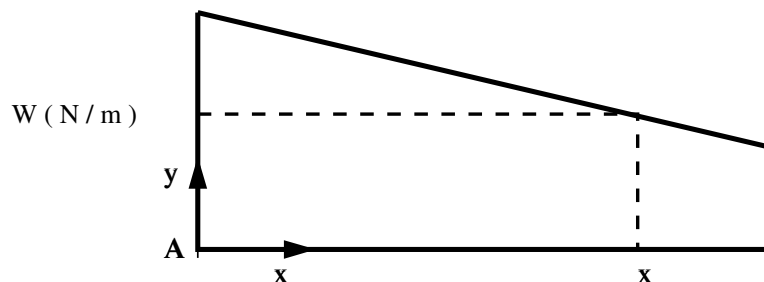
Figure 1: Simply supported beam carrying a triangular load.

The load decreases from W (N/m) at point A to zero at point B. Thus, the total beam loading is $WL/2$.

Part [1a] (4 pts). Starting from first principles of engineering, show that the bending moment at point x is:

$$M(x) = \left[\frac{W}{6L} \right] x (L - x) (2L - x). \tag{1}$$

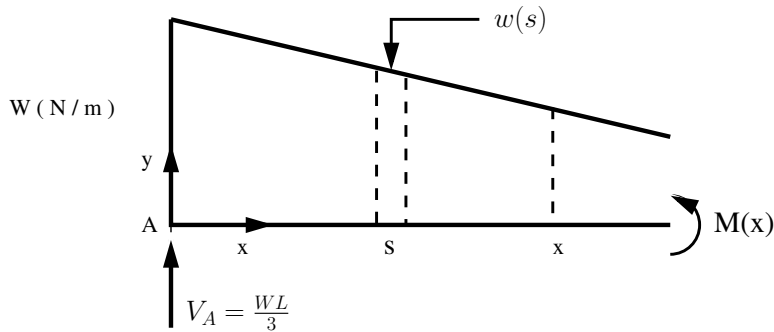
Solution: Cut beam a coordinate x



and then take moments:

$$\begin{aligned}
 M(x) &= \frac{WL}{3}x - W\left(1 - \frac{x}{L}\right)\frac{x^2}{2} - \frac{()}{2} \\
 &= \frac{WL}{3}x - W\frac{x^2}{2} - W\frac{x^3}{6L}. \\
 &= \frac{W}{6L}x(L-x)(2L-x).
 \end{aligned} \tag{2}$$

Alternate Solution: We can get the same result via a one-dimensional convolution:



$$\begin{aligned}
 M(x) &= V_A x - \int_0^x w(s)(x-s) ds \\
 &= \frac{WL}{3}x - W \int_0^x \left(1 - \frac{s}{L}\right)(x-s) ds \\
 &= \frac{W}{6L} [x(L-x)(2L-x)].
 \end{aligned} \tag{3}$$

Part [1b] (4 pts). Show that the elastic curve for beam deflection is given by (notice that in Figure 1, the \$y\$ axis is pointing upwards):

$$y(x) = \left[\frac{WL^4}{360EI} \right] \left(3 \left[\frac{x}{L} \right]^5 - 15 \left[\frac{x}{L} \right]^4 + 20 \left[\frac{x}{L} \right]^3 - 8 \left[\frac{x}{L} \right] \right) \tag{4}$$

Solution: Integrate \$M(x)\$ twice to get \$y(x)\$:

$$EI \left(\frac{6L}{W} \right) \frac{dy}{dx} = L^2 x^2 - \frac{3Lx^3}{3} + \frac{x^4}{4} + A. \tag{5}$$

$$EI \left(\frac{6L}{W} \right) y(x) = \frac{L^2 x^3}{3} - \frac{Lx^4}{4} + \frac{x^5}{20} + Ax + B. \tag{6}$$

Boundary conditions:

$$y(0) = 0 \quad \longrightarrow \quad B = 0. \quad (7)$$

$$y(L) = 0 \quad \longrightarrow \quad L^5 \left[\frac{1}{3} - \frac{1}{4} + \frac{1}{20} \right] + AL = 0 \quad \longrightarrow \quad A = -\frac{2}{15}L^4. \quad (8)$$

Substituting equations 7 and 8 into equation 6, and rearranging terms gives:

$$y(x) = \left[\frac{WL^4}{360EI} \right] \left(3 \left[\frac{x}{L} \right]^5 - 15 \left[\frac{x}{L} \right]^4 + 20 \left[\frac{x}{L} \right]^3 - 8 \left[\frac{x}{L} \right] \right). \quad (9)$$

Part [1c] (2 pts). Show that the maximum beam curvature occurs at $x = \left[1 - \frac{1}{\sqrt{3}} \right] L$.

Solution: At the point of maximum curvature:

$$\text{Max } \phi \quad \longrightarrow \quad \frac{dM}{dx} = 0 \quad \longrightarrow \quad 2L^2 - 6Lx + 3x^2 = 0. \quad (10)$$

The positive root of the quadratic is:

$$x = \left[1 - \frac{1}{\sqrt{3}} \right] L. \quad (11)$$

Question 2: 10 points

The cantilever beam structure shown in Figure 2 carries a uniform load w (N/m) along its entire length.

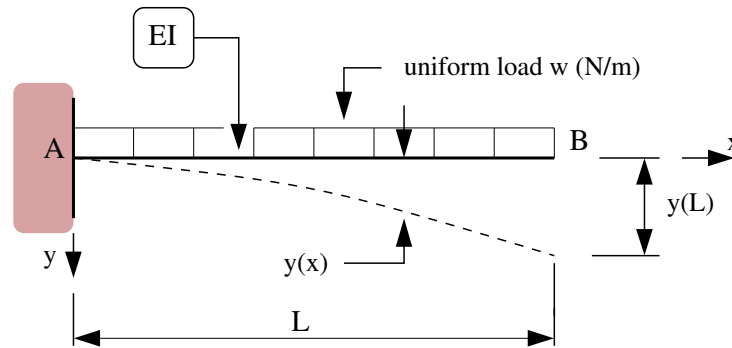


Figure 2: Cantilever beam carrying a uniform load.

The beam is fully fixed at point A and the flexural stiffness EI is constant along the beam. The coordinate system is positioned at point A.

Part [2a] (5 pts) Starting from the differential equation,

$$\frac{d^2 y}{dx^2} = \left[\frac{M(x)}{EI} \right], \quad (12)$$

and appropriate boundary conditions, show that:

$$y(x) = \left(\frac{w}{24EI} \right) (6L^2 x^2 - 4Lx^3 + x^4). \quad (13)$$

Solution: From statics:

$$M(x) = w(L-x) \frac{L-x}{2} = \frac{w}{2} (L-x)^2. \quad (14)$$

Plug equation 14 into 12, then integrate twice:

$$\frac{dy}{dx} = \frac{w}{2EI} \left[L^2 x - Lx^2 + \frac{x^3}{3} + A \right] \quad (15)$$

$$y(x) = \frac{w}{2EI} \left[\frac{L^2 x^2}{2} - \frac{Lx^3}{3} + \frac{x^4}{12} + Ax + B \right]. \quad (16)$$

Boundary conditions:

$$y(0) = 0 \quad \rightarrow \quad B = 0, \quad \text{and} \quad \left. \frac{dy}{dx} \right|_{x=0} \quad \rightarrow \quad A = 0. \quad (17)$$

hence,

$$y(x) = \left(\frac{w}{24EI} \right) (6L^2x^2 - 4Lx^3 + x^4). \quad (18)$$

Part [2b] (5 pts) Using the results of question [2a] as a starting point, compute the support reactions at A and B for the propped cantilever shown in Figure 3

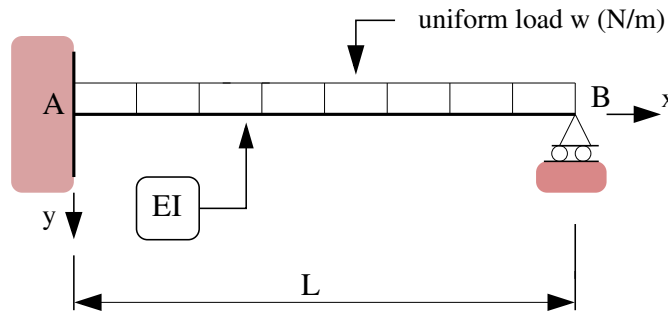
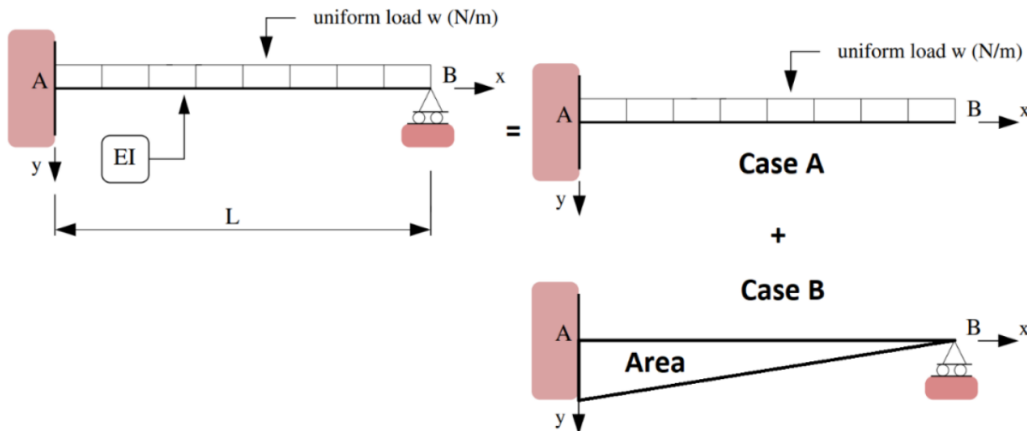


Figure 3: Propped cantilever beam carrying a uniform load.

Solution: Apply principle of superposition (system is linear):



Case A: Substitute $x = L$ into equation 18. This gives:

$$y_A(L) = \frac{w}{24EI} (6L^4 - 4L^4 + L^4) = \frac{wL^4}{8EI} \quad (\text{downward}) \quad (19)$$

Case B: Apply moment area method:

$$y_B(L) = \text{Area} * x_b = \frac{V_b L^3}{3EI} \quad (\text{upward}) \quad (20)$$

Combining displacements in equations 19 and 20, and noting that the net total displacement is zero:

$$\Delta_B = y_A(L) + y_B(L) = 0 \quad \longrightarrow \quad V_B = \frac{3wL}{8}. \quad (21)$$

Hence, $V_A = \frac{5wL}{8}$ and $M_A = \frac{wL^2}{8}$ (counter clockwise).

Question 3: 20 points

Consider the cantilever shown in Figure 4.

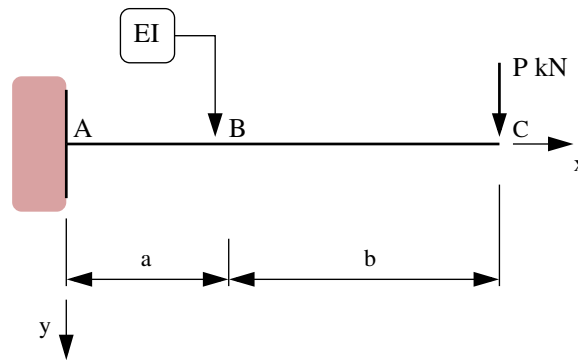


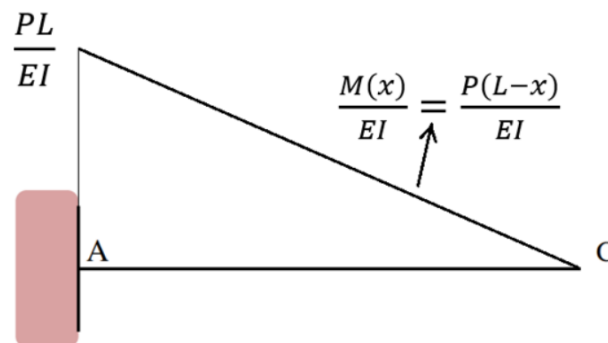
Figure 4: Front elevation view of a cantilever.

The cantilever has constant section properties, EI , along its entire length ($a+b$). A vertical load P (kN) is applied at point C .

Part [3a] (3 pts) Use the method of moment area to show that the vertical deflection at point C is:

$$y_c = \frac{P(a+b)^3}{3EI}. \quad (22)$$

Solution: The curvature is as follows:



Let $L = a + b$, hence $M(x) = P(L - x)$. The beam curvature is:

$$\phi(x) = \frac{M(x)}{EI} = \frac{P(L - x)}{EI}. \quad (23)$$

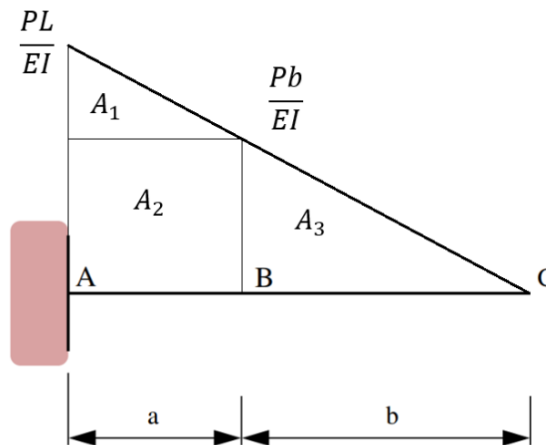
Apply moment area:

$$y(L) = Area * \bar{x}_c = \left(\frac{L PL}{2 EI} \right) \left(\frac{2L}{3} \right) = \frac{PL^3}{3EI} = \frac{P(a+b)^3}{3EI}. \quad (24)$$

Part [3b] (3 pts) Use the method of moment area to show that the vertical deflection at point B is:

$$y_B = y(a) = \frac{Pa^2}{6EI} [3b + 2a]. \quad (25)$$

Solution: We need to compute the first moment of area of the curvature diagram between A and B, evaluated about B:



The areas and centroids are as follows:

$$A_1 = \frac{Pa^2}{2EI}, \quad A_2 = \frac{Pab}{EI}, \quad \bar{x}_1 = \frac{2}{3}a, \quad \bar{x}_2 = \frac{a}{2}. \quad (26)$$

The vertical displacement at B is:

$$y(a) = A_1 \bar{x}_1 + A_2 \bar{x}_2 = \frac{Pa^2}{6EI} [3b + 2a]. \quad (27)$$

Now suppose that a roller support is inserted below point B as follows:

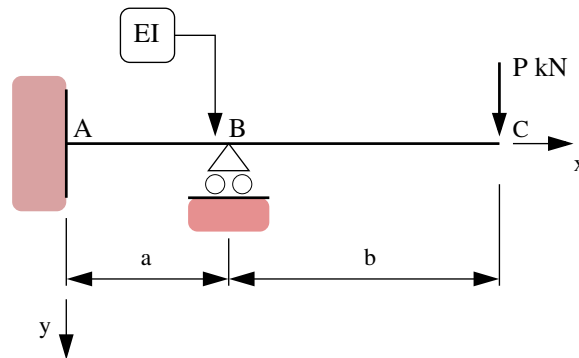
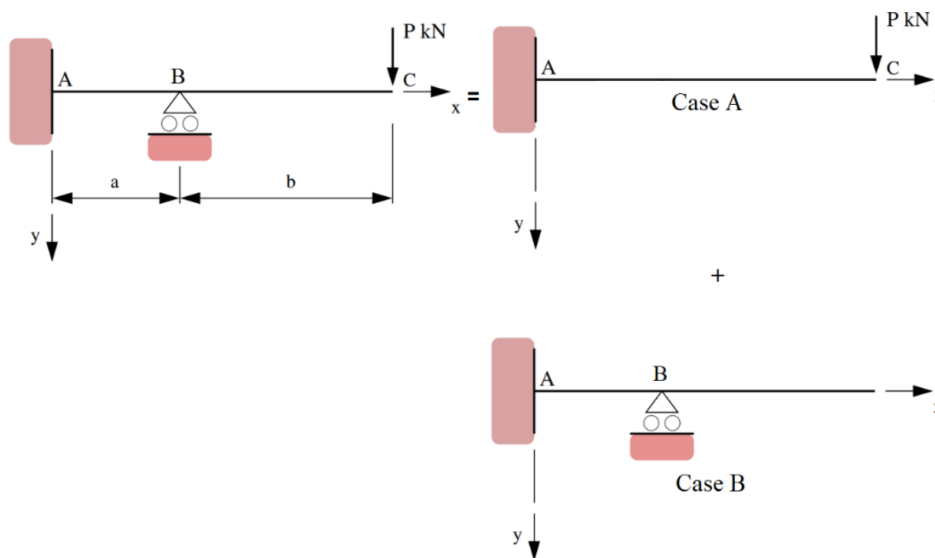


Figure 5: Front elevation view of a cantilever supported by a roller at point B.

Part [3c] (3 pts) Show that the vertical support reaction at B is:

$$V_b = \frac{P}{2} \left[\frac{3b + 2a}{a} \right]. \quad (28)$$

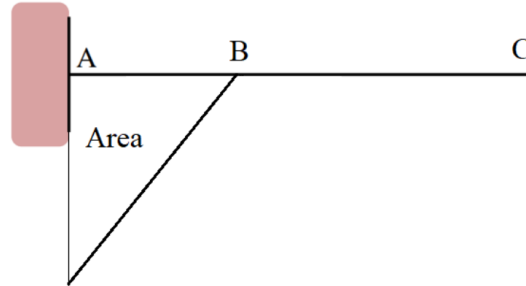
Solution: Using superposition:



Case A: From equation 27,

$$y(a) = \frac{Pa^2}{6EI} [2a + 3b] \quad (\text{downward}). \quad (29)$$

Case B: The upwards deflection due to reaction force V_b is:



$$\Delta_B = \frac{V_B a^3}{3EI} \quad (\text{upward}). \quad (30)$$

The net sum of equations 29 and 30 is 0, i.e.,

$$\frac{Pa^2}{6EI}[2a + 3b] = \frac{V_b a^3}{3EI}, \quad \rightarrow \quad V_b = \frac{P}{2} \left[\frac{3b + 2a}{a} \right]. \quad (31)$$

Part [3d] (3 pts) Hence, derive a simple expression for the bending moment at A.

Solution: Take moments about A:

$$M_A = PL - V_b a = -\frac{Pb}{2} \quad (\text{counter clockwise}). \quad (32)$$

Finally, let's replace the roller support below point B with a spring.

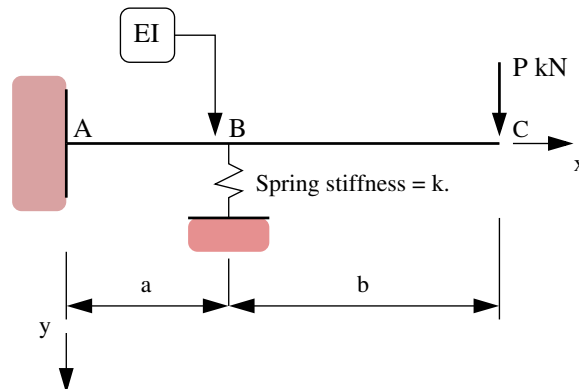


Figure 6: Cantilever supported by a spring at point B.

Part [3e] (4 pts) (4 pts) Show that the support reaction, V_b , is now given by the equation:

$$V_b \left[\frac{1}{k} + \frac{a^3}{3EI} \right] = \frac{Pa^2}{6EI} [3b + 2a]. \quad (33)$$

Solution: Recall

$$\Delta_B = \frac{Pa^2}{6EI} [2a + 3b] - \frac{V_b a^3}{3EI}, \quad (34)$$

Now we have a spring at B: $V_b = k\Delta_B$. Substituting equation 32 into $V_b = k\Delta_B$ and rearranging terms gives:

$$V_b \left[\frac{1}{k} + \frac{a^3}{3EI} \right] = \frac{Pa^2}{6EI} [3b + 2a]. \quad (35)$$

Part [3f] (4 pts) Explain why V_b for spring support is always lower than for roller support (i.e., equation 31).

Solution: You can think of the roller support as being a spring with infinite stiffness, i.e., $k \rightarrow \infty$ and $\frac{1}{k} \rightarrow 0$. Equation 36 simplifies to:

$$V_b = \left[\frac{\frac{Pa^2}{6EI} [3b + 2a]}{\left[\frac{a^3}{3EI} \right]} \right]. \quad (36)$$

Thus, V_b with roller is always larger than with a spring.

Question 4: 10 points

The simple beam shown in Figure 7 has length L and uniform section properties EI . A point load P is applied at distance a from the left-hand support.

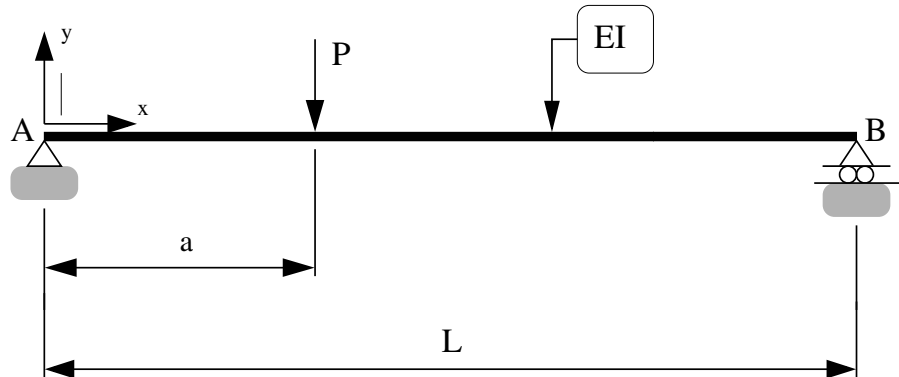
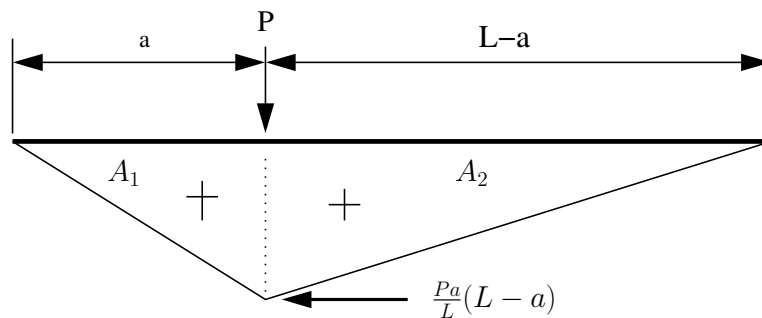


Figure 7: Front elevation view of a simple beam structure.

Part [4a] (2 pts) Draw and label the $M(x)/EI$ diagram in terms of the problem parameters (i.e., P , EI , L and a).

Solution: The $M(x)/EI$ diagrams is as follows:



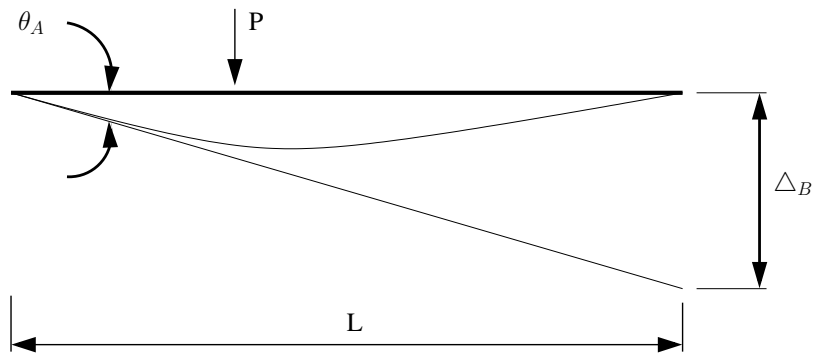
Here:

$$\begin{aligned} A_1 &= \frac{1}{2EI} \frac{Pa^2(L-a)}{L}, & \bar{x}_1 &= L - \frac{2}{3}a. \\ A_2 &= \frac{1}{2EI} \frac{Pa(L-a)^2}{L}, & \bar{x}_2 &= \frac{2}{3}(L-a). \end{aligned} \quad (37)$$

Part [4b] (5 pts) Use the method of moment-area to show that the beam rotation at A is:

$$\theta_A = \left[\frac{P}{6EI} \right] \left[\frac{a(L-a)(2L-a)}{L} \right]. \quad (38)$$

Solution: The moment area diagram is:



From geometry:

$$\theta_A = \frac{\Delta_B}{L}. \quad (39)$$

Applying moment area:

$$\Delta_B = A_1 \bar{x}_1 + A_2 \bar{x}_2 = \frac{1}{2EI} \frac{Pa^2(L-a)}{L} \left(L - \frac{2}{3}a \right) + \frac{1}{2EI} \frac{Pa(L-a)^2}{L} \frac{2}{3}(L-a). \quad (40)$$

Simplifying equation 40:

$$\Delta_B = \left[\frac{P}{6EI} \right] [a(L-a)(2L-a)] = \theta_A L. \quad (41)$$

Note. If \$a = L/2\$, then \$\theta_A = \frac{PL^2}{16EI}\$. It works!

Part [4c] (3 pts) Show that the maximum value of beam rotation at A occurs when:

$$a = L \left[1 - \frac{\sqrt{12}}{6} \right]. \quad (42)$$

Devise a simple strategy to test whether your answer makes sense (or not).

Solution: At the maximum value:

$$\frac{\partial \theta_A}{\partial a} = 0, \quad \longrightarrow \quad 2L^2 - 6La + 3a^2 = 0. \quad (43)$$

The root to equation 16 within the interval $0 < a < L$ is:

$$a = \left[1 - \frac{\sqrt{12}}{6} \right] \approx 0.422L. \quad (44)$$

The corresponding rotation is $0.0641 \frac{PL^2}{EI}$.

Note. As a point of comparison, when $a = L/2$, then $\theta_{a=L/2} = 0.0625 \frac{PL^2}{EI}$.