Department of Civil and Environmental Engineering, Spring Semester, 2026

ENCE 353 Final Exam, Open Notes and Open Book

Name : ________________________

Exam Format and Grading. This take home final exam is open notes and open book. You need to comply with the university regulations for academic integrity.

Answer Question 1. Then answer **two of the three** remaining questions. Partial credit will be given for partially correct answers, so please show all your working.

IMPORTANT: Only the **first three questions** that you answer will be graded, so please **cross out the question you do not want graded** in the table below. Also, before submitting your exam, check that **every page has been scanned correctly**.

Question	Points	Score
1	20	
2	10	
3	10	
4	10	
Total	40	

Question 1: 20 points

COMPULSORY: Method of Virtual Forces, Moment-Area Method. Figure 1 shows an L-shaped frame structure that carries a horizontal load P applied at B.

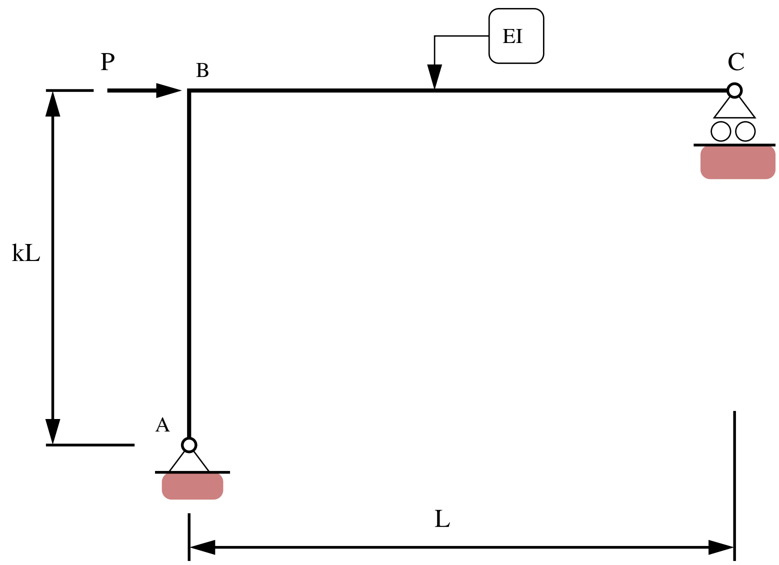

Figure 1: L-shaped frame structure.

The length of the beam segment B-C is L. The height of the column A-B is kL, where k is a positive constant. The L-shaped frame structure has flexural stiffness EI throughout. And the boundary conditions are as shown in Figure 1.

[1a] (3 pts). Draw and label a diagram showing the support reactions at A and C, and the bending moment diagram. Clearly indicate the peak values of bending moment (as a function of the problem parameters) and areas of the frame that are in tension/compression.

[1b] (2 pts). Create a qualitative sketch of displacements, highlighting the curvature of column A-B and beam B-C.

[1c] (6 pts). Use the method of **virtual forces** to show that the horizontal displacement of the frame at B, $\triangle_{bx}$, is given by:

$$\triangle_{bx} = \frac{PL^3}{3EI}\left[k^3 + k^2\right]. \tag{1}$$

Show all of your working.

[1d] (3 pts). Use the method of **virtual forces** to show that the clockwise rotation of the frame at B, θ_b, is given by:

$$\theta_b = \frac{PkL^2}{3EI}. \tag{2}$$

Show all of your working.

[1e] (3 pts). Use the method of **virtual forces** to show that the clockwise rotation of the frame at C, θ_c, is given by:

$$\theta_c = -\frac{PkL^2}{6EI}. \tag{3}$$

Show all of your working.

[1f] (3 pts). Briefly explain (a labeled diagram would be good) how equations 2 and 3 relate to the **method of moment area**.

Question 2: 10 points

OPTIONAL: Derive Elastic Curve for Cantilever Beam Deflection. Figure 2 is a front elevation view of a cantilevered beam carrying a uniform load, w (N/m), plus a single point load P. EI is constant along the cantilever A-B.

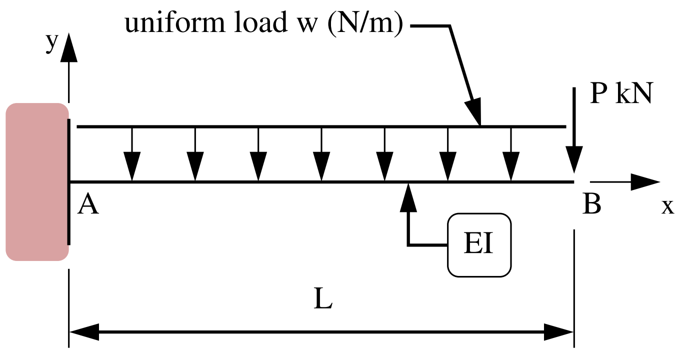

Figure 2: Cantilevered beam carrying a uniform load w (N/m) + single applied load P.

[2a] (2 pts). Briefly explain how the principle of superposition can be used in the analysis of this problem.

[2b] (3 pts). Draw and label the bending moment diagram(s).

[2c] (5 pts) Starting from the differential equation,

$$\frac{d^2y}{dx^2} = \left[\frac{M(x)}{EI}\right], \tag{4}$$

appropriate boundary conditions, derive expressions for: (1) the clockwise rotation of the cantilever at B, and (2) the vertical displacement of the beam at B.

Question 3: 10 points

OPTIONAL: Bending Moment and Curvature in an Elastic Beam. Figure 3 is a front elevation view of a simply supported beam that carries a trapezoid load.

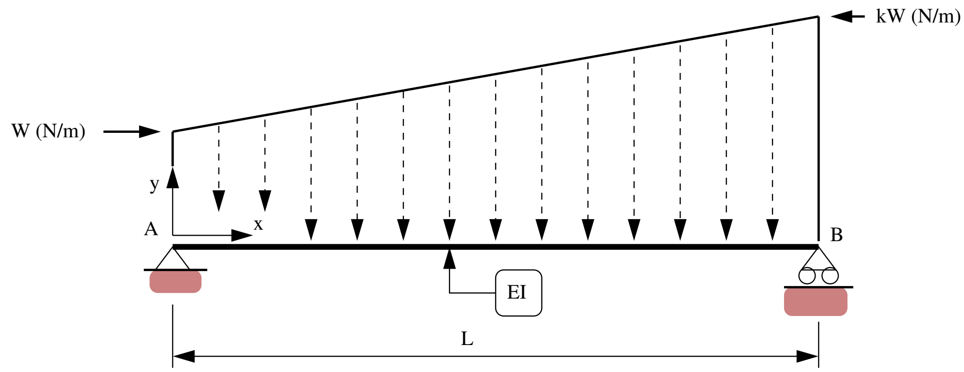

Figure 3: Simply supported beam carrying a trapazoid load.

The load increases from W (N/m) at x = 0, to kW (N/m) at x = L, where k is a non-negative constant. Thus, the total beam loading is $\frac{WL}{2}(1+k)$.

[3a] (2 pts). Starting from first principles of engineering, show that the vertical reactions at A and B are:

$$V_A = \frac{WL}{6}(2+k) \quad \text{and} \quad V_B = \frac{WL}{6}(1+2k)\,. \tag{5}$$

[3b] (3 pts). Show that the bending moment at point x is:

$$M(x) = \frac{WL^2}{6}\left(\frac{x}{L}\right)\left[(2+k) - 3\left(\frac{x}{L}\right) + (1-k)\left(\frac{x}{L}\right)^2\right]. \quad (6)$$

Notice that M(0) = M(L) = 0, regardless of the value of k.

The math for this part is a bit tedious – hence, I suggest you work out a solution on a separate sheet of paper, then write a tidy solution here.

[3c] (3 pts). Hence, show that the location of maximum curvature ϕ in the beam corresponds to the solution of the quadratic equation:

$$3\,(1-k)\,x^2 - 6Lx + (2+k)\,L^2 = 0. \tag{7}$$

[3d] (2 pts). For the case where k = 1 (i.e., a constant uniform loading), use equations 6 and 7 to determine the position and value of the maximum bending moment.

Question 4: 10 points

OPTIONAL: Principle of Virtual Work, Flexibility Matrix. The left-hand side of Figure 4 shows a simple two-bar truss that supports vertical and horizontal loads at node B. The right-hand side of Figure 4 shows the same truss with a third bar added – the latter makes the truss structure statically indeterminate to degree one. All of the truss members have cross section properties AE.

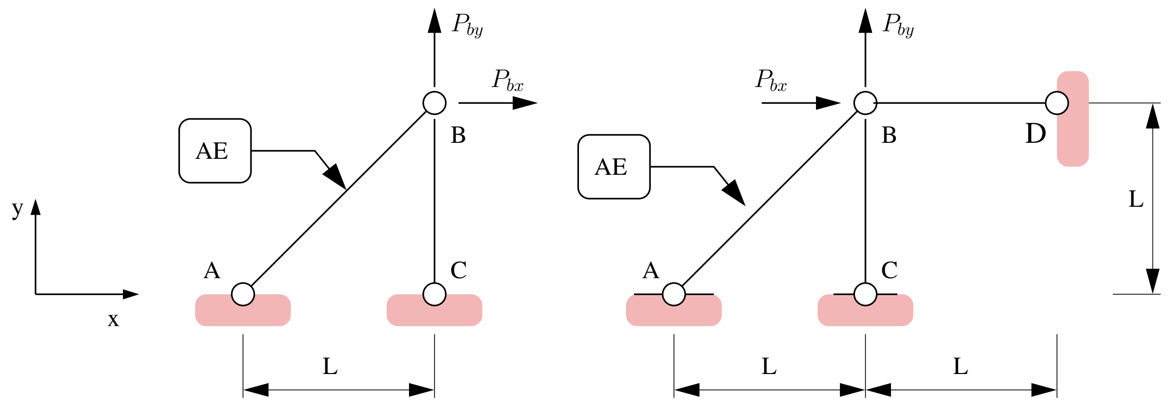

Figure 4: Front elevation view of: (left) A simple two-bar truss, and (right) a simple three-bar truss.

Let's start with the two-bar truss:

[4a] (5 pts) Use the method of **virtual forces** to compute the two-by-two flexibility matrix connecting the horizontal and vertical displacements at node B to the applied loads P_{bx} and P_{by}, L and AE.

$$\begin{bmatrix} \triangle_{bx} \\ \triangle_{by} \end{bmatrix} = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} \begin{bmatrix} P_{bx} \\ P_{by} \end{bmatrix}. \tag{8}$$

Question 4a continued:

Now let's consider the simple three-bar truss:

[4b] (5 pts). Using the method of **virtual forces** and the results of part [4a], or otherwise, derive a formula for the member force **BD** as a function of applied loads P_{bx} and P_{by}.

Show that if P_{by} = 0, the compressive force in member BD is:

$$\text{Member force BD} = \left[\frac{1 + 2\sqrt{2}}{2 + 2\sqrt{2}} \right] P_{bx}. \quad (9)$$