ENCE 353 Solutions to Midterm 2

Question 1: 15 points

Analysis of a Cantilever Beam with Moment Area. Consider the cantilever shown in Figure 1.

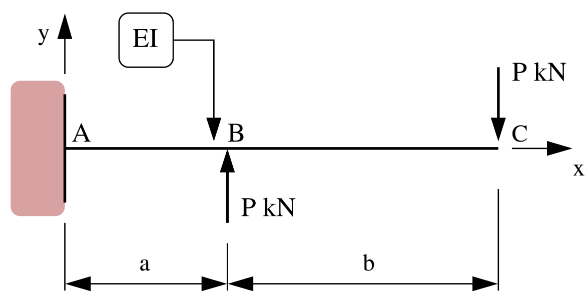
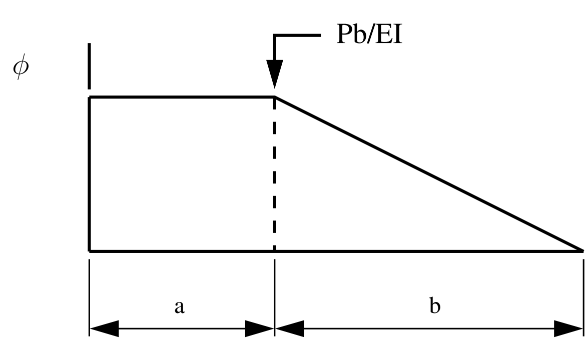


Figure 1: Front elevation view of a cantilever.

The cantilever has constant section properties, EI, along its entire length (a+b). Vertical loads of P (kN) are applied at points B and C. Notice that the y axis is pointing upwards – hence, if we apply the right-hand rule, positive rotations will be anti-clockwise.

Part [1a] (3 pts) Draw and label a diagram showing how the beam curvature varies along its length (i.e., $\phi(x)$).

Solution:



Part [1b] (3 pts) Use the method of moment-area to show that the rotation of point B is:

$$\theta(a) = \left[\frac{-Pba}{EI}\right]. \tag{1}$$

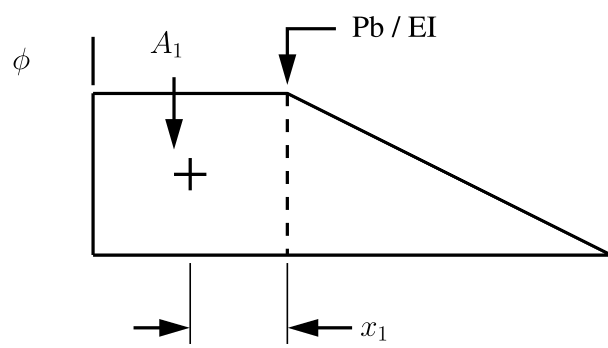
Solution: From the first theorem of moment area, $\theta(a)$ = area of M/EI diagram between x = 0 and x = a. Hence,

$$\theta(a) = -\frac{Pba}{EI}. \tag{2}$$

Part [1c] (4 pts) Use the method of moment-area to show that the vertical displacement of point B is:

$$y(a) = \left[\frac{-Pba^2}{2EI}\right]. \tag{3}$$

Solution: The vertical displacement at B is given by:



Here:
$A_1 = \frac{Pab}{EI}$
$x_1 = \frac{a}{2}$

the first moment of area between A and B evaluated about B, i.e.,

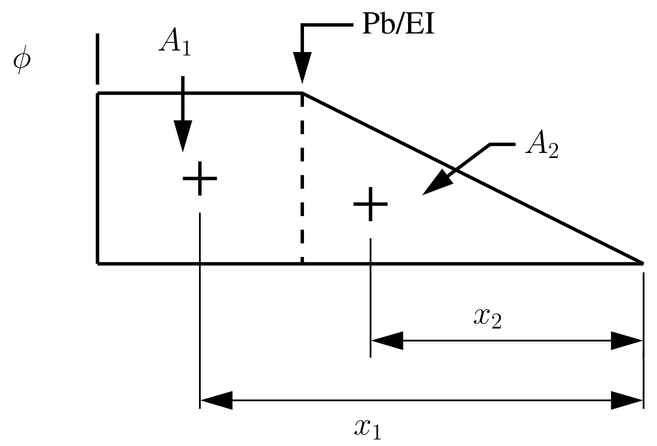
$$y(a) = -A_1 x_1 = -\frac{Pba^2}{2EI}. \tag{4}$$

Part [1d] (5 pts) Use the method of moment-area to show that the vertical displacement of point C is:

$$y(a+b) = \left[\frac{-Pb}{6EI}\right]\left[6ab + 3a^2 + 2b^2\right]. \tag{5}$$

Note: Notice that when a = 0, equation 5 simplifies to the formula we have seen many times in class.

Solution: The vertical displacement at C is given by:



Here:

$A_1 = \frac{Pba}{EI}$

$A_2 = \frac{Pb^2}{2EI}$

$x_1 = b + \frac{a}{2}$

$x_2 = \frac{2}{3}b$

the first moment of area between A and C evaluated about C.

$$y(a+b) = [A_1 x_1 + A_2 x_2] = -\frac{Pab}{EI}\left(b + \frac{a}{2}\right) - \frac{Pb^2}{2EI}\left(\frac{2}{3}b\right) = -\frac{Pb}{6EI}\left[6ab + 3a^2 + 2b^2\right]. \qquad (6)$$

Question 2: 15 points

Elastic Curve for a Cantilever Beam Structure. Consider the cantilever shown in Figure 2.

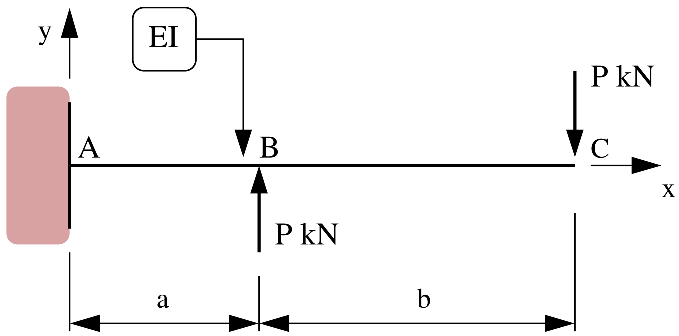


Figure 2: Front elevation view of a cantilever.

The cantilever has constant section properties, EI, along its entire length (a+b). Vertical loads of P (kN) are applied at points B and C.

Part [2a] (3 pts) Write a formula, M(x), for the bending moment along the beam as a function of x.

Solution: M(x) can be expressed as a piecewise function:

$$M(x) = \begin{cases} -Pb & 0 \leq x \leq a \\ -P\left(a + b - x\right) & a \leq x \leq a + b. \end{cases} \tag{7}$$

Part [2b] (4 pts) Starting from the differential equation,

$$\frac{d^2y}{dx^2} = \left[\frac{M(x)}{EI}\right], \tag{8}$$

and appropriate boundary conditions, show that in the interval $0 \leq x \leq a$, the beam displacement and rotation are:

$$y(x) = \left[\frac{-Pbx^2}{2EI}\right] \quad \text{and} \quad \theta(x) = \left[\frac{-Pbx}{EI}\right]. \tag{9}$$

Show all of your working.

Solution: Within the interval $0 \leq x \leq a$,

$$\frac{d^2y}{dx^2} = \frac{-Pb}{EI}. \tag{10}$$

Integrating twice gives:

$$\frac{dy}{dx} = \frac{-Pbx}{EI} + A. \tag{11}$$

and

$$y(x) = \frac{-Pbx^2}{2EI} + Ax + B. \tag{12}$$

Two boundary conditions: $y(0) = \frac{dy}{dx}|_{x=0} = 0$. Hence, A = 0 and B = 0. Thus,

$$y(x) = -\left[\frac{Pbx^2}{2EI}\right] \quad \text{and} \quad \theta(x) = \frac{dy}{dx} = -\left[\frac{Pbx}{EI}\right]. \tag{13}$$

Part [2c] (7 pts) Starting from the differential equation,

$$\frac{d^2y}{dx^2} = \left[\frac{M(x)}{EI}\right], \tag{14}$$

and appropriate boundary conditions, show that within the interval $a \leq x \leq a + b$, the beam displacement is:

$$y(x) = \left[\frac{-P}{6EI}\right]\left[3(a+b)x^2 - x^3 - 3a^2x + a^3\right]. \tag{15}$$

Solution: Boundary conditions:

- At x = a,

$$y(a) = -\frac{Pba^2}{2EI} \quad \text{and} \quad \frac{dy}{dx} = -\frac{Pab}{EI}. \tag{16}$$

- At x = a + b,

$$\frac{d^2y}{dx^2}|_{x=a+b} = 0. \quad \longleftarrow \quad \text{because M(a+b) = 0.} \tag{17}$$

Within our interval of interest $M(x) = -P\,(a+b-x)$. Hence, from equation 14,

$$\frac{d^2y}{dx^2} = -\frac{P}{EI}\,(a+b-x)\,. \tag{18}$$

Integrating twice:

$$-\frac{EI}{P}\frac{dy}{dx} = (a+b)x - \frac{x^2}{2} + A. \tag{19}$$

$$-\frac{EI}{P}y(x) = (a+b)\frac{x^2}{2} - \frac{x^3}{6} + Ax + B. \tag{20}$$

At x = a,

$$-\frac{EI}{P}\frac{dy}{dx} = ba = (a+b)a - \frac{a^2}{2} + A. \tag{21}$$

$$-\frac{EI}{P}y(x) = (a+b)\frac{a^2}{2} - \frac{a^3}{6} + Aa + B. \tag{22}$$

Hence, $A = -\frac{a^2}{2}$ and $B = \frac{a^3}{6}$. Collecting terms:

$$y(x) = \left[\frac{-P}{6EI}\right]\left[3(a+b)x^2 - x^3 - 3a^2x + a^3\right]. \tag{23}$$

Note: Equation 23 is consistent with equations 3 and 5 in Question 1.

$$y(a) = -\frac{Pba^2}{2EI} \quad \longleftarrow \quad \text{Matches moment area.} \tag{24}$$

and

$$y(a+b) = -\frac{Pba^2}{2EI} \quad \longleftarrow \quad \text{Also matches moment area.} \tag{25}$$

Question 3: 10 points

Analysis of a Three-Pinned Parabolic Arch. Consider the three-pinned parabolic arch shown in Figure 3

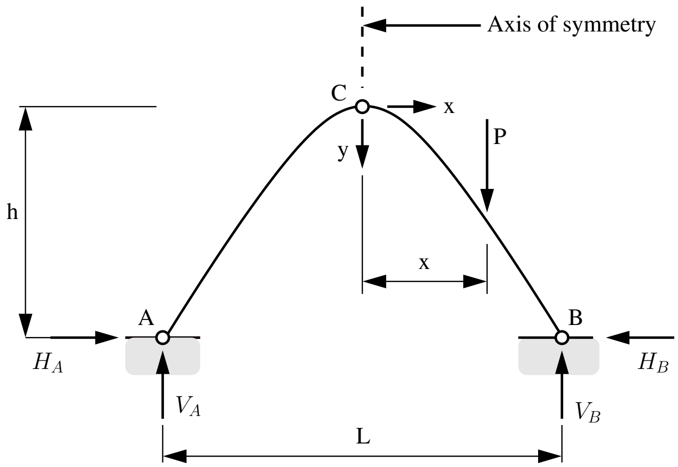


Figure 3: Three-pinned parabolic arch carrying a point load.

The height and width of the arch are h and L, respectively. A point load P is applied at a distance x, $0 \leq x \leq L/2$, from the axis of symmetry.

Part [3a] (5 pts) Starting from first principles of engineering (i.e., equations of equilibrium), show that the vertical and horizontal reaction forces at A and B are:

$$V_A(x) = P\left[\frac{1}{2} - \frac{x}{L}\right] \tag{26}$$

$$V_B(x) = P\left[\frac{1}{2} + \frac{x}{L}\right] \tag{27}$$

$$H_A(x) = H_B(x) = \frac{P}{2h}\left[\frac{L}{2} - x\right] \tag{28}$$

Solution: Taking moments about A:

$$\sum M_A = 0 \quad \longrightarrow \quad P\left(\frac{L}{2} + x\right) = V_b \cdot L. \tag{29}$$

Hence,

$$V_B(x) = P\left(\frac{1}{2} + \frac{x}{L}\right). \tag{30}$$

Summing forces in the vertical direction, $V_A + V_B = P$. Hence,

$$V_A = P - V_B \quad \longrightarrow \quad V_A(x) = P\left(\frac{1}{2} - \frac{x}{L}\right). \tag{31}$$

Finally, take moments about C and sum forces in the horizontal direction:

$$\sum M_C = 0 \quad \longrightarrow \quad Px + H_B = V_B\left(\frac{L}{2}\right). \tag{32}$$

Hence,

$$H_A(x) = H_B(x) = \frac{P}{2h}\left(\frac{L}{2} - x\right). \tag{33}$$

Part [3b] (5 pts) Now suppose that the point load P is replaced by a uniform load w_o (N/m) along the right-hand side of the arch, as shown in Figure 4. By using equations 26 through 28 as a starting point, and noting that a tiny increment of uniform loading can be written P(x) = w_0 dx, show that:

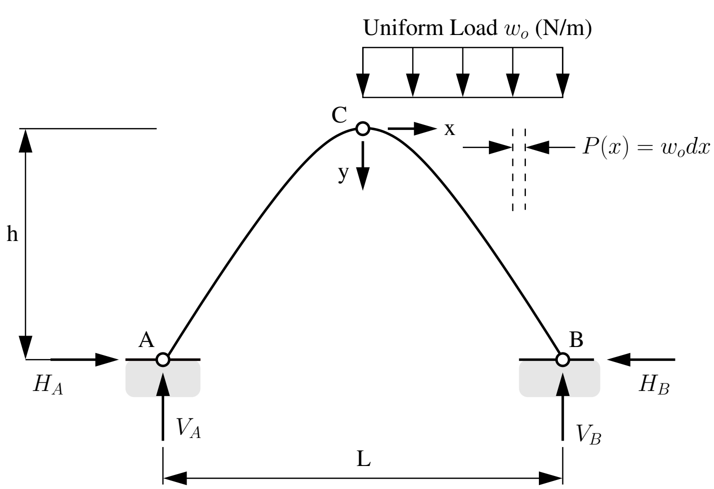


Figure 4: Three-pinned parabolic arch carrying a uniform load.

$$V_A = \left[\frac{w_o L}{8}\right], \quad V_B = \left[\frac{3 w_o L}{8}\right] \quad \text{and} \quad H_A = H_B = \left[\frac{w_o L^2}{16h}\right]. \tag{34}$$

Solution: Let $P(x) = w(x)dx = w_o dx$. The total reaction force will simply be the sum of all the tiny increments, integrated over the interval $0 \leq x \leq \frac{L}{2}$.

$$V_B = \int_0^{L/2} w_o \left[\frac{1}{2} + \frac{x}{L}\right] dx = \frac{3}{8} w_o L. \tag{35}$$

Similarly,

$$V_A = \int_0^{L/2} w_o \left[\frac{1}{2} - \frac{x}{L}\right] dx = \frac{1}{8} w_o L. \tag{36}$$

and

$$H_A = \frac{w_o}{2h} \int_0^{L/2} \left[\frac{L}{2} - x\right] dx = \frac{w_o L^2}{16h}. \tag{37}$$

Note: Equation 37 is exactly one half of the horizontal reaction force caused by a parabolic arch carrying a uniform load across its entire span.