

ENCE 353 Solutions to Final Exam

Question 1: 20 points

COMPULSORY: Covers moment-area, virtual forces, superposition. Figure 5 is a front elevation view of a simple beam structure carrying two external loads P . The beam has section properties EI near the supports and $2EI$ in the center section.

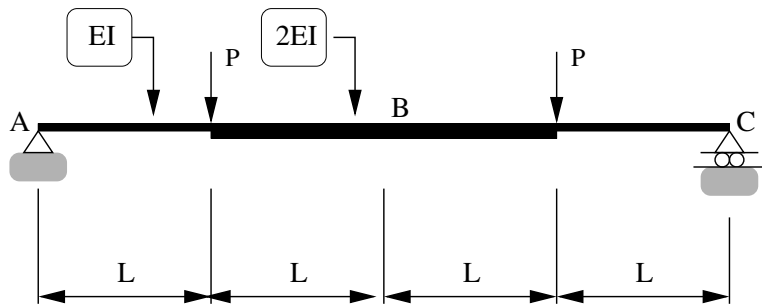
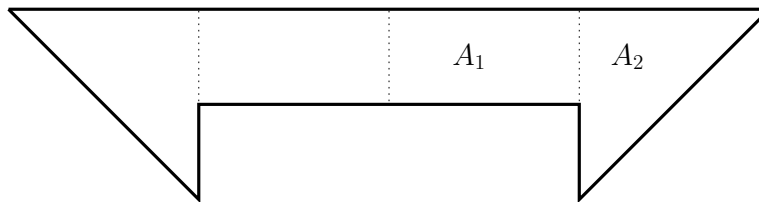


Figure 1: Simple beam structure (symmetric loads P).

Part [1a] (4 pts) Use the method of **moment area** to show that the end rotation at A (measured clockwise) is:

$$\theta_A = \frac{PL^2}{EI}. \tag{1}$$

Solution: This problem has an axis of symmetry at B – as such, $\theta_B = 0$. The M/EI diagram is as follows:



End rotation at A is:

$$\theta_A - \theta_B = \int dx = \frac{1}{2} \frac{PL^2}{EI} + \frac{PL^2}{2EI} = \frac{PL^2}{EI}. \tag{2}$$

Part [1b] (4 pts) Use the method of **moment area** to show that the vertical beam deflection at B is:

$$\Delta_B = \frac{13 PL^3}{12 EI}. \quad (3)$$

Solution: Vertical deflection at B is given by first moment of area of curvature diagram between A and B, evaluated about A, i.e.,

$$\Delta_B = A_1 \bar{x}_1 + A_2 \bar{x}_2. \quad (4)$$

From the curvature diagram:

$$A_1 = \frac{PL^2}{2EI}, \quad \bar{x}_1 = \frac{3}{2}L, \quad A_2 = \frac{PL^2}{2EI}, \quad \bar{x}_2 = \frac{2}{3}L. \quad (5)$$

Substitute equations 6 into 4:

$$\Delta_B = A_1 \bar{x}_1 + A_2 \bar{x}_2 = \frac{PL^3}{2EI} \left[\frac{3}{2} + \frac{2}{3} \right] = \frac{13 PL^3}{12 EI}. \quad (6)$$

Part [1c] (4 pts) A function is said to be even if it has the property $f(x) = f(-x)$ (i.e., it is symmetric about the y axis). And a function is said to be odd if it has the property $g(x) = -g(-x)$ (i.e., it is skew-symmetric about the y axis). Examples of even functions include $y = x^2$ and $y = \cos(x)$. Similarly, $y = x^3$ and $y = \sin(x)$ are two examples of an odd function.

Using concepts from calculus (or otherwise), show that:

$$\int_{-h}^h f(x)g(x)dx = 0. \quad (7)$$

Solution: Split integral (equation 7) into two parts:

$$\int_{-h}^h f(x)g(x)dx = \int_{-h}^0 f(x)g(x)dx + \int_0^h f(x)g(x)dx. \quad (8)$$

Now define:

$$I = \int_0^h f(x)g(x)dx. \quad (9)$$

For the first integral, let $y = -x$. It follows that $dy = -dx$, and $f(x) = f(y)$ and $g(x) = -g(y)$. Hence:

$$\int_{-h}^0 f(x)g(x)dx = - \int_{-h}^0 f(y)g(y)dy = -I \quad (10)$$

Therefore,

$$\int_{-h}^h f(x)g(x)dx = -I + I = 0. \quad (11)$$

This result holds for all even/odd functions $f(x)/g(x)$, not just a few specific examples.

Alternate: Define $h(x) = f(x)g(x)$. Show that the function $h(x)$ is odd. Hence,

$$\int_{-h}^h h(x)dx = \int_{-h}^h f(x)g(x)dx = 0. \quad (12)$$

Part [1d] (4 pts) Figure 2 shows the same beam structure, but now the external loads are rearranged so that one load points down and one load points up.

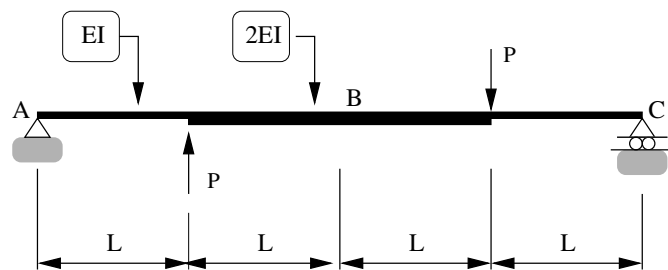
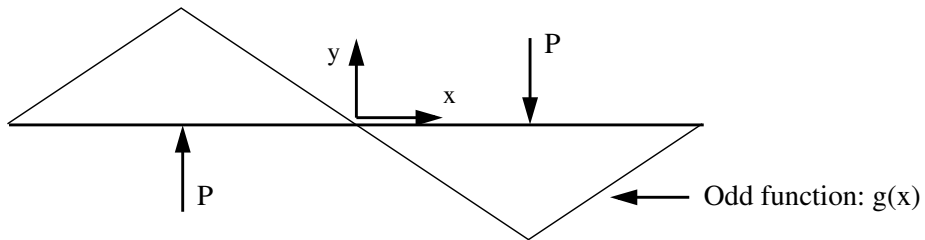


Figure 2: Simple beam structure (skew-symmetric loads P).

Use the method of **virtual forces** and a coordinate system positioned at B to show that the vertical displacement of B is zero, i.e., $\Delta_B = 0$.

Solution: The upper/lower sections of Figure 3 show: (1) the real BMD due to the applied loads, and (2) the BMD caused by a unit virtual load applied at B. Notice that the real BMD is an odd function and the

BMD due to Real Loads



BMD due to virtual unit load at B

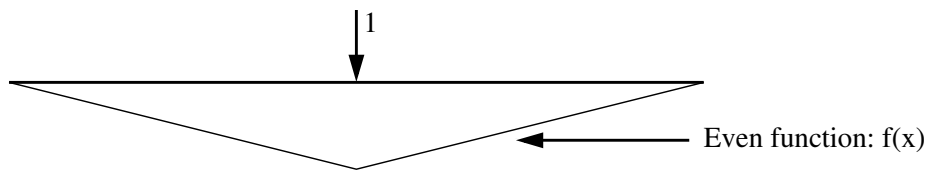


Figure 3: BMDs for real applied loads and unit virtual load.

BMD caused by a unit load applied at B is an even function. Hence, from part 1c,

$$\Delta_B = \int_{-2L}^{2L} \left[\frac{m_1^*(x)}{EI} \right] m_2^{**}(x) dx = 0. \quad (13)$$

Part [1e] (4 pts) Now consider the problem.

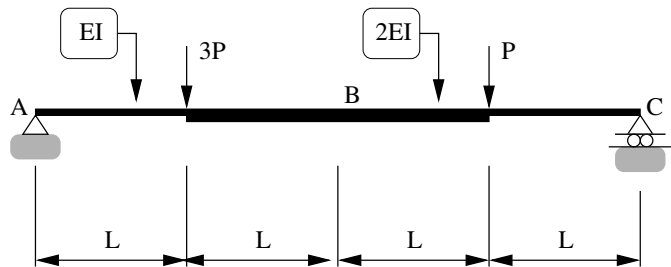
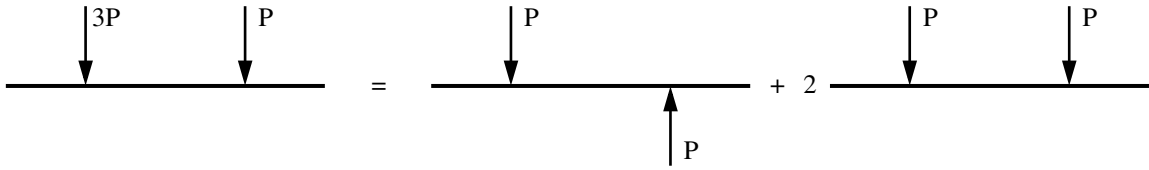


Figure 4: Simple beam structure with external loads 3P and P.

Use your answers from parts [1b] and [1d] to write down an expression for the vertical deflection at B due to the loading pattern shown in Figure 4. Note: You should find this is a one line answer.

Solution: By superposition:



Hence,
$$\Delta_B = 0 + 2\left(\frac{13 PL^3}{12 EI}\right) = \frac{13 PL^3}{6 EI}. \quad (14)$$

Question 2: 10 points

OPTIONAL: Moment-Area Method: The simple beam shown in Figure 5 has length L and uniform section properties EI . A point load P is applied at distance a from the left-hand support.

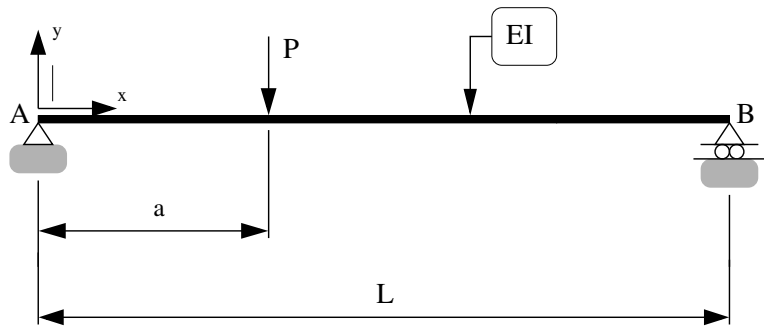
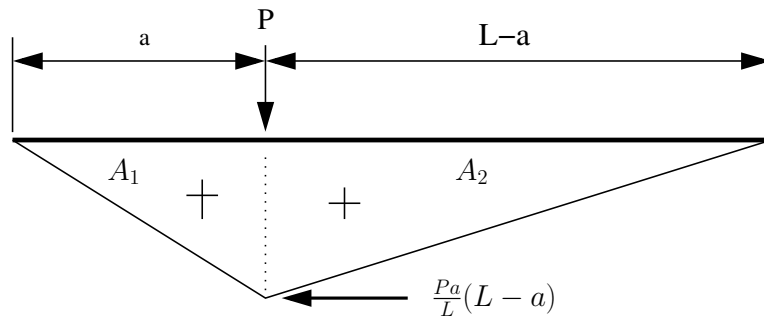


Figure 5: Front elevation view of a simple beam structure.

Part [2a] (2 pts) Draw and label the $M(x)/EI$ diagram in terms of the problem parameters (i.e., P , EI , L and a).

Solution: The $M(x)/EI$ diagrams is as follows:



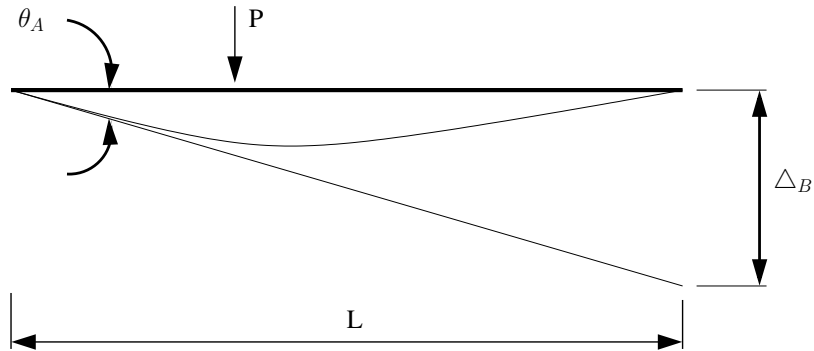
Here:

$$\begin{aligned}
 A_1 &= \frac{1}{2EI} \frac{Pa^2(L-a)}{L}, & \bar{x}_1 &= L - \frac{2}{3}a. \\
 A_2 &= \frac{1}{2EI} \frac{Pa(L-a)^2}{L}, & \bar{x}_2 &= \frac{2}{3}(L-a).
 \end{aligned}
 \tag{15}$$

Part [2b] (5 pts) Use the **method of moment-area** to show that the beam rotation at A is:

$$\theta_A = \left[\frac{P}{6EI} \right] \left[\frac{a(L-a)(2L-a)}{L} \right]. \quad (16)$$

Solution: The moment area diagram is:



From geometry:

$$\theta_A = \frac{\Delta_B}{L}. \quad (17)$$

Applying moment area:

$$\Delta_B = A_1 \bar{x}_1 + A_2 \bar{x}_2 = \frac{1}{2EI} \frac{Pa^2(L-a)}{L} \left(L - \frac{2}{3}a \right) + \frac{1}{2EI} \frac{Pa(L-a)^2}{L} \frac{2}{3}(L-a). \quad (18)$$

Simplifying equation 18:

$$\Delta_B = \left[\frac{P}{6EI} \right] [a(L-a)(2L-a)] = \theta_A L. \quad (19)$$

Note. If \$a = L/2\$, then \$\theta_A = \frac{PL^2}{16EI}\$. It works!

Part [2c] (3 pts) Show that the **maximum value** of beam rotation at A occurs when:

$$a = L \left[1 - \frac{\sqrt{12}}{6} \right]. \quad (20)$$

Solution: At the maximum value:

$$\frac{\partial \theta_A}{\partial a} = 0, \quad \longrightarrow \quad 2L^2 - 6La + 3a^2 = 0. \quad (21)$$

The root to equation 22 within the interval $0 < a < L$ is:

$$a = \left[1 - \frac{\sqrt{12}}{6} \right] \approx 0.422L. \quad (22)$$

The corresponding rotation is $0.0641 \frac{PL^2}{EI}$.

Note. As a point of comparison, when $a = L/2$, then $\theta_{a=L/2} = 0.0625 \frac{PL^2}{EI}$.

Question 3: 10 points

OPTIONAL: Principle of Virtual Work. Figure 6 is a front elevation view of a simple truss that supports vertical loads at nodes C and D. All of the truss members have cross section properties AE .

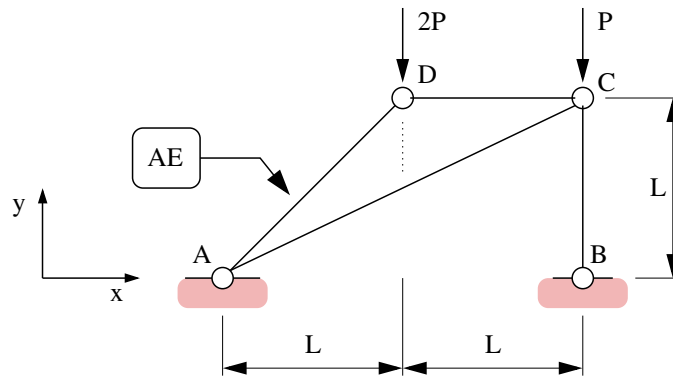


Figure 6: Front elevation view of a simple truss.

Part [3a] (4 pts). Compute the support reactions and distribution of forces throughout the structure.

Solution: From statics:

$$\sum M_A = 0, \quad \rightarrow \quad 2PL + 2PL = 2LV_B \quad \rightarrow \quad V_B = 2P. \quad (23)$$

$$\sum V = 0, \quad \rightarrow \quad V_A + V_B = 3P \quad \rightarrow \quad V_A = P. \quad (24)$$

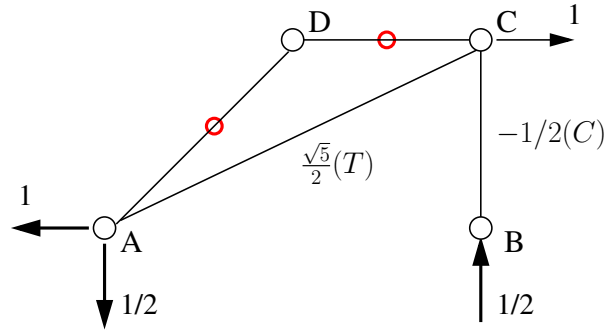
$$\sum H = 0, \quad \rightarrow \quad H_A = 0. \quad (25)$$

Member forces: $\overline{CD} = \overline{BC} = -2P$ (C), $\overline{CA} = \sqrt{5}P$ (T), and $\overline{AD} = -2\sqrt{2}P$ (C).

Part [3b] (6 pts). Use the method of **virtual forces** to show that the horizontal deflection at node C is:

$$\Delta = \frac{PL}{AE} \left[1 + \frac{5\sqrt{5}}{2} \right]. \quad (26)$$

Solution: Apply unit force in the horizontal direction at node C:



Displacement in the horizontal direction is given by:

$$\Delta_x = \sum_{i=1}^N \left(\frac{F_i f_h L_i}{A_i E_i} \right). \quad (27)$$

Tabulate displacements:

| Member | L/AE | F | f_h | $\sum_{i=1}^N \left(\frac{F_i f_h L_i}{A_i E_i} \right)$ |
|--------|----------------|---------------|--------------|---|
| 1 | $\sqrt{2}L/AE$ | $-2\sqrt{2}P$ | 0 | 0 |
| 2 | L/AE | $-2P$ | 0 | 0 |
| 3 | $\sqrt{5}L/AE$ | $\sqrt{5}P$ | $\sqrt{5}/2$ | $\frac{5\sqrt{5}}{2} \frac{PL}{AE}$ |
| 4 | L/AE | $-2P$ | $-1/2$ | $\frac{PL}{AE}$ |
| Total | | | | $\left(1 + \frac{5\sqrt{5}}{2} \right) \frac{PL}{AE}$ |

Summary:

$$\Delta_x = \left(1 + \frac{5\sqrt{5}}{2} \right) \frac{PL}{AE}. \quad (28)$$

Question 4: 10 points

OPTIONAL: Analysis of Forces in a Cable Structure. Figure 7 is a front elevation view of a cable structure and bridge deck that carries a snow loading.

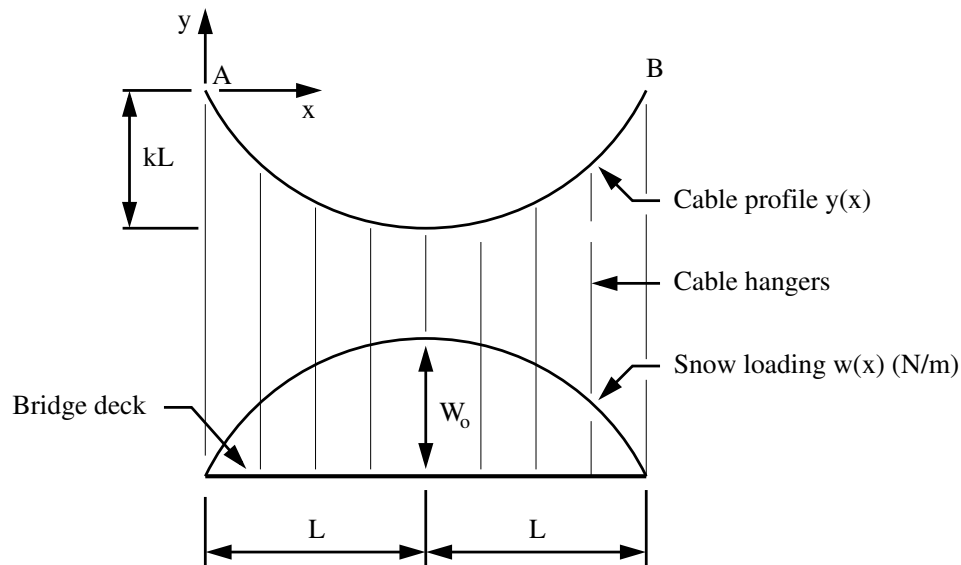


Figure 7: Cable structure carrying a snow loading.

The bridge deck and cable system have total span $2L$. The cable sag is kL , where $0 < k < 1$. The distribution of snow loading follows a sinusoidal shape:

$$w(x) = W_o \sin\left(\frac{\pi x}{2L}\right) \quad (N/m). \quad (29)$$

Part [4a] (4 pts) Starting from first principles of engineering (i.e., the differential equation for cable behavior) show that the horizontal component of cable force is:

$$H = \frac{W_o}{kL} \left(\frac{2L}{\pi}\right)^2 \quad (30)$$

Solution: Behavior of the cable system is given by solutions to the differential equation:

$$\frac{d^2 y}{dx^2} = \frac{W_o}{H} \sin\left(\frac{\pi x}{2L}\right) \quad (31)$$

Integrating equation 32 twice:

$$\begin{aligned}\frac{dy}{dx} &= \frac{-W_o}{H} \frac{2L}{\pi} \cos\left(\frac{\pi x}{2L}\right) + A, \\ y(x) &= \frac{-W_o}{H} \left(\frac{2L}{\pi}\right)^2 \sin\left(\frac{\pi x}{2L}\right) + Ax + B.\end{aligned}\tag{32}$$

Notice that the coordinate system is positioned at the top- left-hand support. The displacement boundary conditions are as follows:

$$y(0) = y(2L) = 0, \quad \longrightarrow \quad B = 0.\tag{33}$$

Also, from symmetry:

$$\left.\frac{dy}{dx}\right|_{x=L} = \frac{-W_o}{H} \frac{2L}{\pi} \cos\left(\frac{\pi}{2}\right) + A = 0, \quad \longrightarrow \quad A = 0.\tag{34}$$

Finally, at $x = L$:

$$y(L) = -kL = \frac{-W_o}{H} \left(\frac{2L}{\pi}\right)^2 \sin\left(\frac{\pi}{2}\right)\tag{35}$$

Rearranging terms:

$$H = \frac{W_o}{kL} \left(\frac{2L}{\pi}\right)^2\tag{36}$$

Part [4b] (3 pts) Determine the vertical and horizontal components of cable force at the supports A and B.

Solution: We begin by noting that:

$$\text{Total snow loading} = \int_0^{2L} w(x) = \int_0^{2L} W_o \sin\left(\frac{\pi x}{2L}\right) dx = \frac{4W_o L}{\pi}.\tag{37}$$

At the support reactions:

$$\left.\frac{dy}{dx}\right|_{x=0} = \frac{V}{-H} = \frac{W_o}{-H} \frac{2L}{\pi} \cos(0).\tag{38}$$

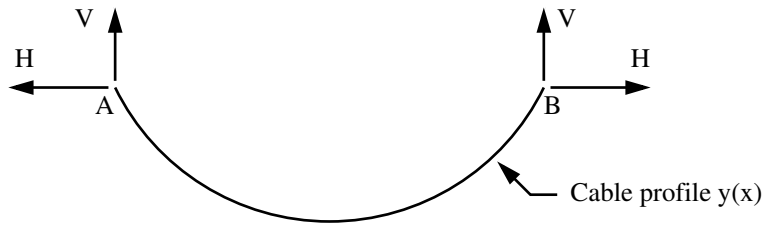


Figure 8: Horizontal and vertical support reactions.

Plugging equation 36 into 38 and rearranging terms:

$$H = \frac{W_o}{kL} \left(\frac{2L}{\pi} \right)^2 \quad \text{and} \quad V = \frac{2W_oL}{\pi}. \quad (39)$$

Notice that the total snow loading = $2V$, the cable system is in equilibrium.

Part [4c] (3 pts) Show that the maximum force in the cable, T_{max} , is:

$$T_{max} = \frac{2W_oL}{\pi} \left[1 + \left(\frac{2}{k\pi} \right)^2 \right]^{1/2} \quad (40)$$

Solution: The maximum cable force occurs at the supports:

$$T_{max} = [H^2 + V^2]^{1/2} = W_o \left(\frac{2L}{\pi} \right) \left[1 + \left(\frac{2}{k\pi} \right)^2 \right]^{1/2} \quad (41)$$

Question 5: 10 points

OPTIONAL: Structural Analysis of a T-shaped Beam Structure. Consider the T-shaped beam structure shown in Figure 9.

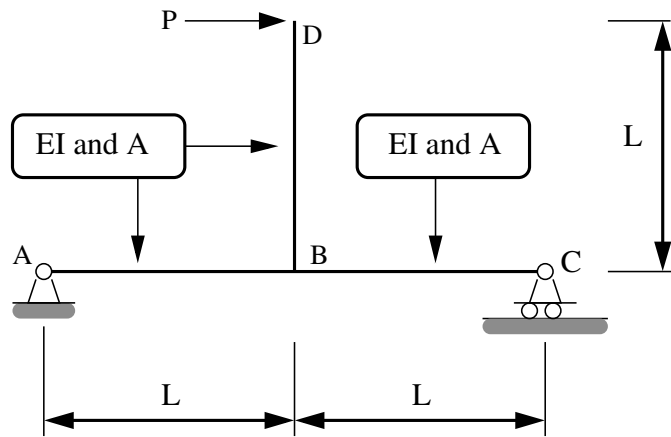
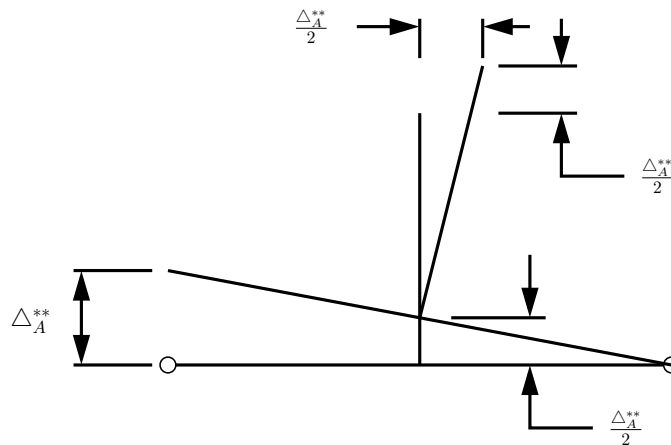


Figure 9: Front elevation view of a simple T-shaped beam structure.

Part [5a] (3 pts) Use the method of **virtual displacements** to compute the vertical reaction at A.

Solution: Impose virtual displacement Δ_A at A:



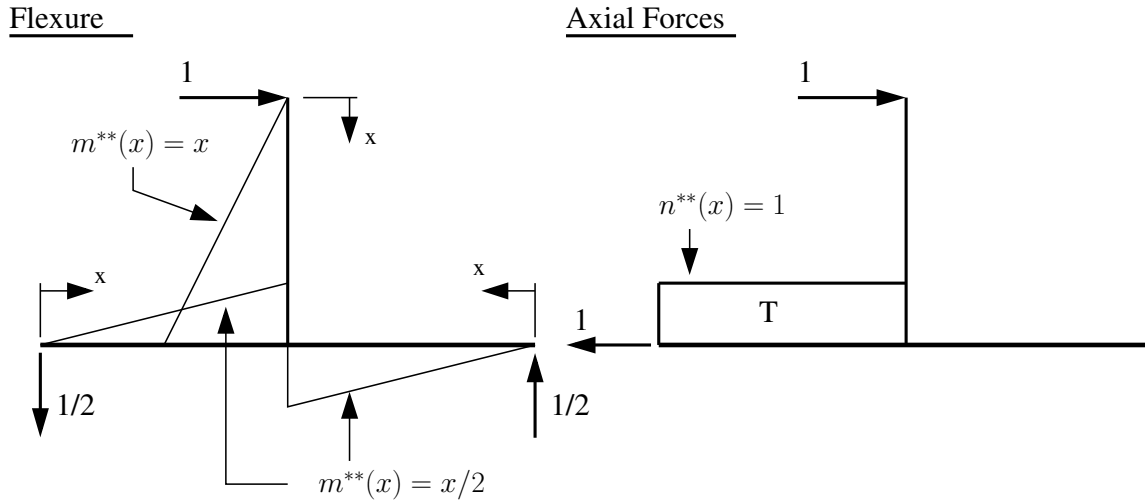
Evaluate energy balance:

$$\sum EWD = 0, \quad \rightarrow \quad V_A \Delta_A^{**} + P \left(\frac{\Delta_A^{**}}{2} \right) = 0, \quad \rightarrow \quad V_A = -P/2. \quad (42)$$

Part [5b] (7 pts) Use the method of **virtual forces** to compute a formula for the horizontal displacement at

D due to flexure of the individual segments, e.g., A-B, B-C, B-D) plus axial extension of A-B.

Solution: Apply a unit point load at D. The bending moment and axial force diagrams are as follows:



Apply method of virtual forces:

$$\begin{aligned}
 \Delta_D &= \frac{1}{EI} \int_0^L M_c^*(x) m_c^{**}(x) dx + \frac{2}{EI} \int_0^L M_b^*(x) m_b^{**}(x) dx + \frac{1}{EA} \int_0^L N_b^*(x) n_b^{**}(x) dx. \\
 &= \frac{P}{EI} \int_0^L x^2 dx + \frac{2P}{EI} \int_0^L \left(\frac{x}{2}\right)^2 dx + \frac{P}{AE} \int_0^L 1 dx. \\
 &= \frac{PL^3}{2EI} + \frac{PL}{AE}.
 \end{aligned} \tag{43}$$