Arch Structures

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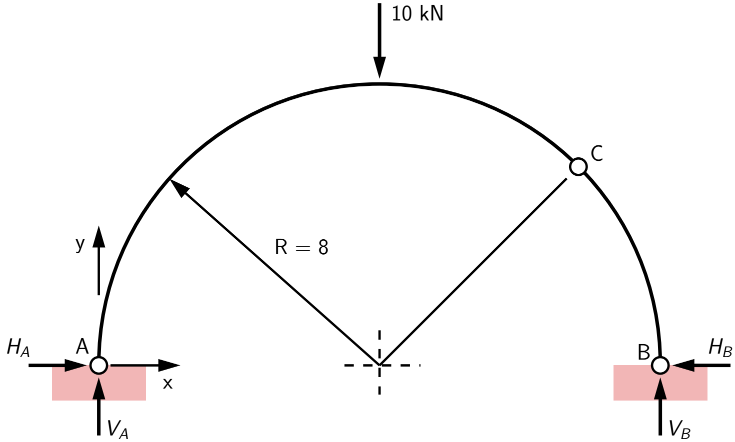
Overview

Analysis

Part 1: Circular Arch

Analysis of Circular Arch

Problem Setup:



Analysis of Circular Arch

Equations of Equilibrium:

$$\sum F_y = 0, \qquad V_A + V_B = 0, \tag{1}$$

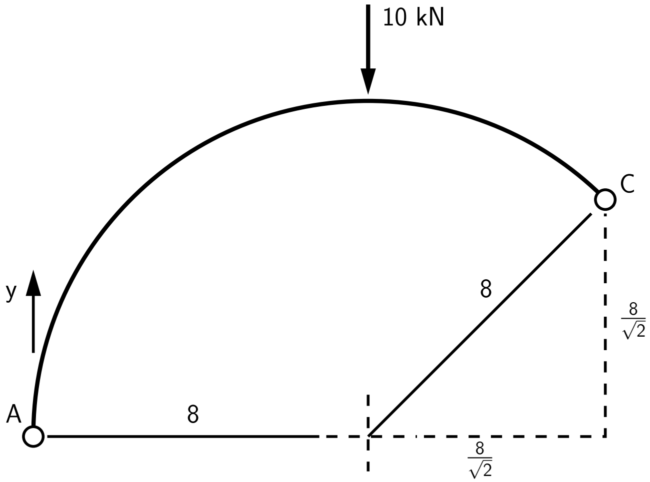
$$\sum M_A = 0, \qquad 10R - V_B(2R) = 0, \tag{2}$$

$$\sum F_x = 0, \qquad H_A = H_B \quad \text{(does not help)}. \tag{3}$$

From equations 1 and 2, $V_A = V_B = 5.0$ kN. We also know $M_C = 0$. For the left-hand substructure:

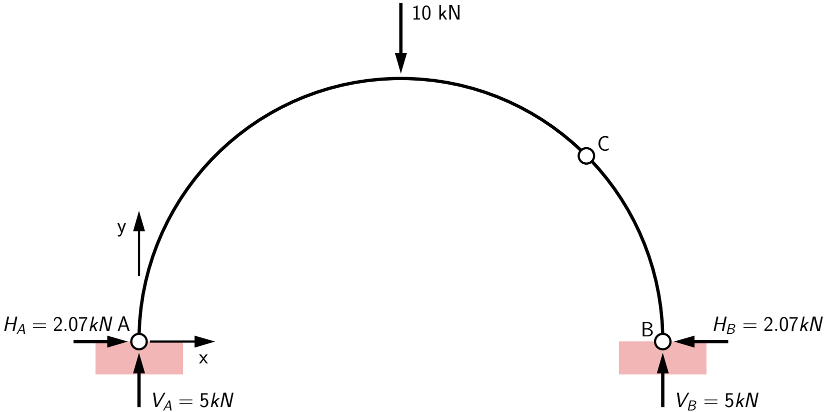
$$10 + H_A \frac{8}{\sqrt{2}} = V_A \left(8 + \frac{8}{\sqrt{2}}\right) \longrightarrow H_A = H_B = 2.07 \quad \text{kN}. \tag{4}$$

Analysis of Circular Arch



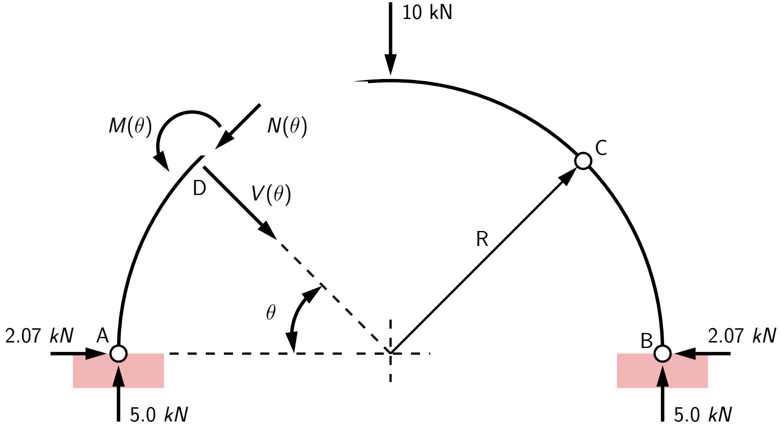
Analysis of Circular Arch

Summary of Support Reactions:



Analysis of Circular Arch

Bending Moments and Shear Forces: Polar coordinates (R,θ):



Analysis of Circular Arch

Bending Moments and Shear Forces: $\sum M_D = 0$,

$$M(\theta) = 5R\left[1 - \cos(\theta)\right] - 2.07R\sin(\theta). \tag{5}$$

$$V(\theta) = 5\sin(\theta) - 2.07\cos(\theta). \tag{6}$$

Axial Force:

$$N(\theta) = -\left[5\cos(\theta) + 2.07\sin(\theta)\right]. \tag{7}$$

Note: N($\theta = 0$) = -5 kN, so the system is in equilibrium.

Analysis of Circular Arch

Relationship between Shear Forces and Bending Moments:

Question: Does V $= \frac{dM}{dx}$ still work? Well, sort of ...

In polar coordinates, the distance x measured around the circumference is:

$$x = R\theta \tag{8}$$

Applying the chain rule:

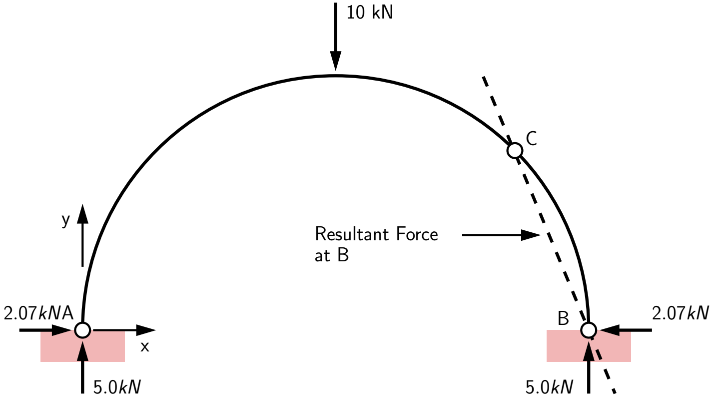
$$V(\theta) = \frac{dM}{dx} = \frac{dM}{dx}\frac{d\theta}{d\theta} = \frac{dM}{d\theta}\frac{d\theta}{dx} = \frac{dM}{d\theta}\frac{1}{R}. \tag{9}$$

Hence, we need to check:

$$V(\theta) = \frac{1}{R}\frac{dM}{d\theta}. \longrightarrow \text{It works!!} \tag{10}$$

Analysis of Circular Arch

Observation: There are no external loads acting on segment B-C of the arch. Hence, the resultant force at B must pass through the hinge at C.

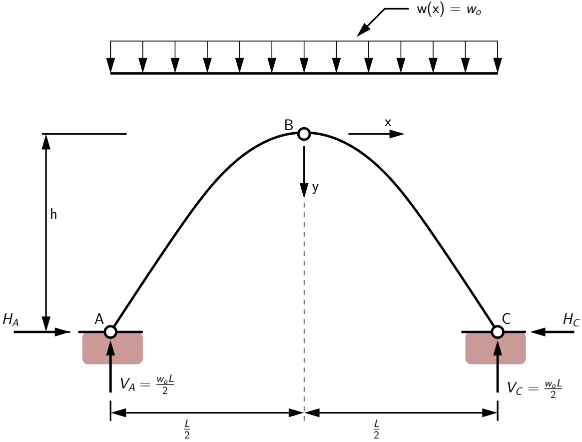


Analysis

Part 2: Parabolic Arch

Analysis of Parabolic Arch

Problem Setup: Let y(x) = k x^2.



Analysis of Parabolic Arch

Boundary Condition:

$$y = h \quad \text{at } x = \pm\frac{L}{2} \quad \longrightarrow \quad k = \frac{4h}{L^2}. \tag{11}$$

Also, note:

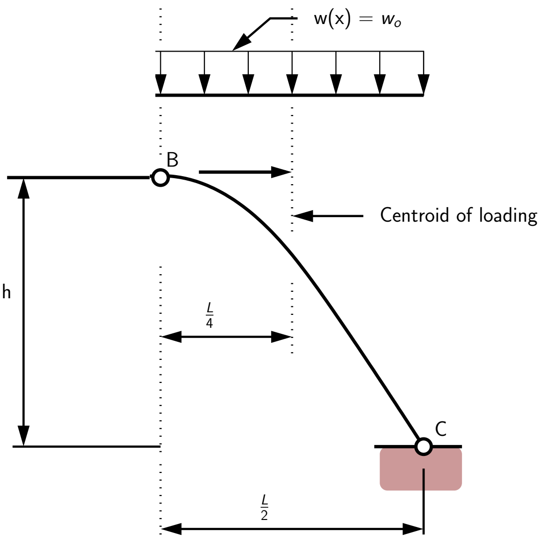
$$y(x) = \left[\frac{4h}{L^2}\right] x^2 \quad \longrightarrow \quad \frac{dy}{dx} = \left[\frac{8h}{L^2}\right] x. \tag{12}$$

Horizontal and Vertical Reactions:

$$\sum V = 0 \quad \longrightarrow \quad V_A = V_C = \frac{w_o L}{2}. \tag{13}$$

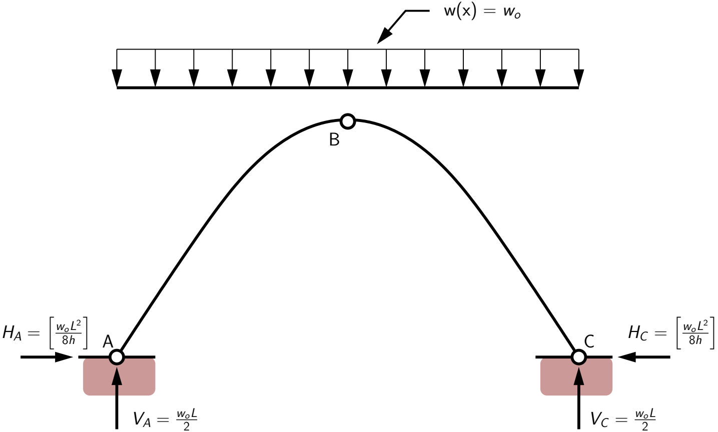
$$\sum M_B = 0 \quad \longrightarrow \quad H_C = \left[\frac{w_o L^2}{8h}\right]. \tag{14}$$

Analysis of Parabolic Arch



Analysis of Parabolic Arch

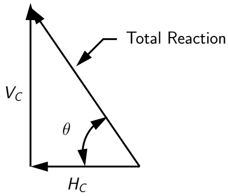
Summary of Reactions:



Analysis of Parabolic Arch

Reaction Magnitude:

$$R = \frac{w_o L}{2} \left[1 + \frac{L^2}{16h^2} \right]^{1/2}. \qquad (15)$$



Reaction Direction:

$$\tan(\theta) = \frac{V_C}{H_C} = \frac{4h}{L}. \qquad (16)$$

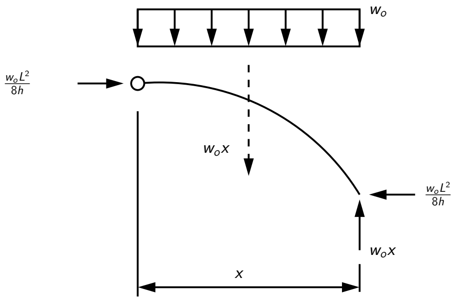
At $x = \frac{L}{2}$:

$$\frac{dy}{dx} = \left(\frac{8h}{L^2} \right) \frac{L}{2} = \left(\frac{4h}{L} \right). \qquad (17)$$

Equations 16 and 17 are identical $\longrightarrow$ pure compression (no bending, no shear) at the foundation support.

Analysis of Parabolic Arch

Analysis of an Arbitrary Section:



Bending Moment:

$$M(x) = w_o x \frac{x}{2} - \frac{w_o L^2}{8h} y = \frac{w_o x^2}{2} - \frac{w_o x^2}{2} = 0. \qquad (18)$$

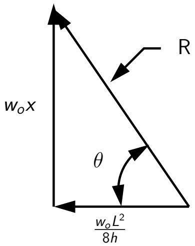
Analysis of Parabolic Arch

Analysis of an Arbitrary Section:

Resultant of the ground reaction forces:

$$R = w_o\left[x^2 + \left(\frac{L^2}{8h}\right)^2\right]^{1/2} \qquad (19)$$

$$\tan(\theta) = \left(\frac{8xh}{L^2}\right) \qquad (20)$$



Slope of curve section:

$$\frac{dy}{dx} = 2kx = \left[\frac{8hx}{L^2}\right]. \qquad (21)$$

Equation 20 equals 21 $\longrightarrow$ pure compression throughout!

Analysis of Parabolic Arch

Practical Considerations: To eliminate bending moments, loading patterns need to be uniformly distributed.

To make this happen, put holes in the bridge ...

Humpback Bridge

Holes create uniform loading

Clever, huh!