Solutions to Homework 5

Question 1: 10 points

Problem Statement. Figure 1 shows a simple three-bar truss. The bar elements have section properties EA throughout. Horizontal and vertical loads P are applied at node C.

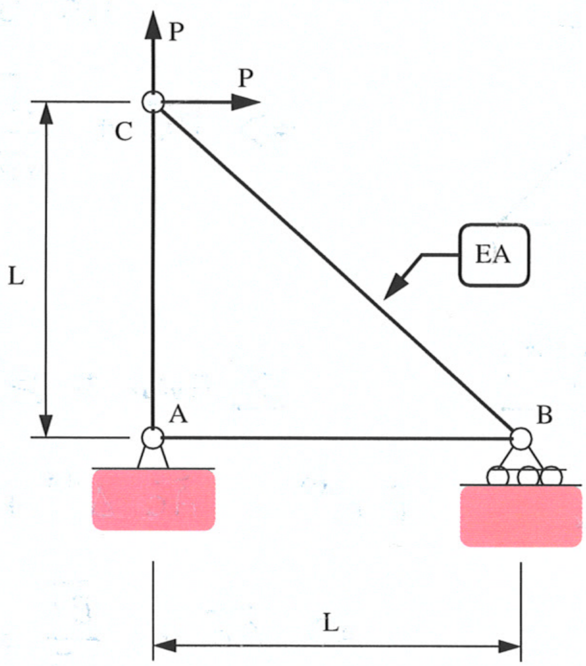
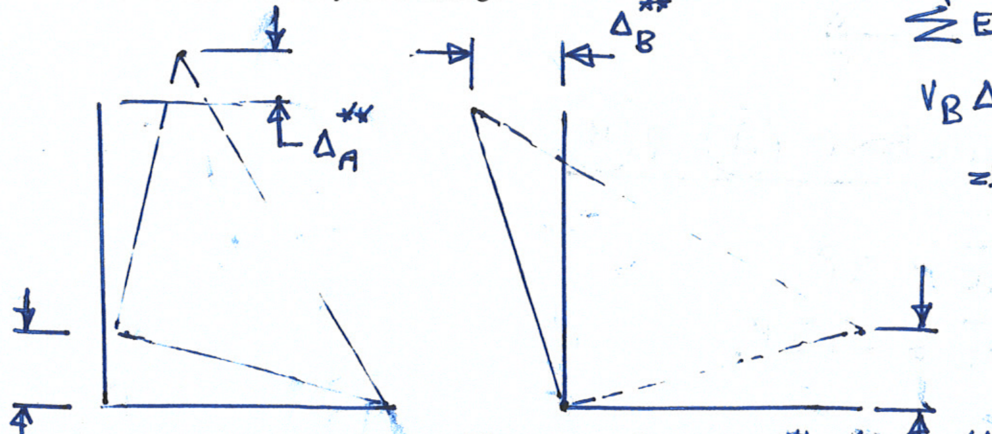


Figure 1: Simple three-bar truss.

Part [1a]. (5 pts). Use the **method of virtual displacements** to compute the vertical reaction forces at nodes A and B. Show all of your working.



$\sum EWD = 0$

$V_B \Delta_B^{**} + P(-\Delta_B^{**}) = 0$

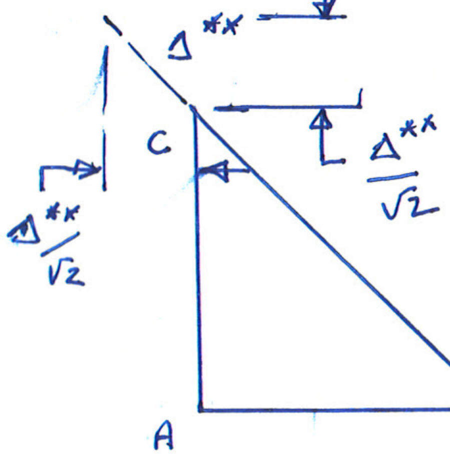
$\Rightarrow \underline{V_B = P.}$

$\sum EWD = 0 \quad V_A \Delta_A^{**} + \Delta_A^{**} P + \Delta_A^{**} P = 0$

$\Rightarrow V_A = -2P$

Part [1b]. (5 pts). Use the **method of virtual displacements** to compute the member forces AC and BC. Show all of your working.

Extend $\overline{BC}$ by Δ^{**}:

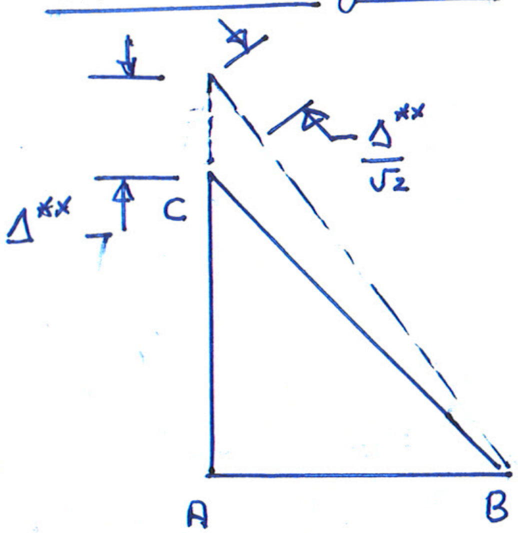


$$\sum_1' EWD = \sum_1' IWD.$$

$$\left(\frac{\Delta^{**}}{\sqrt{2}}\right)(-P) + \left(\frac{\Delta^{**}}{\sqrt{2}}\right)P = \overline{BC}\,\Delta^{**} + \overline{AC}\left(\frac{\Delta^{**}}{\sqrt{2}}\right)$$

$$\Rightarrow \overline{BC} + \frac{\overline{AC}}{\sqrt{2}} = 0 \longrightarrow \text{(A)}$$

Extend $\overline{AC}$ by Δ^{**}:



$$\sum_1' EWD = \sum_1' IWD.$$

$$P\,\Delta^{**} = \overline{AC}\,\Delta^{**} + \overline{BC}\left(\frac{\Delta^{**}}{\sqrt{2}}\right)$$

$$\Rightarrow \overline{AC} + \frac{\overline{BC}}{\sqrt{2}} = P. \longrightarrow \text{(B)}$$

Combining equations (A) & (B)

$\overline{AC} = 2P$

$\overline{BC} = -\sqrt{2}P.$

Question 2: 10 points

Problem Statement. The cantilevered beam structure shown in Figure 2 supports a uniformly distributed load w (N/m) between points C and D.

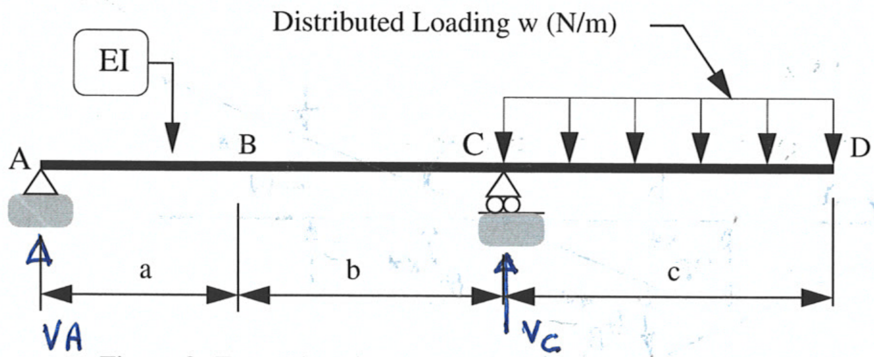
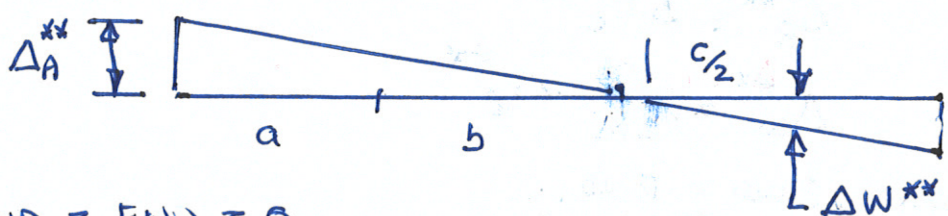


Figure 2: Front elevation view of a simple beam structure.

Part [2a]. (4 pts) Use the method of **virtual displacements** to compute formulae for the vertical reactions at A and C. Show all of your working.

Virtual Displacement at A.



From geometry:

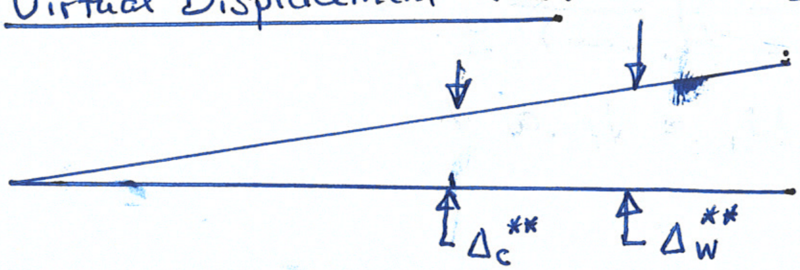
$$\Delta_W^{**} = \frac{c}{2(a+b)} \cdot \Delta_A^{**}$$

IWD = EWD = 0

$$EWD = V_A \cdot \Delta_A^{**} + \frac{wc^2}{2(a+b)} \Delta_A^{**} = 0$$

$$\Rightarrow V_A = \frac{-wc^2}{2(a+b)}. \quad \text{—— (A)}$$

Virtual Displacement at C.



From geometry:

$$\Delta_w^{**} = \frac{a+b+c/2}{(a+b)} \Delta_c^{**}$$

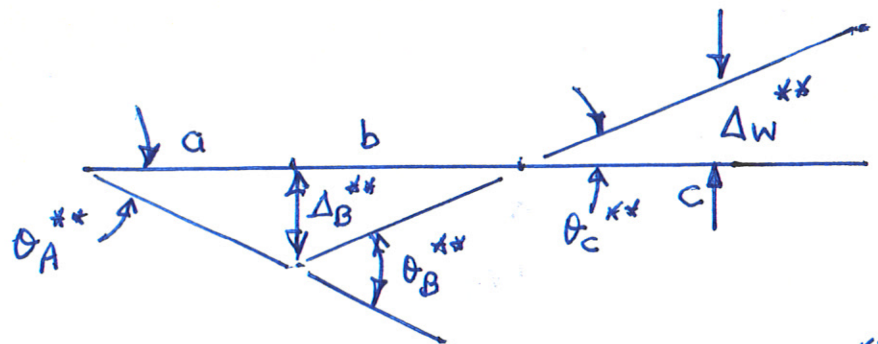
EWD = 0

$$\Rightarrow \Delta_c^{**} V_c - wc \, \Delta_c^{**} = 0$$

$$\Rightarrow V_c = \frac{wc(a+b+c/2)}{(a+b)}. \quad \text{— (B)}$$

Part [2b]. (6 pts) Use the method of **virtual displacements** to compute a formula for the bending moment at B. Show all of your working.

Apply virtual rotation at B.



From geometry: $\theta_B^{**} = \theta_A^{**} + \theta_c^{**} = \dfrac{\Delta_B^{**}}{a} + \dfrac{\Delta_B^{**}}{b}$

$$\Delta_B^{**} = a\,\theta_A^{**}$$

$$\Delta_B^{**} = b\,\theta_c^{**}$$

$$\Delta_W^{**} = \frac{c}{2b} \cdot \Delta_B^{**}.$$

Energy Balance: IWD = EWD.

$$M_B\,\theta_B^{**} + Wc\,\Delta_W^{**} = 0$$

$$\Rightarrow M_B^{**}\left[\frac{1}{a} + \frac{1}{b}\right]\Delta_B^{**} = \frac{-Wc^2}{2b}\Delta_B^{**}$$

$$\Rightarrow M_B = \left(\frac{ab}{a+b}\right)\left[\frac{-Wc^2}{(2b)}\right]. \checkmark$$

Note: From statics: $M_B = V_A \cdot a$ ✓✓.

Question 3: 10 points

Problem Statement. Consider the articulated cantilever beam structure shown in Figure 3.

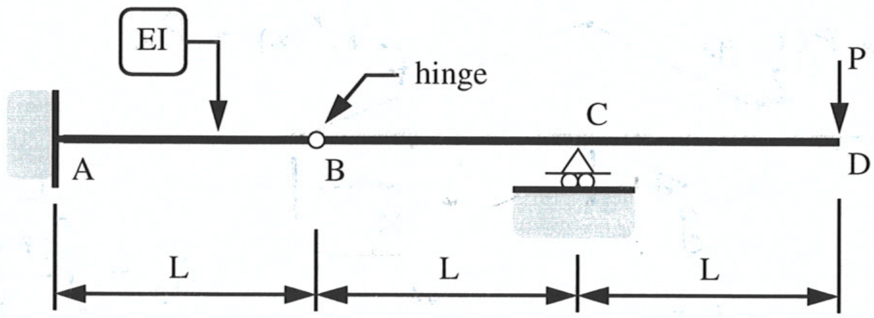
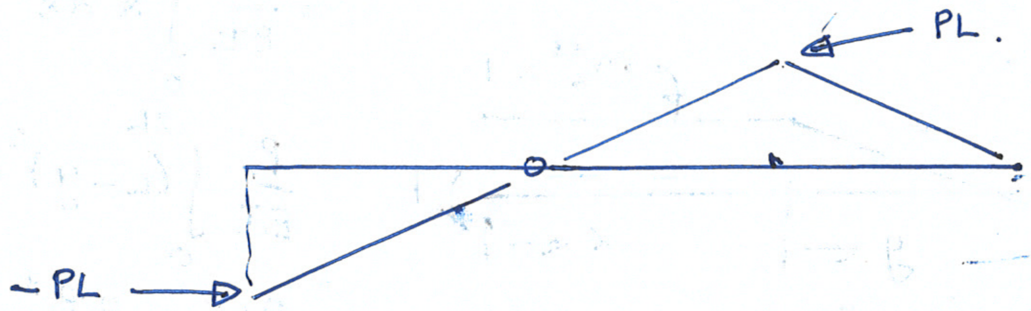


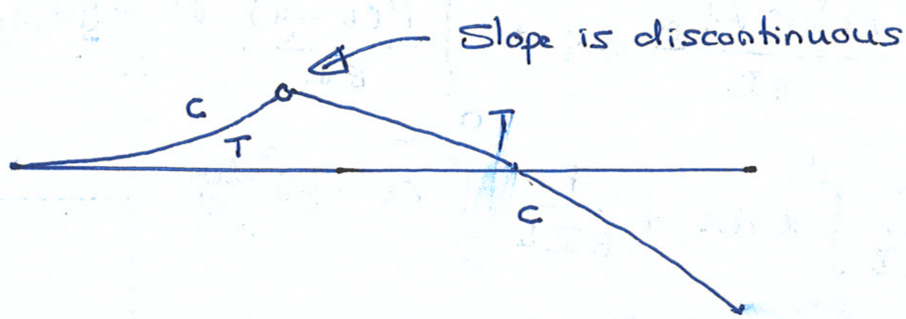
Figure 3: Elevation view of articulated cantilever beam structure.

At Point A, the cantilever is fully fixed (no movement) to a wall. Point B is a hinge. Both members have cross section properties EI. A single point load **P** (N) is applied at node D as shown in the figure.

Part [3a]. (2 pts). Draw and label the bending moment diagram for this problem.

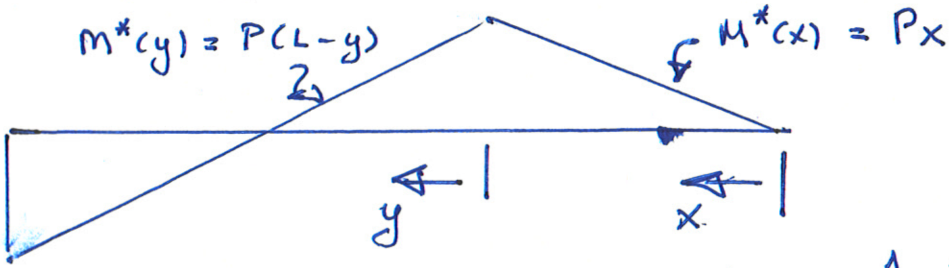


Part [3b]. (2 pts). Qualitatively sketch the deflected shape. Indicate regions of tension/compression, and any points where slope of the beam is discontinuous.

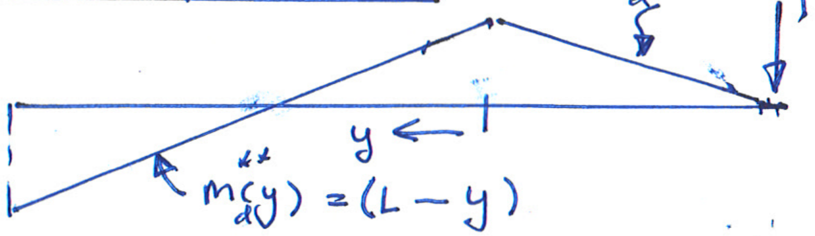


Part [3c]. (6 pts). Use the method of **virtual forces** to compute the **vertical displacement** and **end rotation** of the beam at D. Show all of your working.

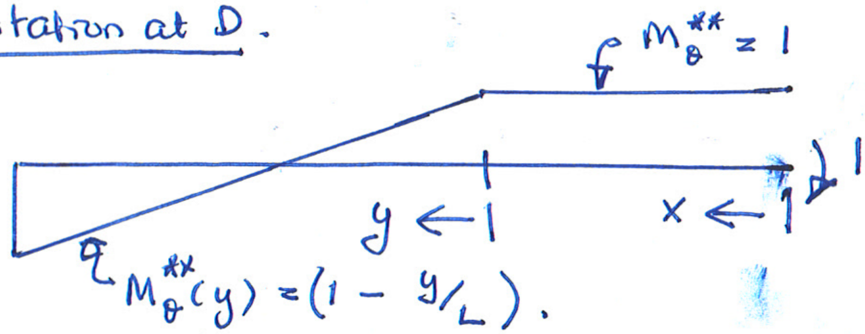
BMD due to load at D:



Vertical deflection at D:



Rotation at D.



$$\Delta_D = \int_0^L \frac{M^*(x)\, M_d^{**}(x)}{EI}\,dx + \int_0^{2L} \frac{M^*(y)\, M_d^{**}(y)}{EI}\,dy.$$

$$= \frac{P}{EI}\int_0^L x^2\,dx + \frac{P}{EI}\int_0^{2L} (L-y)^2\,dy$$

$$= \frac{PL^3}{3EI} + \frac{2}{3}\frac{PL^3}{EI} = \frac{PL^3}{EI}.$$

$$\theta_D = \int_0^L \frac{M^*(x)\, M_\theta^{**}}{EI}\,dx + \int_0^{2L} \frac{P(L-y)}{EI}\cdot\frac{(L-y)}{L}\,dy.$$

$$= \frac{P}{EI}\int_0^L x\,dx + \frac{P}{EIL}\int_0^{2L} (L-y)^2\,dy$$

$$= \frac{PL^2}{2EI} + \frac{2PL^2}{3EI}$$

$$\theta_D = \frac{7}{6}\frac{PL^2}{EI}$$

$$\boxed{\Delta_D = \frac{PL^3}{EI},\quad \theta_D = \frac{7}{6}\frac{PL^2}{EI}}$$

Question 4: 10 points

Problem Statement. Figure 4 is a front elevation view of a simple truss that supports vertical loads at nodes C and D.

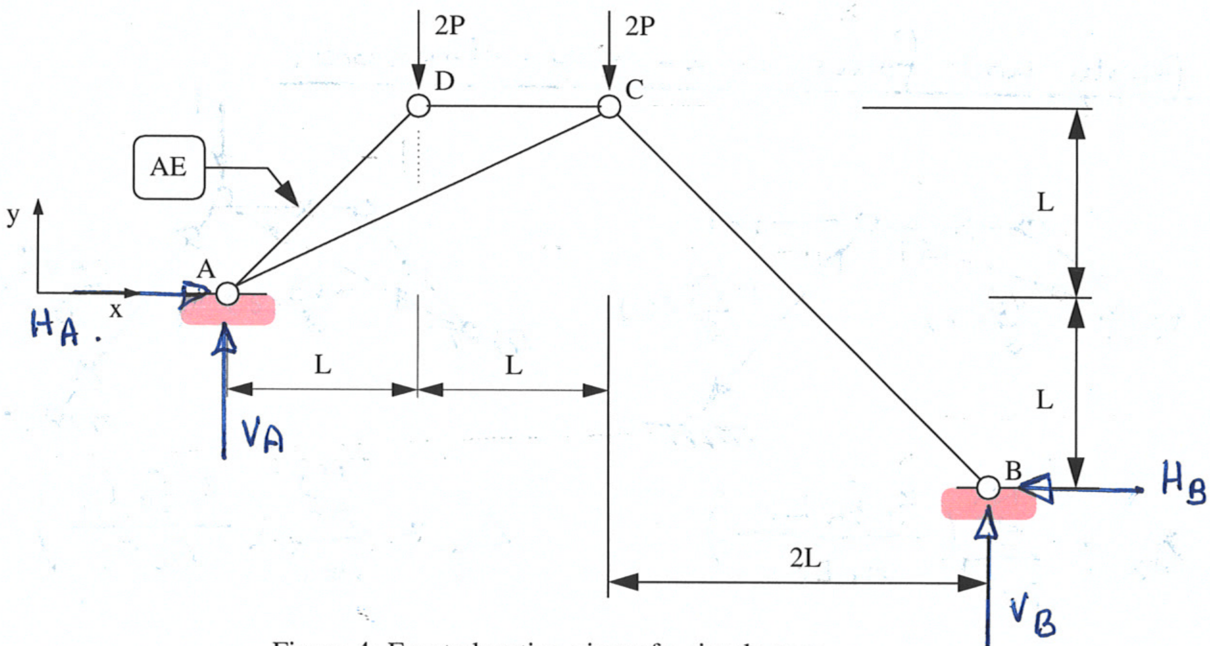


Figure 4: Front elevation view of a simple truss.

All of the truss members have cross section properties AE.

Part [4a]. (5 pts). Compute the support reactions and distribution of forces throughout the structure.

$\sum M_A = 0 \quad -2PL - 4PL - H_B L + 4L V_B = 0$

$\Rightarrow 4 V_B = H_B + 6P.$

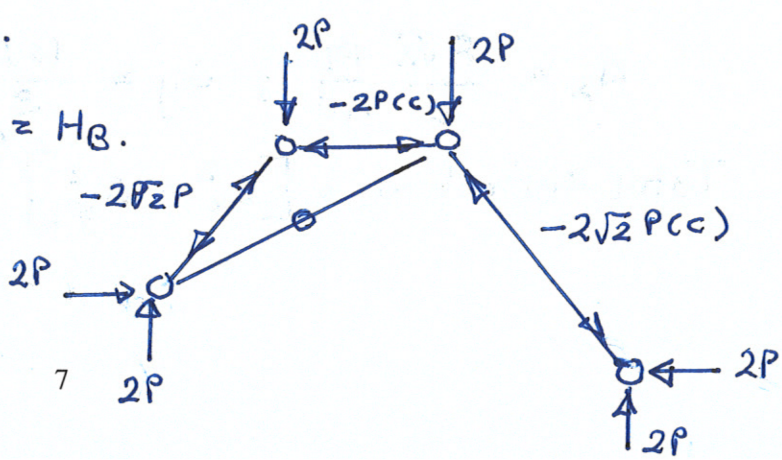
$\sum F_x = 0 \quad H_A = H_B$

$\sum F_y = 0 \quad V_A + V_B = 4P.$

Equilibrium at node B; $V_B = H_B$.

$\Rightarrow \quad V_A = V_B = 2P$

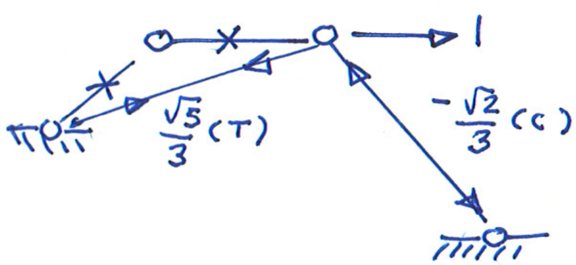
$H_A = H_B = 2P.$

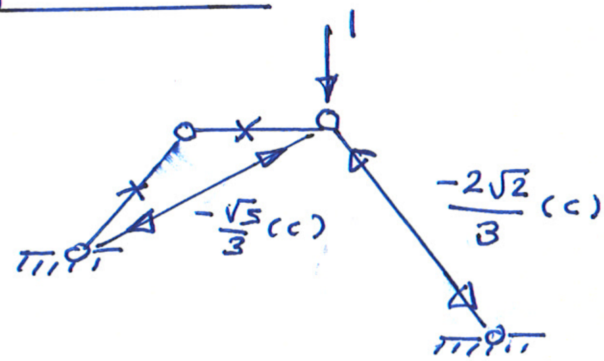


Part [4b]. (5 pts). Use the method of **virtual forces** to show that the total deflection at node C is:

$$\triangle = \frac{PL}{AE}\left[\frac{8\sqrt{10}}{3}\right]. \tag{1}$$

Apply unit forces in x- & y-directions:





$$\Delta_x = \sum_{i=1}^{N}\left(\frac{F_i f_h L_i}{A_i E_i}\right) \qquad \Delta_y = \sum_{i=1}^{N}\left(\frac{F_i f_v L_i}{A_i E_i}\right)$$

Tabulate displacements:

Member	L/AE	F	f_h	f_v	$\frac{F f_h L}{AE}$	$\frac{F f_v L}{AE}$
AC	$\sqrt{3}L/AE$	0	$\sqrt{5}/2$	$-\sqrt{5}/3$	0	0
AD	$\sqrt{2}L/AE$	$-2\sqrt{2}P$	0	0	0	0
BC	$2\sqrt{2}L/AE$	$-2\sqrt{2}P$	$-\sqrt{2}/3$	$-\frac{2\sqrt{2}}{3}$	$8\sqrt{2}/3\ \frac{PL}{AE}$	$\frac{16\sqrt{2}}{3}\frac{PL}{AE}$
CD	L/AE	$-2P$	0	0	0	0

$$\Delta_x = \frac{8\sqrt{2}}{3}\frac{PL}{AE}\ ; \quad \Delta_y = \frac{16\sqrt{2}}{3}\cdot\frac{PL}{AE}.$$

$$\text{Total deflection} = \left[\Delta_x^2 + \Delta_y^2\right]^{1/2} = \frac{8\sqrt{10}}{3}\frac{PL}{AE}.$$

Question 5: 20 points

Problem Statement. The T-shaped beam structure shown in Figure 5 has flexural stiffness EI throughout.

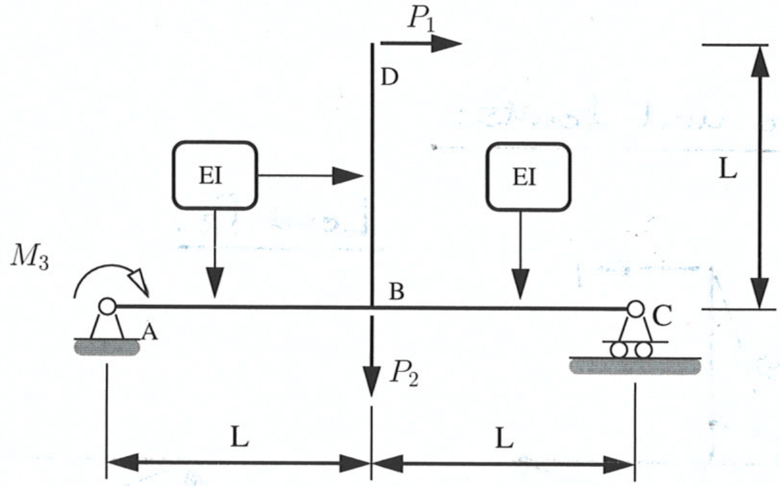
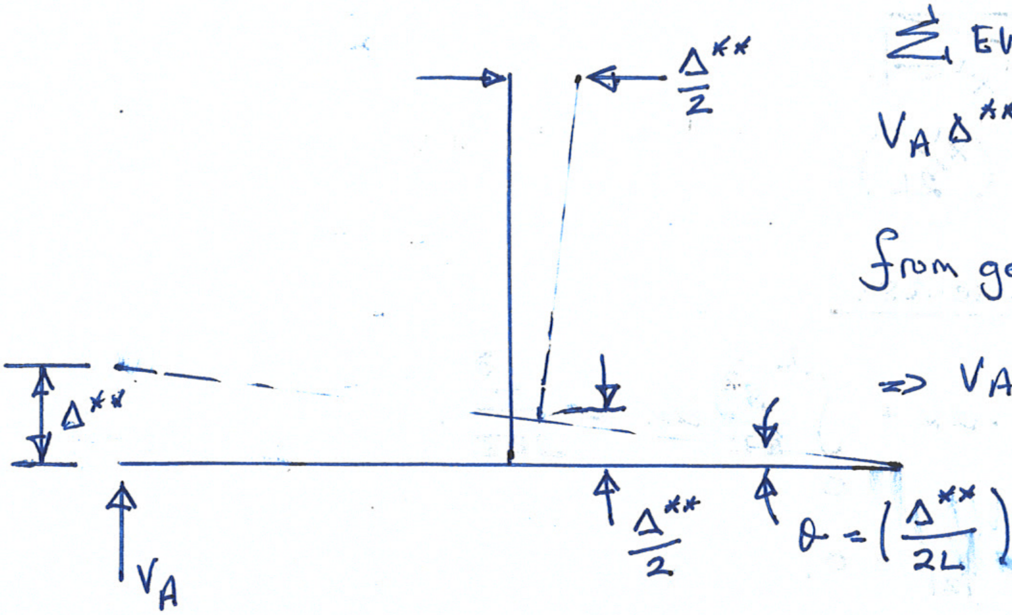


Figure 5: Front elevation view of a T-shaped beam.

Part [5a]. (5 pts). Use the **method of virtual displacements** to compute the vertical reaction force at node A.



$\sum_i EWD = 0$

$V_A \Delta^{**} + M_3\theta + \frac{P_1 \Delta^{**}}{2} - P_2 \frac{\Delta^{**}}{2} = 0$

from geometry: $\theta = \frac{\Delta^{**}}{2L}$

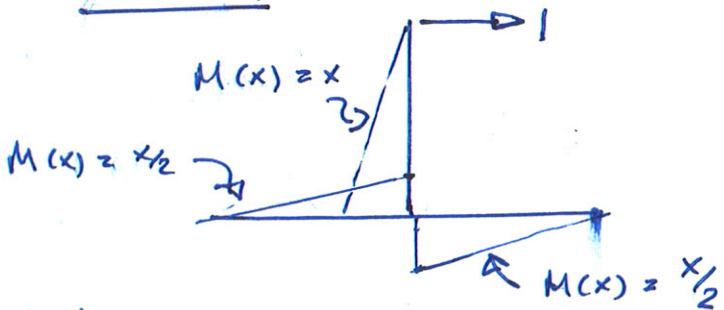
$\Rightarrow V_A = \frac{P_2}{2} - \frac{P_1}{2} - \frac{M_3}{2L}$ — (A)

Part [5b]. (15 pts). Use the **method of virtual forces** to compute the flexibility matrix:

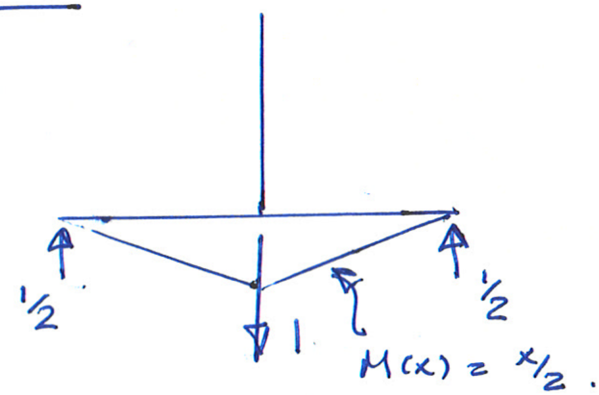
$$\begin{bmatrix} \triangle_{dx} \\ \triangle_{by} \\ \theta_A \end{bmatrix} = \begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \\ M_3 \end{bmatrix}. \tag{2}$$

① BMD due to unit loads:

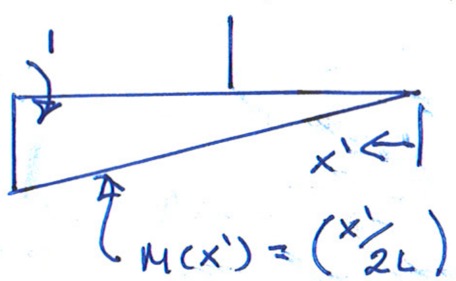
Load P_1.



Load P_2.



Moment M_3:



② Flexibility Coefficients:

$$f_{11} = 2\int_0^L \left(\frac{x}{2}\right)^2 \frac{1}{EI} \cdot dx + \int_0^L \frac{x^2}{EI} dx = \frac{L^3}{2EI}.$$

From symmetry: $f_{12} = f_{21} = 0$.

$$f_{31} = \int_0^L -\frac{x}{2}\left(1 - \frac{x}{2L}\right)\frac{1}{EI} dx + \int_0^L \left(\frac{x'}{2}\right)\left(\frac{x'}{2L}\right)\frac{1}{EI} dx'$$

$$= \frac{-L^2}{12EI}.$$

Part [5b]. continued: ...

$$f_{22} = 2\int_0^L \left(\frac{x}{2}\right)^2 \frac{1}{EI}\,dx = \frac{L^3}{6EI}.$$

$$f_{32} = \int_0^L \left(\frac{x}{2}\right)\left(1-\frac{x}{2L}\right)\frac{1}{EI}\,dx + \int_0^L \frac{x'}{2}\cdot\frac{x'}{2L}\cdot\frac{1}{EI}\,dx'$$

$$= \frac{L^2}{4EI},$$

$$f_{33} = \int_0^{2L} \left(\frac{x}{2L}\right)^2 \cdot \frac{1}{EI}\,dx = \frac{2}{3}\frac{L}{EI}.$$

Flexibility Matrix

$$f = \begin{bmatrix} L^3/2EI & 0 & -L^2/12EI \\ 0 & L^3/6EI & L^2/4EI \\ -L^2/12EI & L^2/4EI & \frac{2}{3}\cdot\frac{L}{EI} \end{bmatrix}$$

Question 6: 10 points

Problem Statement. Consider the supported cantilevered beam structure shown in Figure 6.

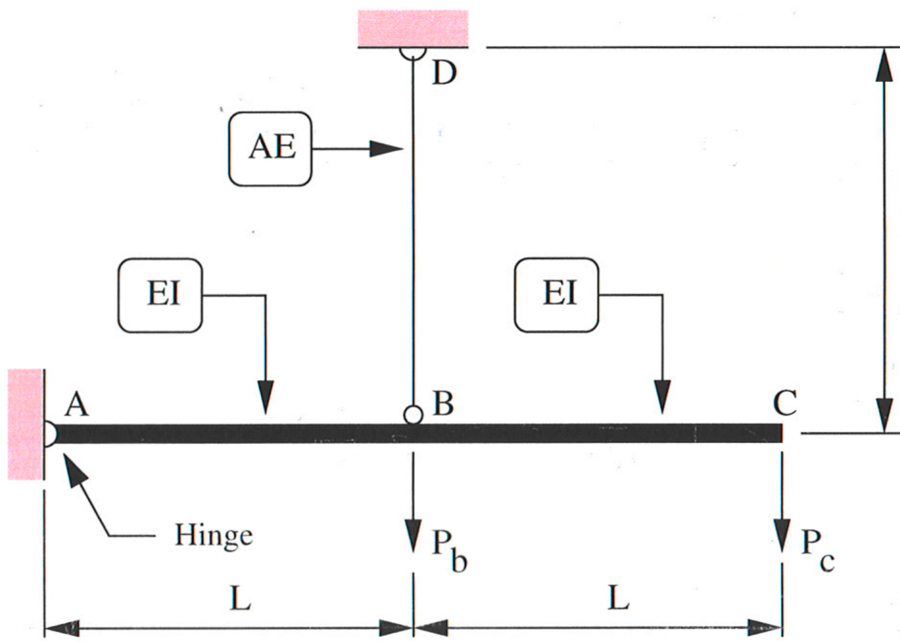
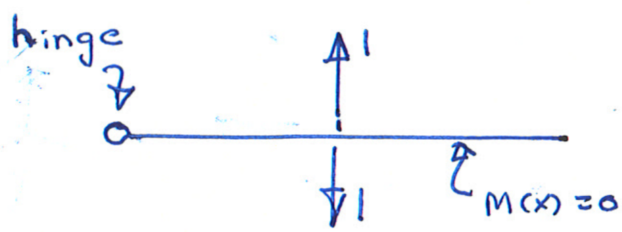


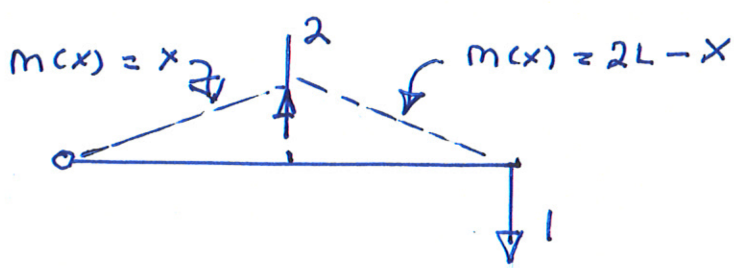
Figure 6: Front elevation view of a supported cantilevered beam structure.

Use the principle of **virtual forces** to compute the two-by-two flexibility matrix connecting displacements at points B and C to applied loads P_b and P_c, i.e.,

$$\begin{bmatrix} \triangle_b \\ \triangle_c \end{bmatrix} = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} \begin{bmatrix} P_b \\ P_c \end{bmatrix}. \tag{3}$$

Apply unit force at B. Apply unit force at C.





Question 6. continued: ...

$$f_{11} = \underbrace{\int_0^{2L} \frac{M_1^2(x)}{EI}\,dx}_{0} + \int_0^{L} \frac{f_1^2(x)}{AE}\,dx = \left[\frac{L}{AE}\right]$$

$$f_{22} = \underbrace{\int_0^{2L} \frac{M_2^2(x)}{EI}\,dx}_{I_1} + \underbrace{\int_0^{L} \frac{f_2^2(x)}{AE}\,dx}_{I_2}$$

$$\left.\begin{aligned} I_1 &= 2\int_0^{L} \frac{x^2}{EI}\,dx = \frac{2}{3}\frac{L^3}{EI}. \\ I_2 &= \int_0^{L} \frac{2.2}{AE}\,dx = \frac{4L}{AE} \end{aligned}\right\} f_{22} = \frac{2}{3}\frac{L^3}{EI} + \frac{4L}{AE}.$$

$$f_{12} = \underbrace{\int_0^{2L} \frac{M_1(x)\,M_2(x)\,dx}{EI}}_{\to\, 0!} + \int_0^{L} \frac{f_1(x)\,f_2(x)\,dx}{AE} = \frac{2L}{AE}.$$

Flexibility Matrix:

$$\begin{bmatrix} \Delta_b \\ \Delta_c \end{bmatrix} = \begin{bmatrix} L/AE & 2L/AE \\ 2L/AE & \frac{2}{3}\frac{L^3}{EI} + \frac{4L}{AE} \end{bmatrix} \begin{bmatrix} P_b \\ P_c \end{bmatrix}.$$