Department of Civil and Environmental Engineering, Fall Semester, 2024

ENCE 353 Final Exam, Open Notes and Open Book

Name : Austin.

Exam Format and Grading. This take home final exam is open notes and open book. You need to comply with the university regulations for academic integrity.

Answer Question 1. Then answer **three of the four** remaining questions. Partial credit will be given for partially correct answers, so please show all your working.

IMPORTANT: Only the **first four questions** that you answer will be graded, so please **cross out the question you do not want graded** in the table below. Also, before submitting your exam, check that **every page has been scanned correctly**.

Question	Points	Score
1	20	
2	10	
3	10	
4	10	
5	10	
Total	50	

Question 1: 20 points

COMPULSORY: Method of Virtual Displacements, Method of Virtual Forces, Flexibilty Matrix. The beam structure shown in Figure 1 has flexural stiffness EI throughout.

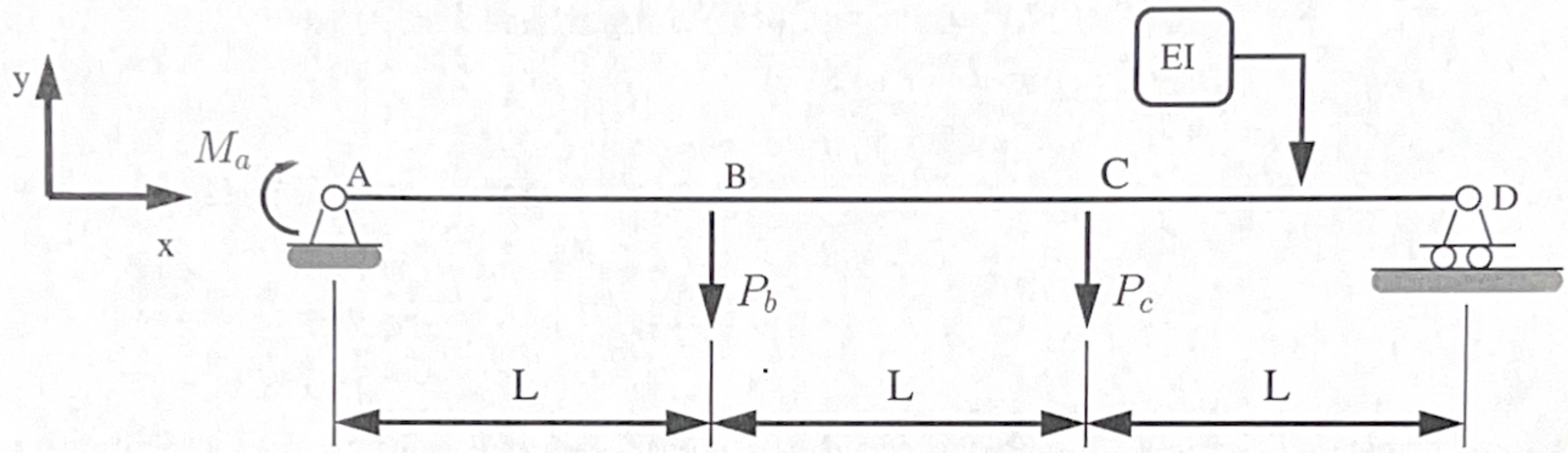
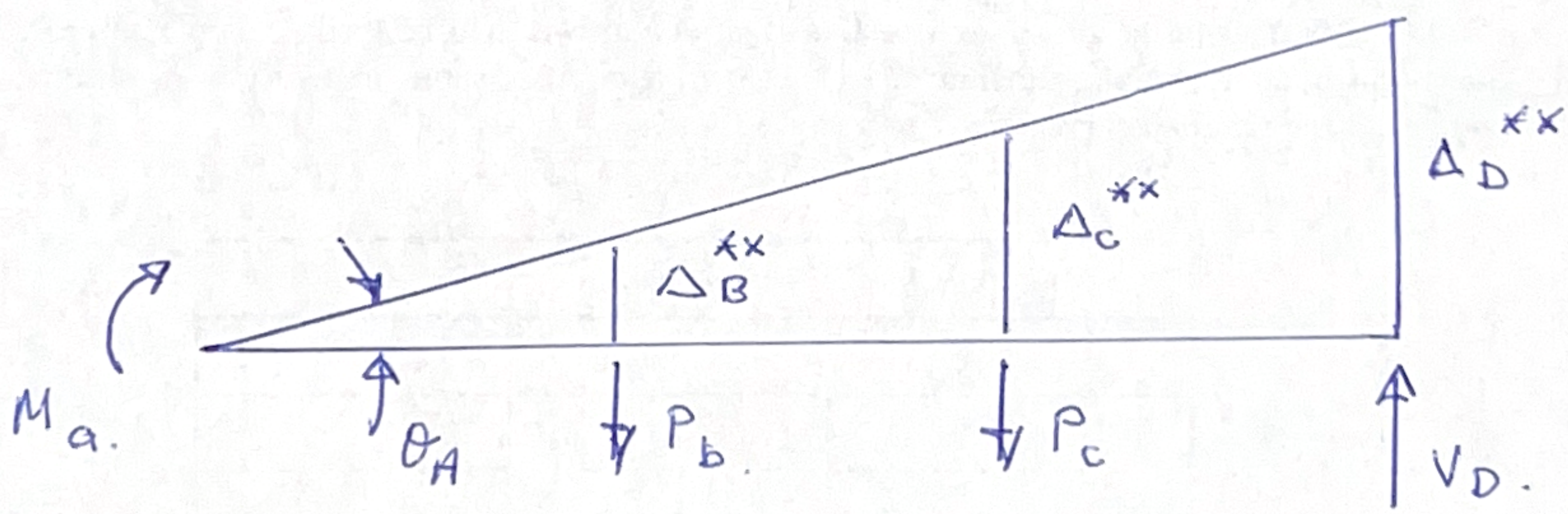


Figure 1: Front elevation view of a long beam.

[1a] (5 pts). Use the **method of virtual displacements** to compute the vertical reaction force at node D.



From geometry:

$$\theta_A = \frac{\Delta_D^{**}}{3L} \; ; \quad \Delta_B^{**} = \frac{\Delta_D^{**}}{3} \; ; \quad \Delta_C^{**} = \frac{2}{3} \Delta_D^{**}$$

Energy balance:

IWD = 0

$$\text{EWD} = 0 \rightarrow -P_b \Delta_B^{**} - P_c \Delta_C^{**} - M_A \theta_A + V_D \Delta_D^{**} = 0$$

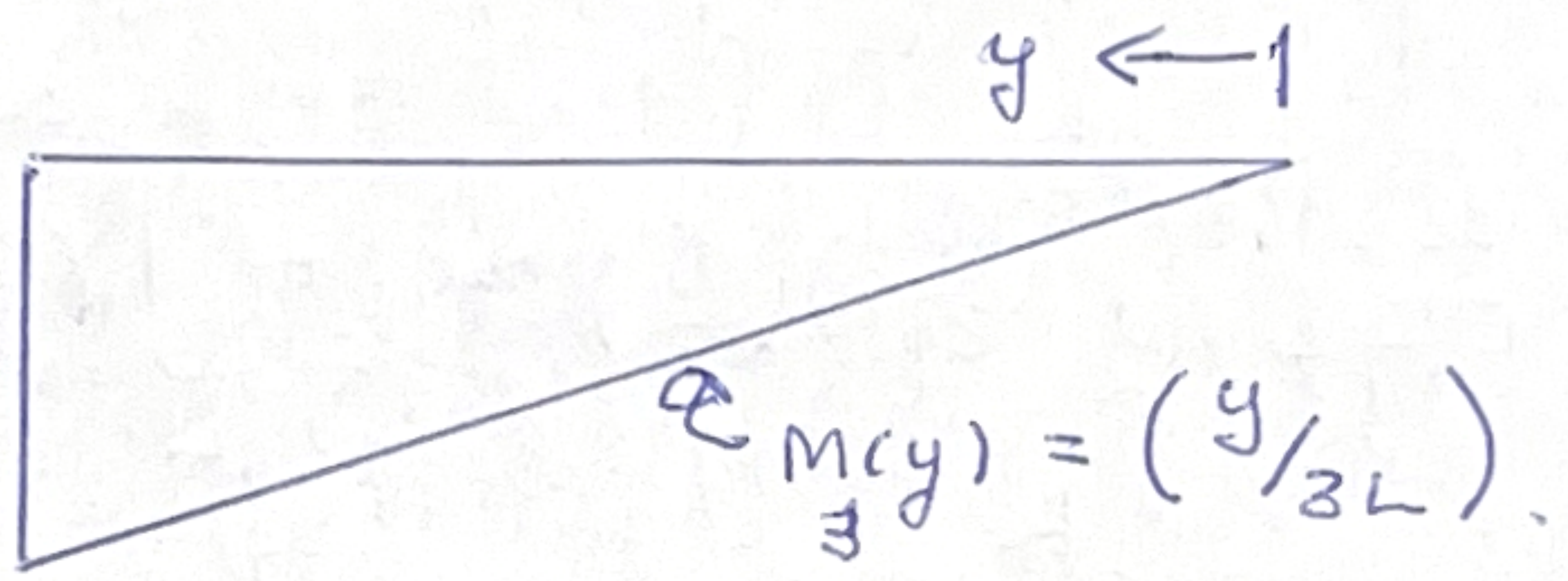
$$\Rightarrow V_D = \frac{P_b}{3} + \frac{2 P_c}{3} + \frac{M_A}{3L} \quad \text{———} \quad (A)$$

[1b] (15 pts). Use the **method of virtual forces** to compute the flexibility matrix:

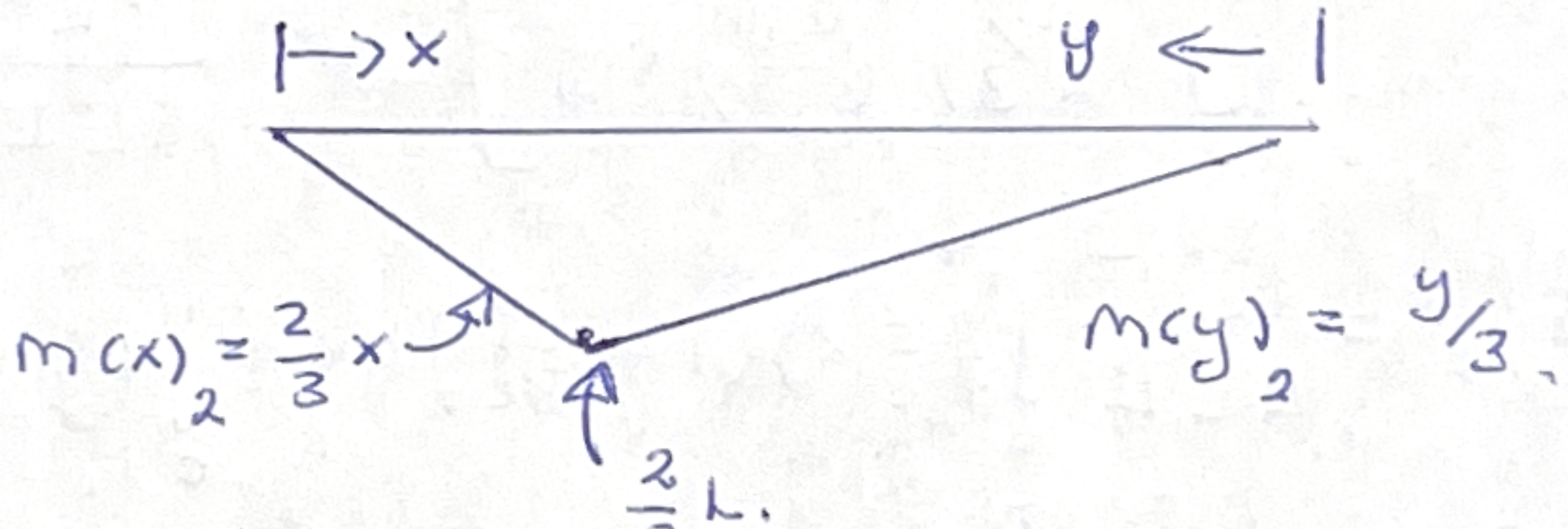
$$\begin{bmatrix} \Delta_{by} \\ \Delta_{cy} \\ \theta_a \end{bmatrix} = \begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{bmatrix} \begin{bmatrix} P_b \\ P_c \\ M_a \end{bmatrix}. \tag{1}$$

Apply unit loads / moments at A, B & C:

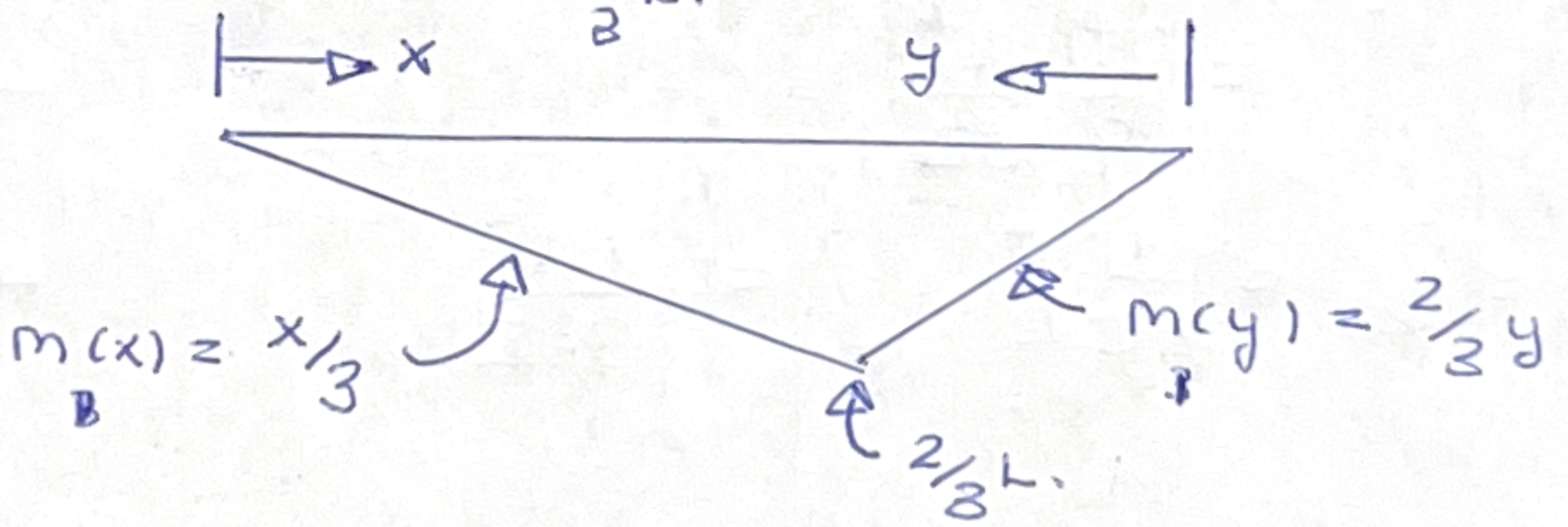
$M_A = 1$:



$P_B = 1$:



$P_c = 1$:



Flexibility Coefficients:

$$f_{11} = \frac{1}{EI}\left[\int_0^L \left(\frac{2}{3}x\right)^2 dx + \int_0^{2L} (y/3)^2 dy \right]$$

$$= \frac{1}{EI}\left[\frac{4L^3}{27} + \frac{8L^3}{27} \right] = \frac{12L^3}{27EI}.$$ ______ (A)

$$f_{22} = \frac{12L^3}{27EI} \text{ (by symmetry).}$$ ______ (B)

Question 1b continued ...

$$f_{33} = \frac{1}{EI}\left[\int_0^{3L}\left(\frac{y}{3L}\right)^2 dy\right] = \frac{L}{EI}. \qquad \text{——— (C)}$$

$$f_{12} = f_{21} = \frac{1}{EI}\left[\int_0^{3L} M_2(\cdot)\, M_1(\cdot)\, dx\right]$$

$$= \frac{1}{EI}\left[\int_0^{L}\left(\frac{2}{3}x\right)\left(\frac{x}{3}\right)dx + \int_L^{2L}\left(\frac{x}{3} - L\right)\left(\frac{x}{3}\right)dx + \int_0^{L}\left(\frac{y}{3}\right)\left(\frac{2y}{3}\right)dy\right] = \frac{7L^3}{18EI} \qquad \text{——— (D)}$$

$$f_{31} = f_{13} = \frac{1}{EI}\left[\int_0^{L}\left(\frac{2}{3}x\right)\left(1 - \frac{x}{3L}\right)dx + \int_0^{2L}\left(\frac{y}{3}\right)\left(\frac{y}{3L}\right)dy\right]$$

$$= \frac{5}{9}\frac{L^2}{EI} \qquad \text{——— (E)}$$

$$f_{32} = f_{23} = \frac{1}{EI}\left[\ldots\right] = \frac{4L^2}{9EI}. \qquad \text{——— (F)}$$

Summary:

$$\begin{bmatrix} \Delta_{By} \\ \Delta_{Cy} \\ \theta_A \end{bmatrix} = \begin{bmatrix} \frac{4L^3}{9EI} & \frac{7L^3}{18EI} & \frac{5L^2}{9EI} \\ \frac{7L^3}{18EI} & \frac{4L^3}{9EI} & \frac{4L^2}{9EI} \\ \frac{5L^2}{9EI} & \frac{4L^2}{9EI} & \frac{L}{EI} \end{bmatrix} \cdot \begin{bmatrix} P_b \\ P_c \\ M_a \end{bmatrix}$$

Question 2: 10 points

OPTIONAL: Simple Three-Pinned Arch. Figure 2 is a front elevation view of a simple three-pinned arch. Vertical and horizontal loads 2P are applied at nodes B and D.

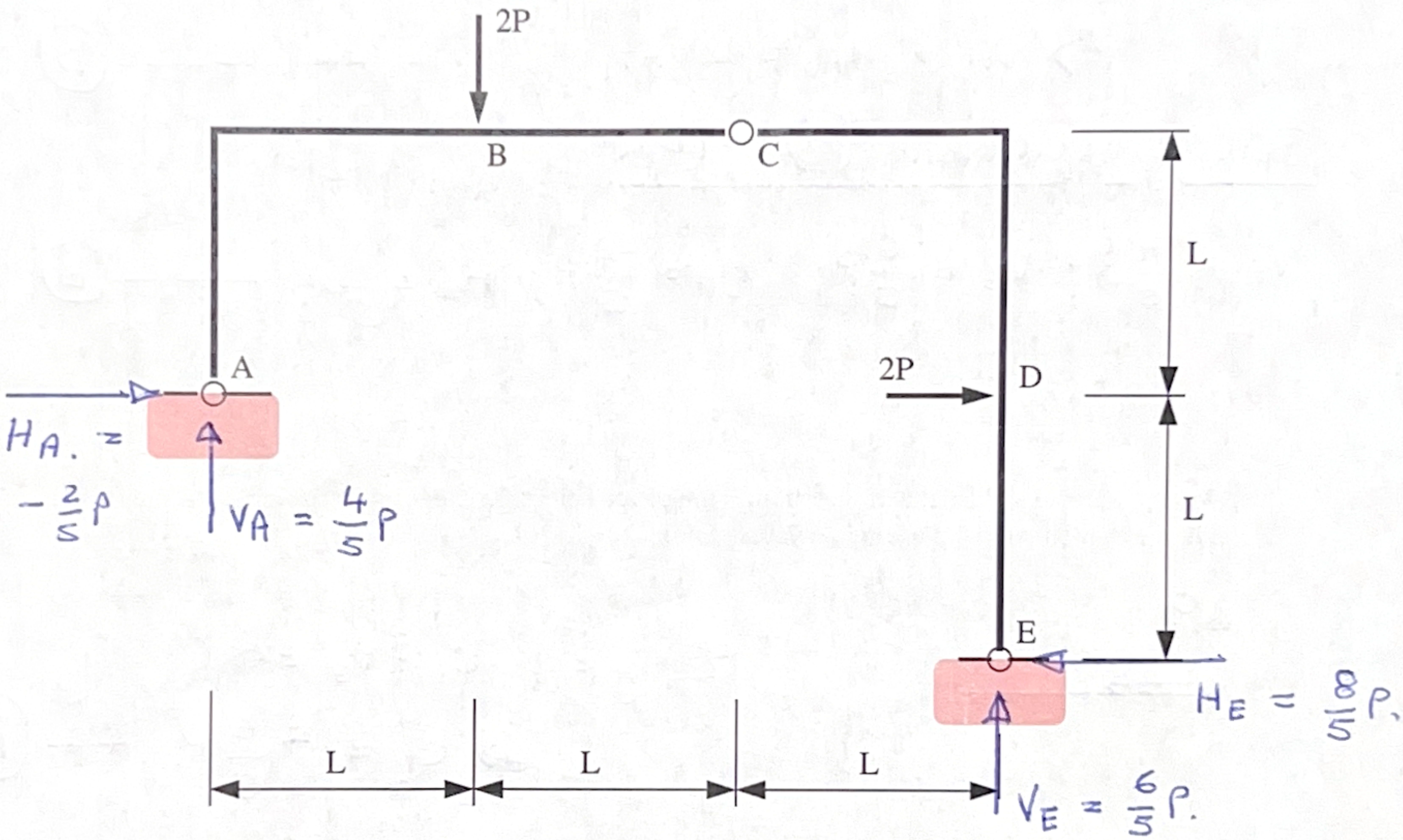


Figure 2: Front elevation view of a simple three-pinned arch.

[2a] (4 pts) Compute the vertical and horizontal components of reaction force at supports A and E as a function of P.

$\Sigma H = 0 \qquad -H_A + H_E = 2P.$ ——— (A)

$\Sigma V = 0 \qquad V_A + V_E = 2P$ ——— (B)

ΣM_C for RHS:

$2PL + V_E \cdot L = H_E \cdot 2PL$

$\rightarrow 2P + V_E = 2H_E$ ——— (C)

Queation 2a continued:

$\sum M_C = 0$ for LHS:

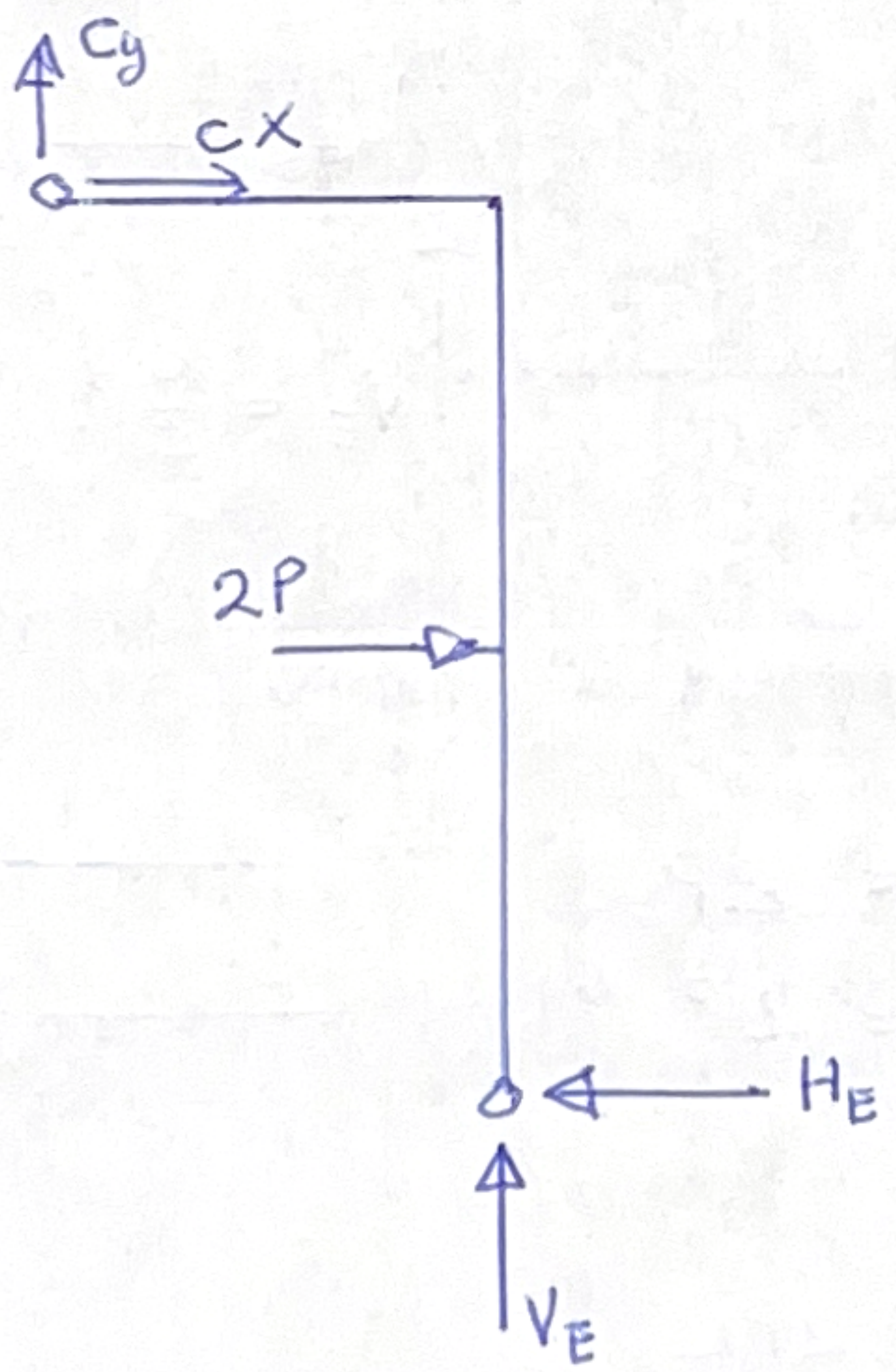
$$2PL + H_A \cdot L = V_A \, 2L$$

$$\Rightarrow 2P + H_A = 2V_A. \quad \text{——} \; (D)$$

<u>Solving equations (A) – (D):</u>

$$V_A = \frac{4}{5}P; \quad V_E = \frac{6}{5}P; \quad H_A = \frac{-2}{5}P; \quad H_E = \frac{8}{5}P. \quad \text{——} \; (E)$$

[2b] (3 pts) Compute and axial and shear forces transfered across the hinge at C. You can annotate Figure 2 if you think it will help to explain your solution.



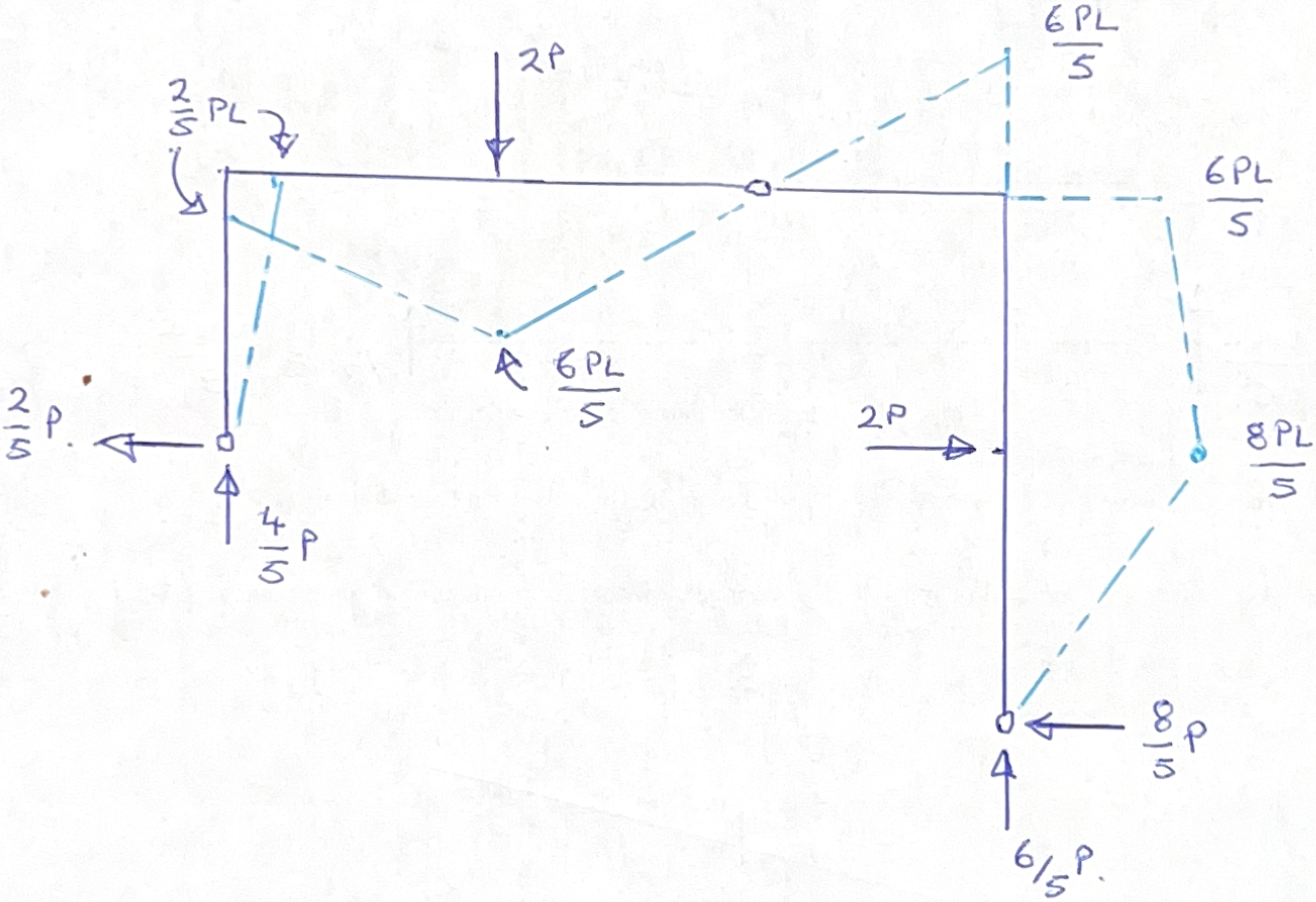
$\sum F_y = 0 \quad C_y + V_E = 0.$

$$\Rightarrow C_y = -\frac{6}{5}P. \quad \text{——} \; (F)$$

$\sum F_x = 0 \quad C_x + 2P = H_E$

$$C_x = -\frac{2}{5}P. \quad \text{——} \; (G)$$

[2c] (3 pts) Draw the bending moment diagram.



Question 3: 10 points

OPTIONAL: Elastic Curve for Beam Deflections. Figure 3 is a front elevation view of a simply supported beam that carries a triangular load.

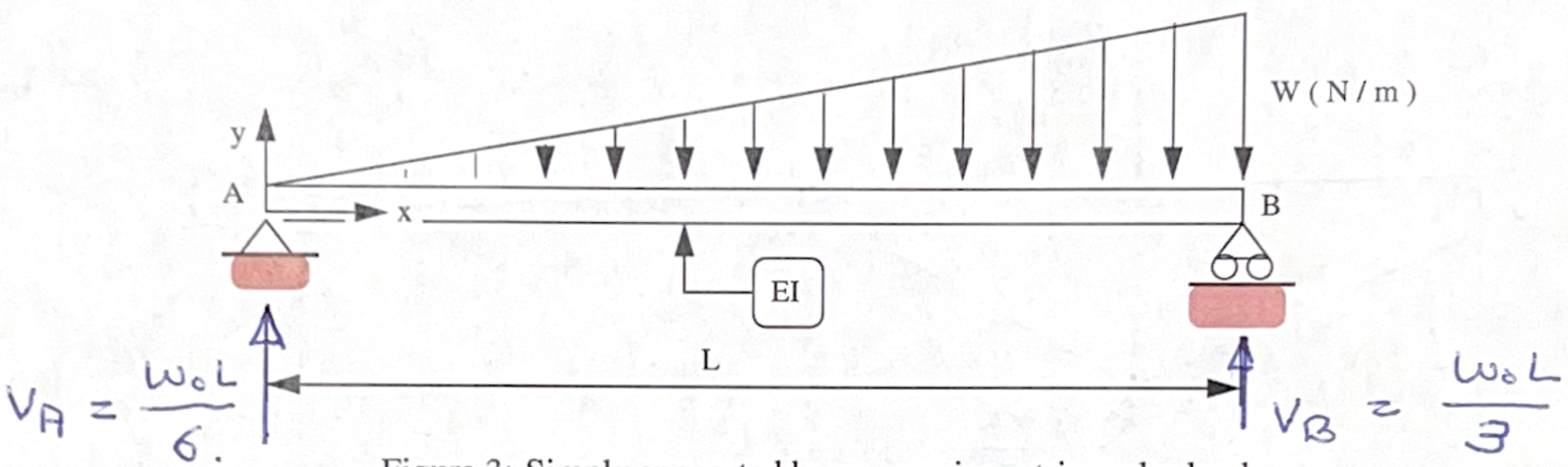
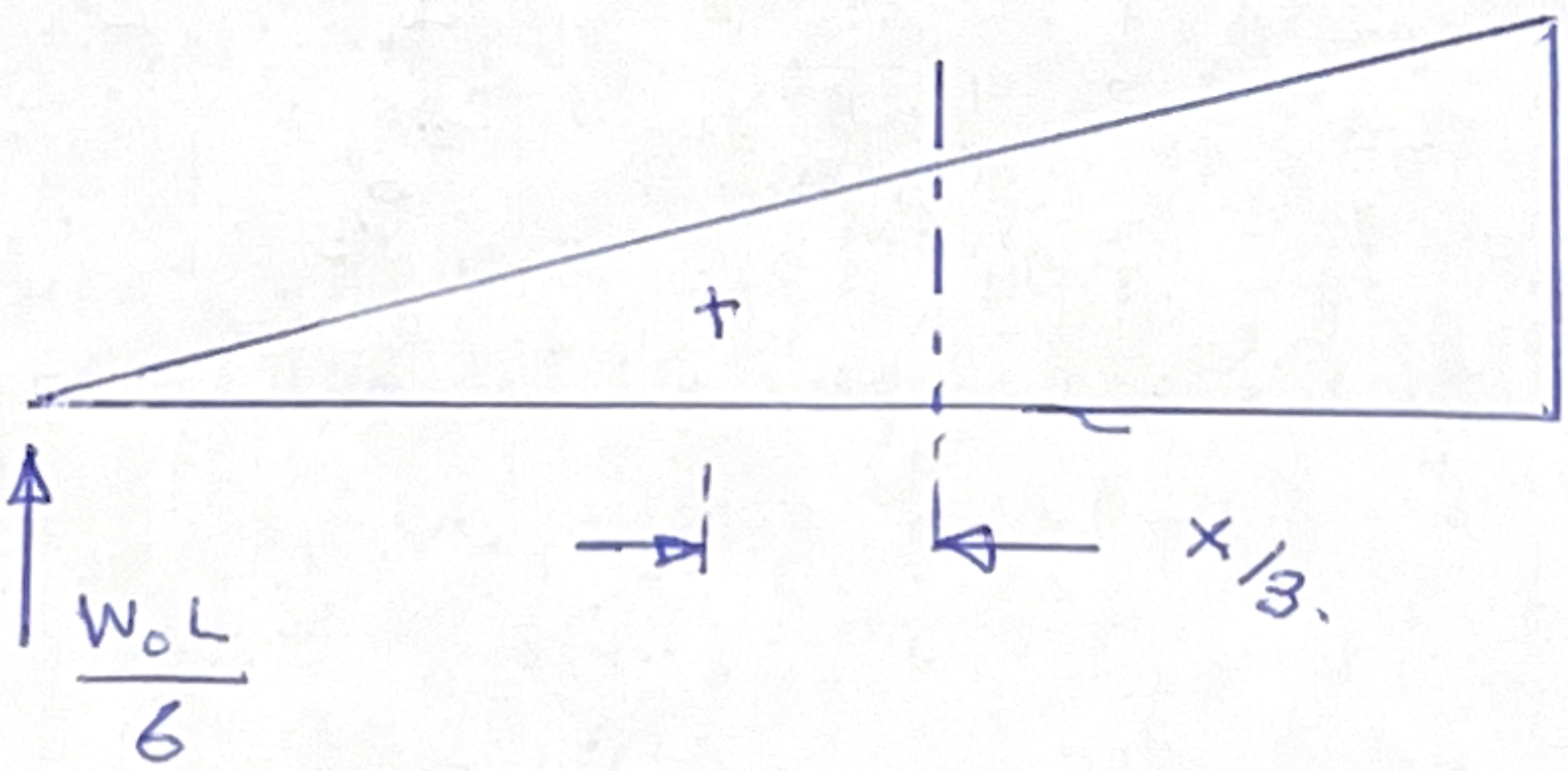


Figure 3: Simply supported beam carrying a triangular load.

The load increases from zero at point A to W (N/m) at point B. Thus, the total beam loading is WL/2.

[3a] (4 pts). Starting from first principles of engineering, show that the bending moment at point x is:

$$M(x) = \left[\frac{W}{6L}\right] x \left(L^2 - x^2\right). \tag{2}$$



$$M(x) = \left(\frac{W_0 L}{6}\right) x - \frac{1}{2} \frac{W_0 x^2}{L} \cdot \left(\frac{x}{3}\right)$$

$$= \left(\frac{W_0}{6L}\right) x \left(L^2 - x^2\right).$$ A

[3b] (4 pts). Show that the elastic curve for beam deflection is given by (notice that in Figure 2, the y axis is pointing upwards):

$$y(x) = \left[\frac{-W}{6LEI}\right]\left[\frac{L^2x^3}{6} - \frac{x^5}{20} - \frac{14L^4x}{120}\right] \tag{3}$$

$$\frac{d^2y}{dx^2} = \frac{-M(x)}{EI} \longrightarrow EI\frac{d^2y}{dx^2} = \frac{-W_0}{6L}(L^2x - x^3)$$

Integrating twice & applying boundary conditions:

$$y(0) = 0 \longrightarrow B = 0$$

$$y(L) = 0 \longrightarrow A = \left[\frac{L^5}{20} - \frac{L^5}{6}\right]\frac{1}{L} = \frac{-14L^4}{120}.$$

Hence,

$$y(x) = \frac{-W_0}{6EIL}\left[\frac{L^2x^3}{6} - \frac{x^5}{20} - \frac{14L^4x}{120}\right]. \text{ --- (B)}$$

[3c] (2 pts). Show that the maximum beam curvature occurs at x = $L/\sqrt{3}$.

$$\phi = \frac{M(x)}{EI}, \quad \text{Max } \phi \longrightarrow \frac{dM}{dx} = 0$$

$$\longrightarrow L^2 - 3x^2 = 0$$

$$\longrightarrow x = L/\sqrt{3} \text{ --- (C)}$$

Question 4: 10 points

OPTIONAL: Computing Displacements with the Method of Virtual Forces. Figure 4 is a front elevation view of a dog-leg cantilever beam carrying a clockwise moment M (N.m) at point C.

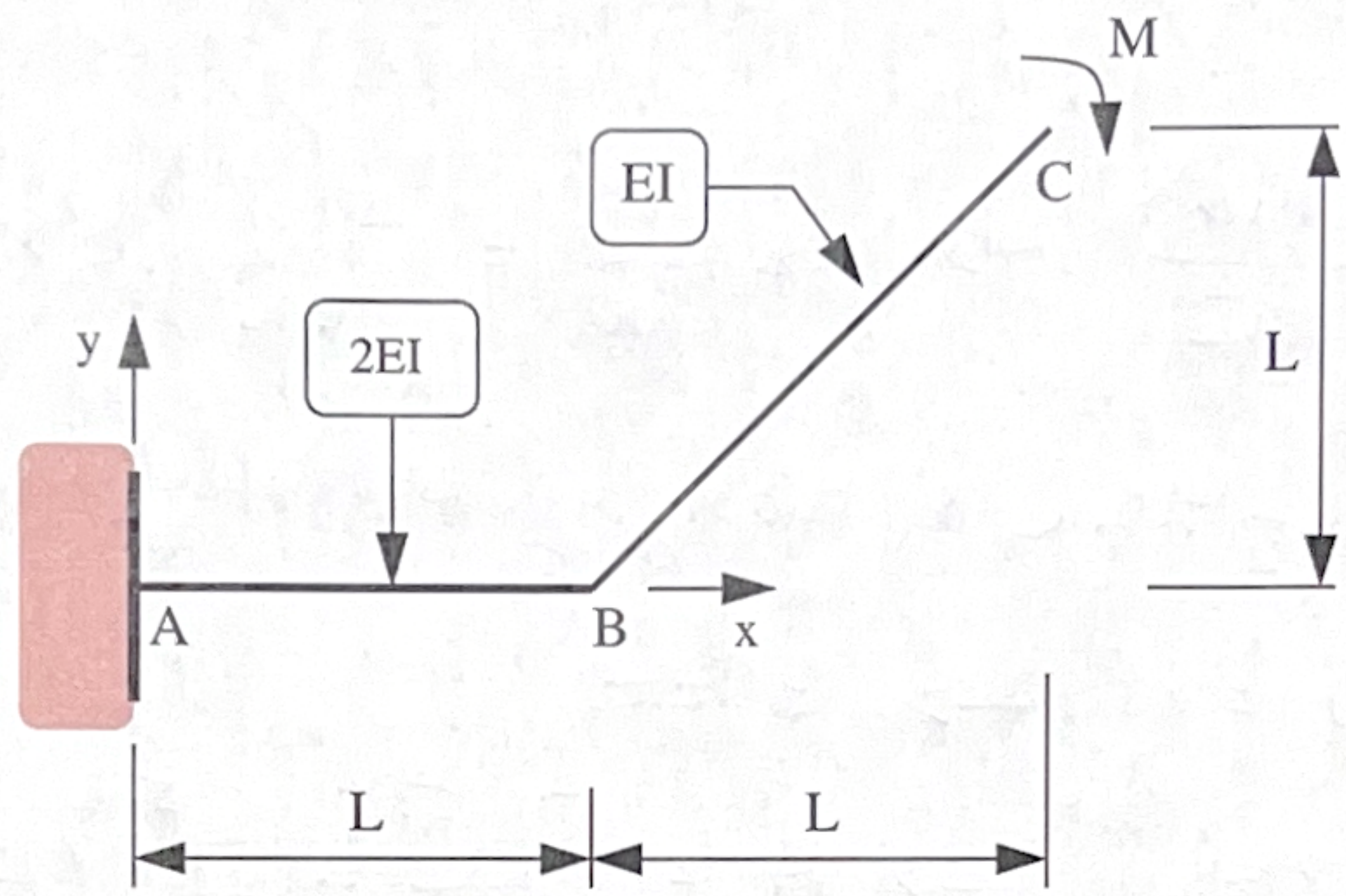


Figure 4: Dog-leg cantilever beam carrying end moment M (N.m).

The flexural stiffness 2EI is constant along A-B, and EI along B-C. The axial stiffness EA is very high and, as such, axial displacements can be ignored in the analysis.

[4a] (5 pts) Use the method of **virtual forces** to show that the clockwise rotation of the beam at point C is:

$$\theta_c = \frac{ML}{EI}\left[\frac{1+2\sqrt{2}}{2}\right]. \tag{4}$$

$$\theta_c = \int_A^B \frac{M}{2EI}\,dx + \int_B^C \frac{M}{EI}\,dx$$

$$= \frac{ML}{2EI} + \frac{M\sqrt{2}L}{EI}$$

$$= \frac{ML}{EI}\left[\frac{1+2\sqrt{2}}{2}\right]$$

A

[4b] (5 pts) Use the method of **virtual forces** to show that the vertical displacement at C (measured downwards) is:

$$y_c = \frac{ML^2}{EI}\left[\frac{1}{\sqrt{2}} + \frac{3}{4}\right]. \tag{5}$$

Show all of your working.

$$y_c = \underbrace{\int_0^{\sqrt{2}L} \frac{M}{EI}\left(\frac{x}{\sqrt{2}}\right) dx}_{\text{Element B-C}} + \underbrace{\int_0^{L} \frac{M}{2EI}(2L - x)\, dx}_{\text{Element A-B.}}$$

$$= \frac{M}{\sqrt{2}EI}\left[\frac{1}{2}x^2\right]_0^{\sqrt{2}L} + \frac{M}{2EI}\left[2Lx - \frac{x^2}{2}\right]_0^{L}$$

$$= \frac{ML^2}{EI}\left[\frac{1}{\sqrt{2}} + \frac{3}{4}\right] \qquad \text{B}$$

Question 5: 10 points

OPTIONAL: Use Principle of Virtual Work to Compute Displacements. Consider the articulated cantilever beam structure shown in Figure 5.

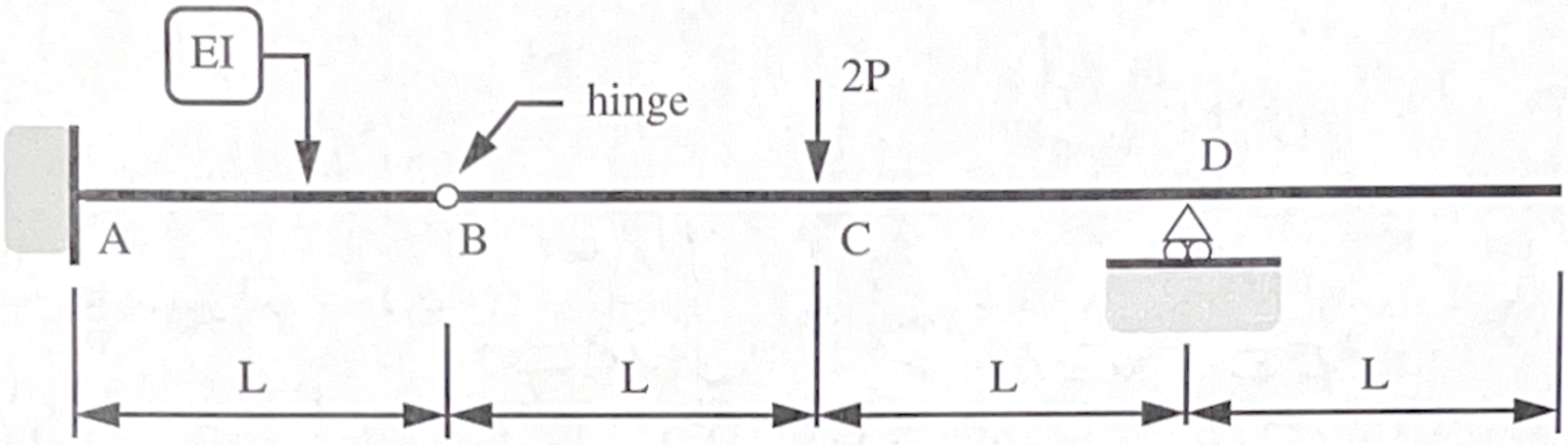
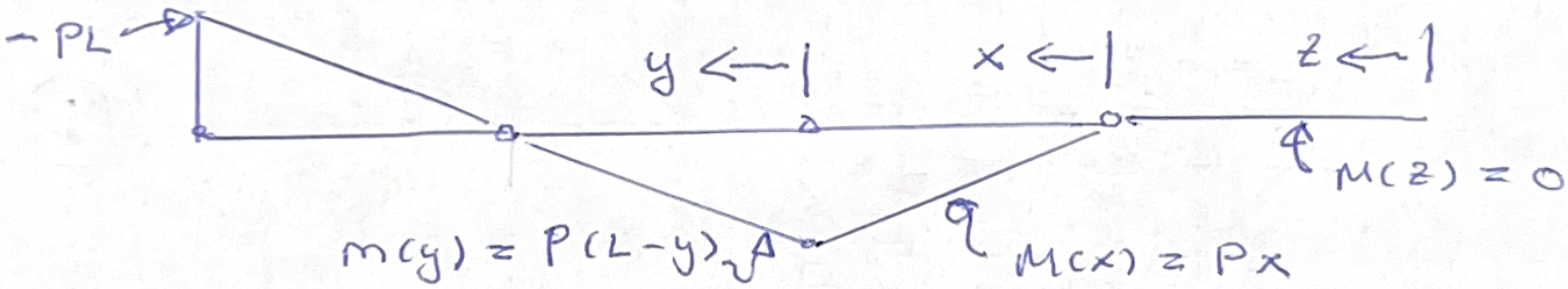


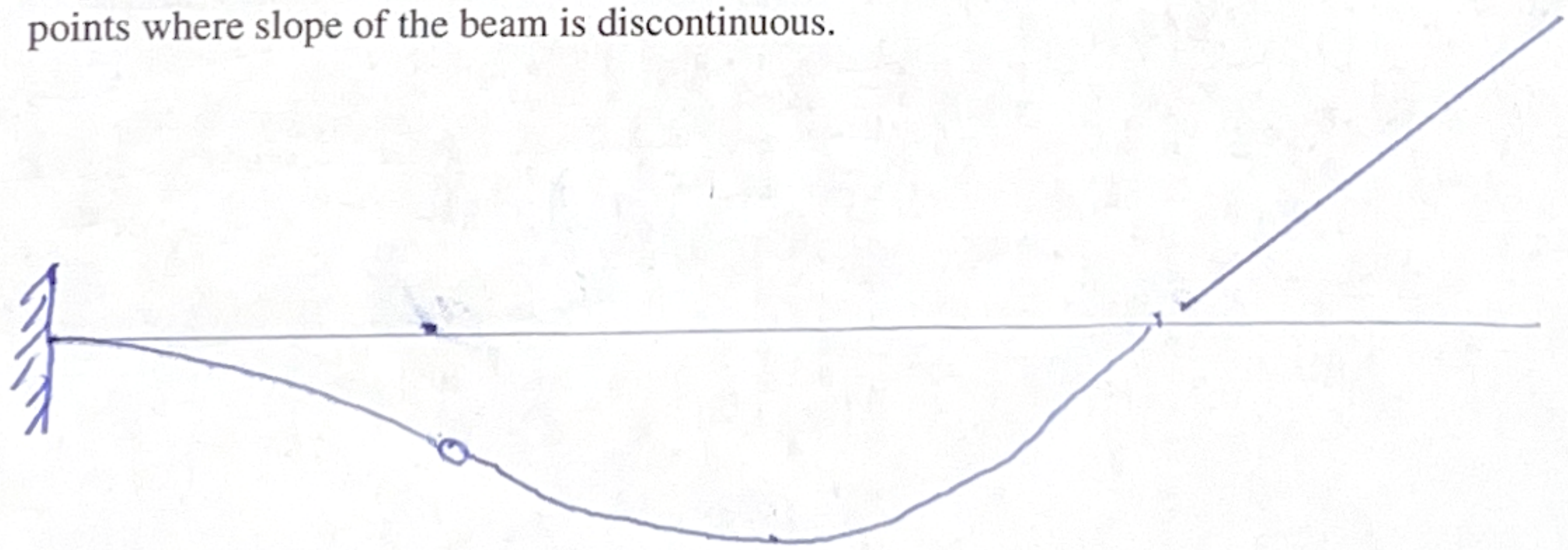
Figure 5: Elevation view of articulated cantilever beam structure.

At Point A, the cantilever is fully fixed (no movement) to a wall. Point B is a hinge. Both members have cross section properties EI. A single point load **2P** (N) is applied at node C as shown in the figure.

[5a] (2 pts). Draw and label the bending moment diagram for this problem.



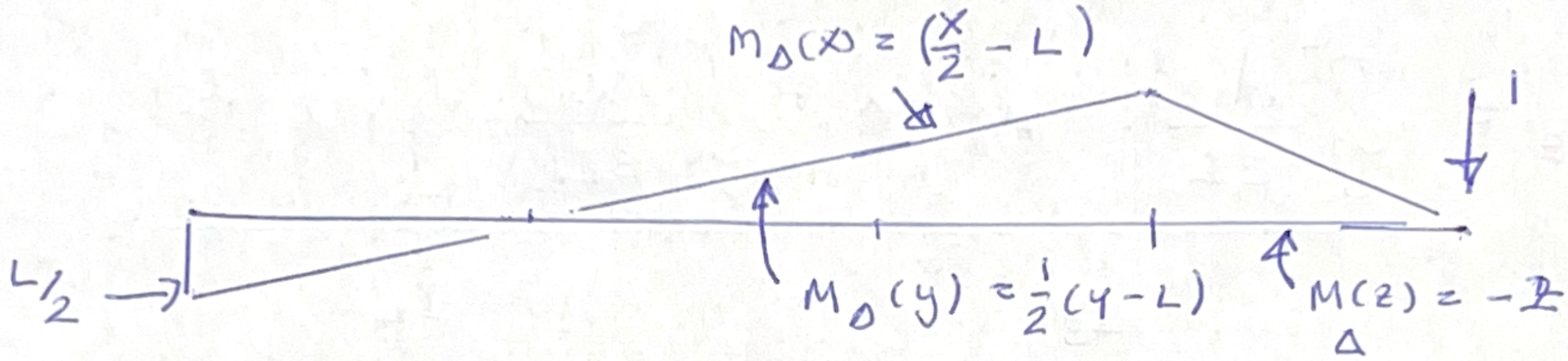
[5b] (2 pts). Qualitatively sketch the deflected shape. Indicate regions of tension/compression, and any points where slope of the beam is discontinuous.



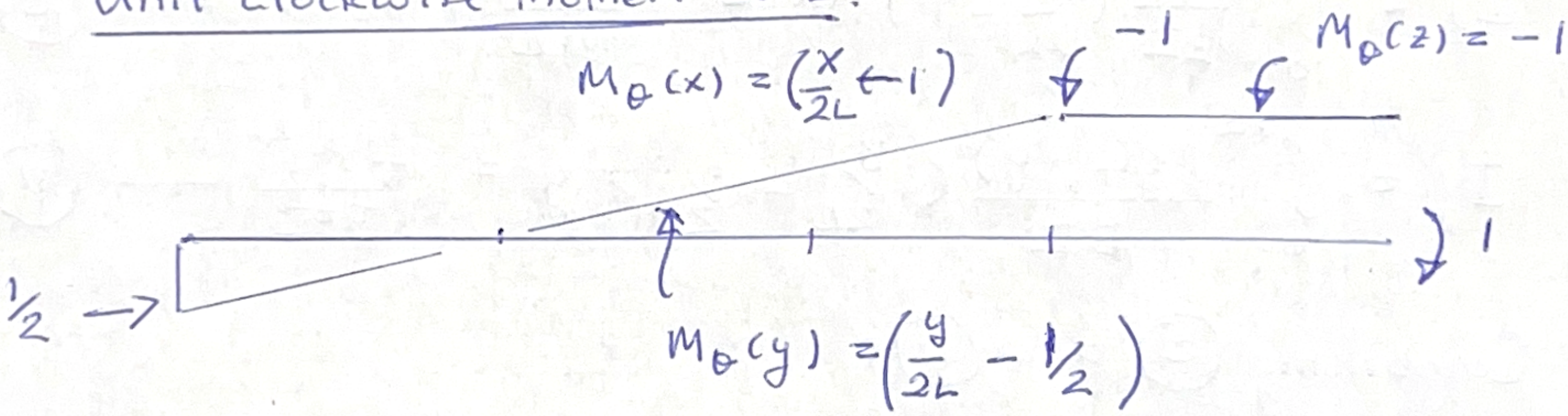
[5c] (6 pts). Use the method of **virtual forces** to compute the **vertical displacement** and **end rotation** of the beam at E.

Show all of your working.

Unit Point load at E.



Unit clockwise moment at E.



Vertical Deflection at E.

$$\Delta_E = \underbrace{\int_0^L \frac{M(x)}{EI} \cdot M_\Delta(x)\, dx}_{I_1} + \underbrace{\int_0^{2L} \frac{M(y)}{EI} \cdot M_\Delta(y)\, dy}_{I_2}.$$

$$I_1 = \int_0^L \frac{Px}{EI}\left(\frac{x}{2} - L\right) dx = \frac{-PL^3}{3EI}. \quad \text{———} \quad (A)$$

$$I_2 = \int_0^{2L} \frac{P(L-y)}{EI}\left(\frac{y}{2} - \frac{L}{2}\right) dy = \frac{-PL^3}{3EI} \quad \text{———} \quad (B)$$

Question [5c] continued:

Combining A and B: $\Delta_E = \frac{2}{3}\frac{PL^3}{EI}\uparrow$

Clockwise Rotation at E.

$$\theta_E = \underbrace{\int_0^L \frac{M(x)}{EI} \cdot M_\theta(x)\,dx}_{I_3} + \underbrace{\int_0^{2L} \frac{M(y)}{EI} \cdot M_\theta(y)\,dy}_{I_4}.$$

$$I_3 = \int_0^L \frac{Px}{EI}\left(\frac{x}{2L} - 1\right)dx = \frac{-PL^2}{3EI}. \quad \text{——— C}$$

$$I_4 = \int_0^{2L} \frac{P(L-y)}{EI}\left(\frac{y}{2L} - \frac{1}{2}\right)dy = \frac{-PL^2}{3EI} \quad \text{——— D}$$

Combining C and D:

Clockwise rotation $\theta_E = \frac{-2}{3}\frac{PL^2}{EI}$.