Department of Civil and Environmental Engineering, Spring Semester, 2024

ENCE 353 Final Exam, Open Notes and Open Book

Name : Austin.

Exam Format and Grading. This take home final exam is open notes and open book. You need to comply with the university regulations for academic integrity.

Answer Question 1. Then answer **three of the five** remaining questions. Partial credit will be given for partially correct answers, so please show all your working.

IMPORTANT: Only the **first four questions** that you answer will be graded, so please **cross out the two questions you do not want graded** in the table below. Also, before submitting your exam, check that **every page has been scanned correctly**.

Question	Points	Score
1	20	
2	10	
3	10	
4	10	
5	10	
6	10	
Total	50	

Question 1: 20 points

COMPULSORY: Method of Virtual Displacements, Method of Virtual Forces, Flexibilty Matrix. The T-shaped beam structure shown in Figure 1 has flexural stiffness EI throughout.

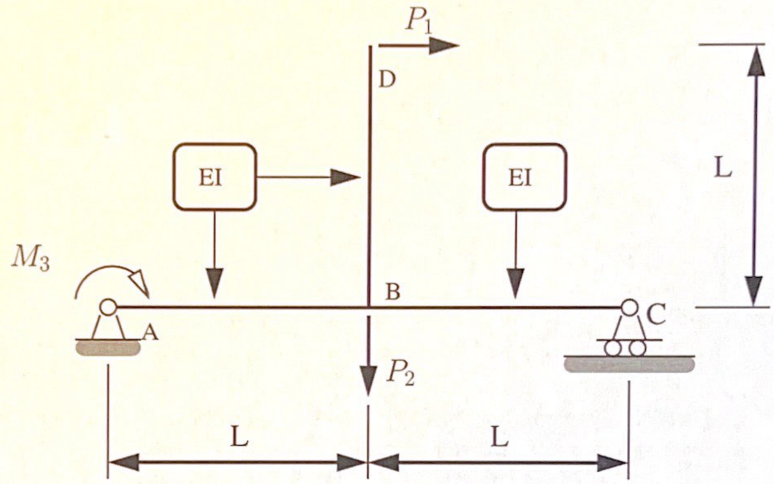
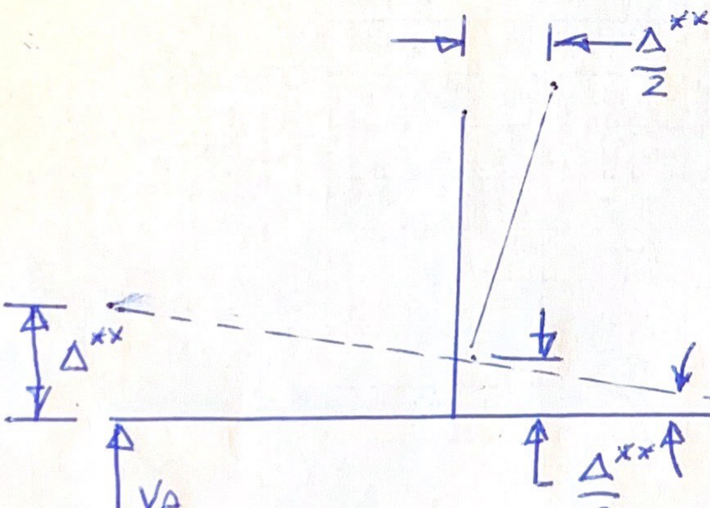


Figure 1: Front elevation view of a T-shaped beam.

[1a] (5 pts). Use the **method of virtual displacements** to compute the vertical reaction force at node A.



$\sum' EWD = 0$

$\Rightarrow V_A \Delta^{**} + M_3 \theta + P_1 \frac{\Delta^{**}}{2} - P_2 \frac{\Delta^{**}}{2} = 0$

$\theta = \frac{\Delta^{**}}{2L}.$

from geometry $\theta = \frac{\Delta^{**}}{2L}$

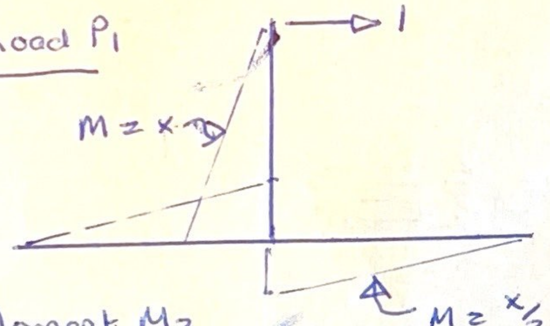
$\Rightarrow V_A = \frac{P_2}{2} - \frac{P_1}{2} - \frac{M_3}{2L}$

[1b] (15 pts). Use the **method of virtual forces** to compute the flexibility matrix:

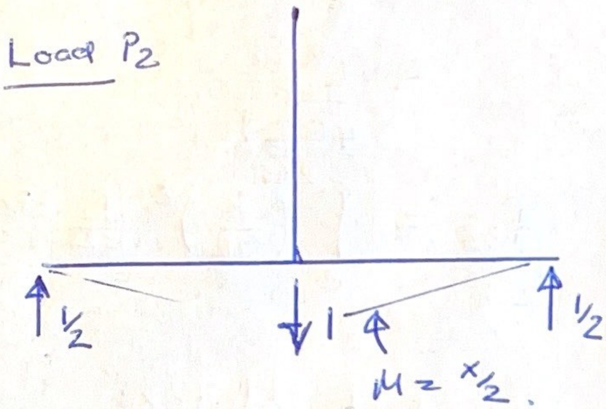
$$\begin{bmatrix} \triangle_{dx} \\ \triangle_{by} \\ \theta_A \end{bmatrix} = \begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \\ M_3 \end{bmatrix}. \tag{1}$$

BMD due to unit loads:

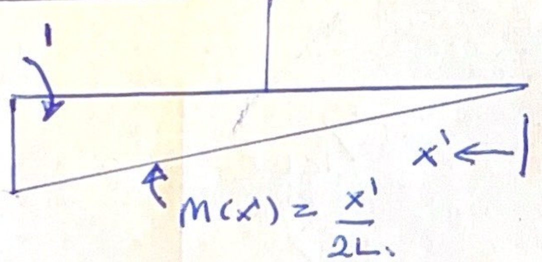
Load P_1



Load P_2



Moment M_3.



Flexibility coefficients:

$$f_{11} = 2\int_0^L \left(\frac{x}{2}\right)^2 \frac{1}{EI}\,dx + \int_0^L \frac{x^2}{EI}\,dx = \frac{L^3}{2EI}.$$

From symmetry: $f_{12} = f_{21} = 0$.

$$f_{31} = \int_0^L \frac{-x}{2}\left(1 - \frac{x}{2L}\right)\frac{1}{EI}\,dx + \int_0^L \left(\frac{x'}{2}\right)\left(\frac{x'}{2L}\right)\frac{1}{EI}\,dx'$$

$$= \frac{-L^2}{12EI}$$

Question 1b continued ...

$$f_{22} = 2\int_0^L \left(\frac{x}{2}\right)^2 \frac{1}{EI}\,dx = \frac{L^3}{6EI}.$$

$$f_{32} = \int_0^L \left(\frac{x}{2}\right)\left(1-\frac{x}{2L}\right)\frac{1}{EI}\,dx + \int_0^L \frac{x'}{2}\cdot\frac{x'}{2L}\cdot\frac{1}{EI}\,dx'$$

$$= \frac{L^2}{4EI}.$$

$$f_{33} = \int_0^{2L} \left(\frac{x}{2L}\right)^2 \cdot \frac{1}{EI}\,dx = \frac{2}{3}\frac{L}{EI}.$$

<u>Flexibility Matrix:</u>

$$f = \begin{bmatrix} L^3/2EI & 0 & -L^2/12EI \\ 0 & L^3/6EI & L^2/4EI \\ -L^2/12EI & \frac{L^2}{4EI} & \frac{2}{3}\frac{L}{EI} \end{bmatrix}$$

Question 2: 10 points

OPTIONAL: Method of Virtual Displacements. Figure 2 shows a simple three-bar truss. The bar elements have section properties EA throughout. Horizontal and vertical loads P are applied at node C.

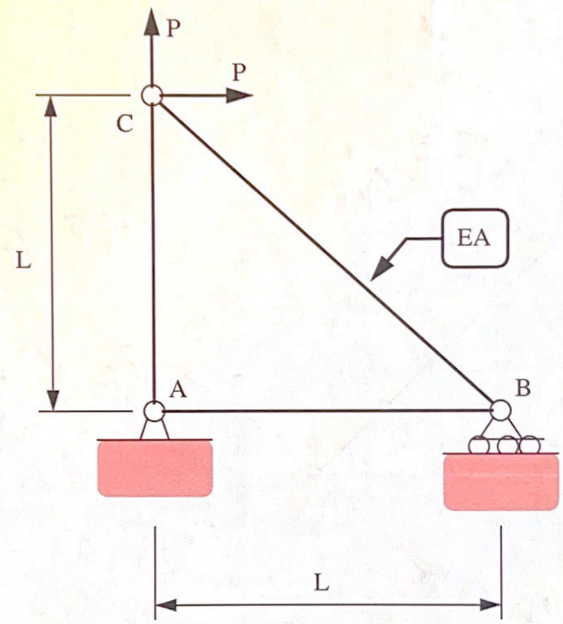
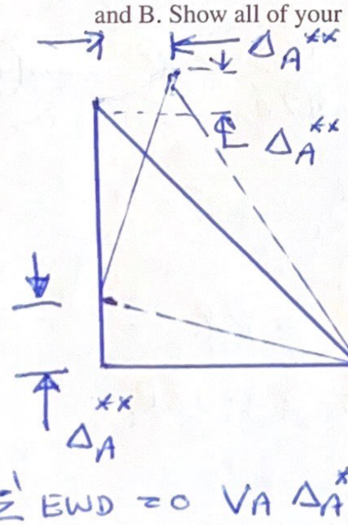
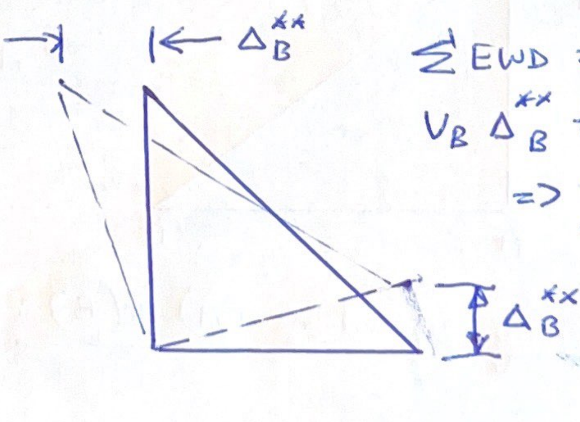


Figure 2: Simple three-bar truss.

[2a] (5 pts). Use the **method of virtual displacements** to compute the vertical reaction forces at nodes A and B. Show all of your working.





$\sum EWD = 0$

$V_B \Delta_B^{**} + P(-\Delta_B^{**}) = 0$

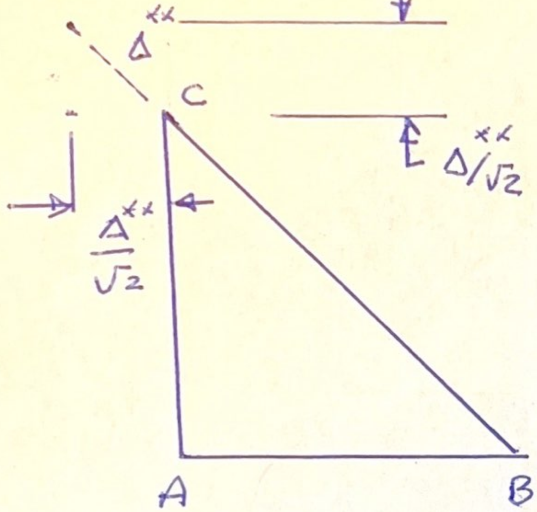
$\Rightarrow \underline{V_B = P.}$

$\sum EWD = 0 \quad V_A \Delta_A^{**} + \Delta_A^{**} P + \Delta_A^{**} P = 0$

$\Rightarrow \underline{V_A = -2P.}$

[2b] (5 pts). Use the **method of virtual displacements** to compute the member forces AC and BC. Show all of your working.

Extend $\overline{BC}$ by Δ^{**}

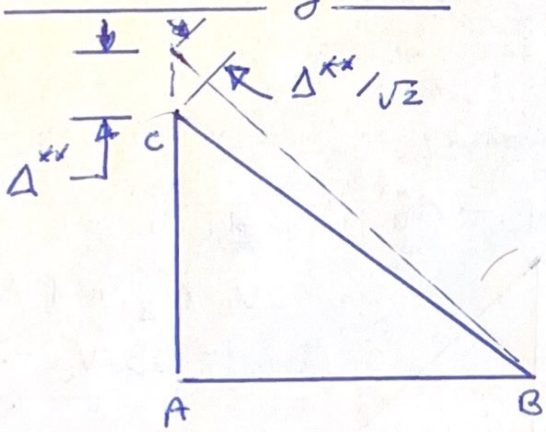


$$\sum^{1} EWD = \sum_{1}^{1} IWD.$$

$$\frac{\Delta^{**}}{\sqrt{2}}(-P) + \frac{\Delta^{**}}{\sqrt{2}}P = \overline{BC}\,\Delta^{**} + \overline{AC}\left(\frac{\Delta^{**}}{\sqrt{2}}\right)$$

$$\Rightarrow \overline{BC} + \frac{\overline{AC}}{\sqrt{2}} = 0 \quad \text{—— (A)}$$

Extend $\overline{AC}$ by Δ^{**}



$$\sum^{1} EWD = \sum_{1}^{1} IWD$$

$$P\,\Delta^{**} = \overline{AC}\,\Delta^{**} + \overline{BC}\,\frac{\Delta^{**}}{\sqrt{2}}$$

$$\Rightarrow \overline{AC} + \frac{\overline{BC}}{\sqrt{2}} = P \quad \text{—— (B)}$$

Combining equations (A) & (B):

$$\overline{AC} = 2P$$

$$\overline{BC} = -\sqrt{2}P.$$

Question 3: 10 points

OPTIONAL: Derive Elastic Curve for Cantilever Beam Deflection. Figure 3 is a front elevation view of a cantilevered beam carrying a uniform load, w (N/m), plus a single point load P. EI is constant along the cantilever A-B.

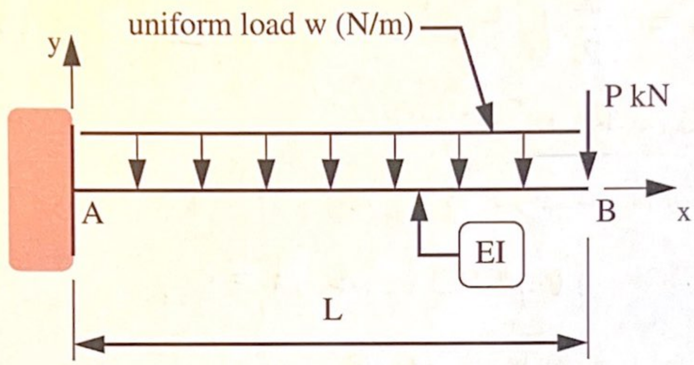
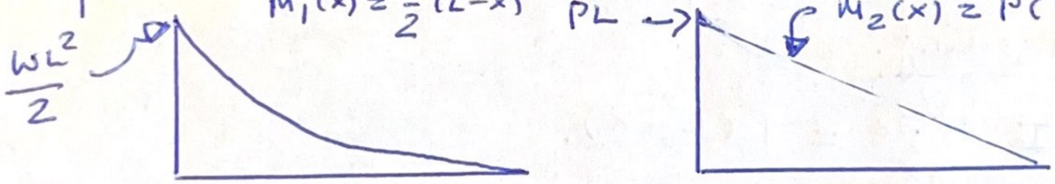


Figure 3: Cantilevered beam carrying a uniform load w (N/m) + single applied load P.

[3a] (4 pts). Draw and label the bending moment diagram for this problem. Decompose into two parts:

$M_1(x) = \frac{w}{2}(L-x)^2$ $\quad$ $M_2(x) = P(L-x)$.



[3b] (6 pts) Starting from the differential equation,

$$\frac{d^2y}{dx^2} = \left[\frac{M(x)}{EI}\right], \qquad (2)$$

appropriate boundary conditions, derive expressions for: (1) the clockwise rotation of the cantilever at B, and (2) the vertical displacement of the beam at B.

$$EI\frac{d^2y}{dx^2} = M_1(x) + M_2(x) = \frac{w}{2}(L^2 - 2Lx + x^2) + P(L-x).$$

<u>Uniform loading:</u>

$$EI\frac{dy_1}{dx} = \frac{w}{2}\int_0^L (L^2 - 2Lx + x^2)\,dx$$

Question [3b] continued:

$$= \frac{W}{2}\left[L^2x - Lx^2 + \frac{x^3}{3}\right] + A$$

$$EI\, y_1(x) = \frac{W}{2}\left[\frac{L^2x^2}{2} - \frac{Lx^3}{3} + \frac{x^4}{12}\right] + Ax + B.$$

Boundary conditions.

$y_1(0) = 0 \rightarrow B = 0$

$dy_1/dx = 0$ at $x = 0 \rightarrow A = 0.$

Displacement / Rotation at B.

$$\left.\frac{dy_1}{dx}\right|_{x=L} = \frac{wL^3}{6EI}, \quad y_1(L) = \frac{wL^4}{8EI}.$$

Point Loading.

$$EI\frac{d^2y_2}{dx} = P(L-x).$$

$$\Rightarrow EI\, y_2(x) = \frac{PLx^2}{2} - \frac{Px^3}{6} + Ax + B.$$

Boundary conditions:

$y_2(0) = 0 \rightarrow B = 0$

$dy_2/dx\,|_{x=0} = 0 \rightarrow A = 0$

$$EI\, y_2(x) = \frac{Px^2}{6}(3L - x).$$

Clockwise Rotation at B.

$$\theta_B = \frac{WL^3}{6EI} + \frac{PL^2}{2EI}$$

Vertical Displacement at B.

$$y(L) = \frac{WL^4}{8EI} + \frac{PL^3}{3EI}.$$

Question 4: 10 points

OPTIONAL: Bending Moment and Curvature in an Elastic Beam. Figure 4 is a front elevation view of a simply supported beam that carries a trapezoid load.

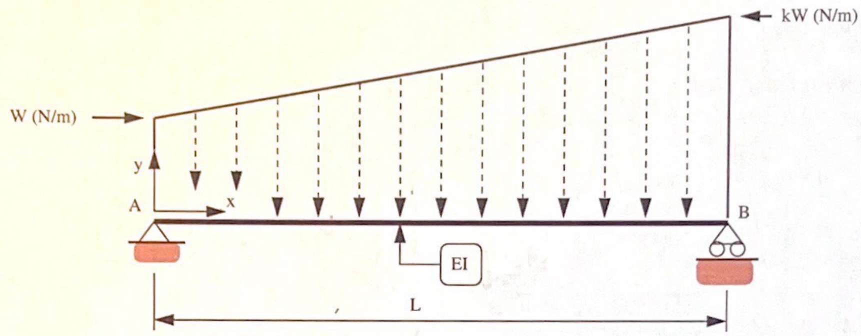
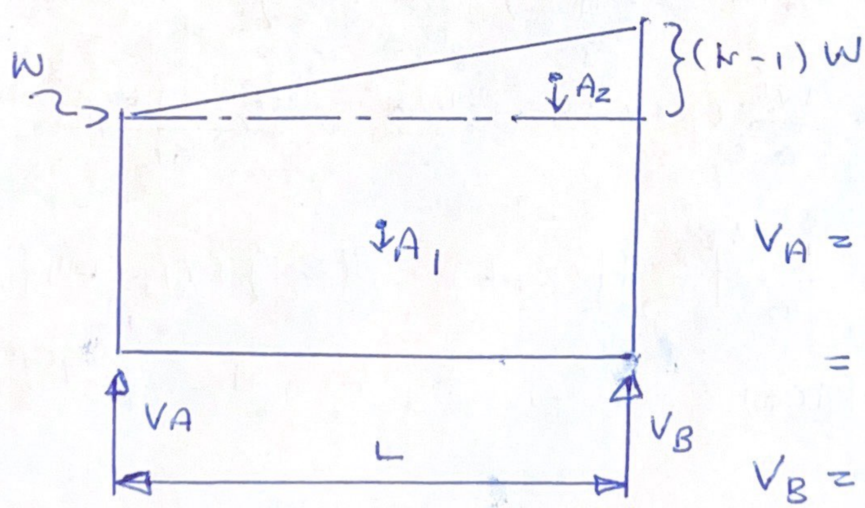


Figure 4: Simply supported beam carrying a trapazoid load.

The load increases from W (N/m) at x = 0, to kW (N/m) at x = L, where k is a non-negative constant. Thus, the total beam loading is $\frac{WL}{2}(1+k)$.

[4a] (2 pts). Starting from first principles of engineering, show that the vertical reactions at A and B are:

$$V_A = \frac{WL}{6}(2+k) \quad \text{and} \quad V_B = \frac{WL}{6}(1+2k)\,. \tag{3}$$



$A_1 = WL.$

$A_2 = \frac{(k-1)WL}{2}.$

$V_A = A_1/2 + A_2/3$

$= \frac{WL}{6}[2+k].$

$V_B = A_1/2 + \frac{2A_2}{3}$

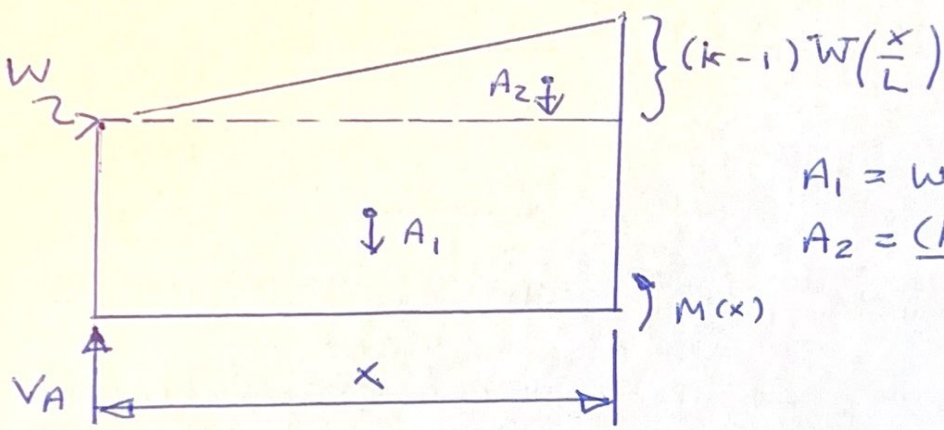
$= \frac{WL}{6}[1+2k].$

[4b] (3 pts). Show that the bending moment at point x is:

$$M(x) = \frac{WL^2}{6}\left(\frac{x}{L}\right)\left[(2+k) - 3\left(\frac{x}{L}\right) + (1-k)\left(\frac{x}{L}\right)^2\right]. \tag{4}$$

Notice that M(0) = M(L) = 0, regardless of the value of k.

The math for this part is a bit tedious – hence, I suggest you work out a solution on a separate sheet of paper, then write a tidy solution here.



$$A_1 = Wx$$

$$A_2 = \frac{(k-1)Wx^2}{2L}.$$

$$M(x) = V_A \cdot x - A_1\left(\frac{x}{2}\right) - A_2\left(\frac{x}{3}\right)$$

$$= \frac{WL}{6}(2+k)x - \frac{Wx^2}{2} - \frac{(k-1)Wx^3}{6L}.$$

$$= \frac{WL^2}{6}\left[\frac{x}{L}\right]\left[(2+k) - 3\left(\frac{x}{L}\right) + (1-k)\left(\frac{x}{L}\right)^2\right] - (A)$$

Notice: $M(0) = M(L) = 0$, regardless of k!

[4c] (3 pts). Hence, show that the location of maximum curvature ϕ in the beam corresponds to the solution of the quadratic equation:

$$3\,(1-k)\,x^2 - 6Lx + (2+k)\,L^2 = 0. \tag{5}$$

$$\text{Max } \phi \implies \frac{dM}{dx} = 0$$

$$\frac{dM}{dx} = \left(\frac{w}{6L}\right)\left[(2+k)L^2 - 6Lx + 3(1-k)x^2\right] = 0$$

$$\implies 3(1-k)\,x^2 - 6Lx + (2+k)L^2 = 0 \quad ✓✓.$$

(B)

[4d] (2 pts). For the case where k = 1 (i.e., a constant uniform loading), use equations 4 and 5 to determine the position and value of the maximum bending moment.

Let $k = 1$. Plug into equation (B):

$$-6Lx + 3L^2 = 0$$

$$\implies x = L/2 \longleftarrow \text{Midpoint of beam.}$$

$$M(L/2) = \frac{wL^2}{6}\left[\tfrac{1}{2}\left(2 - 3/2 + 1/4 + k\left(1-(1/2)^2\right)\right)\right]$$

$$= \frac{wL^2}{8} \quad ✓✓$$

Question 5: 10 points

OPTIONAL: Use Principle of Virtual Work to Compute Displacements. Consider the articulated cantilever beam structure shown in Figure 5.

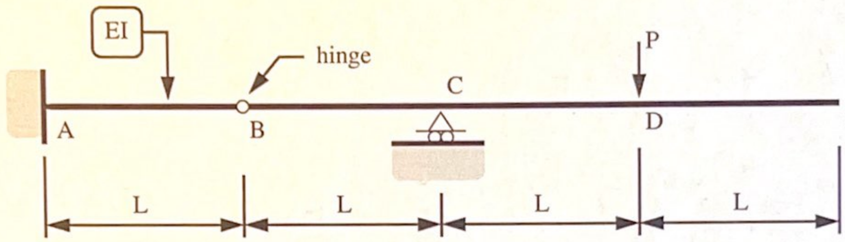
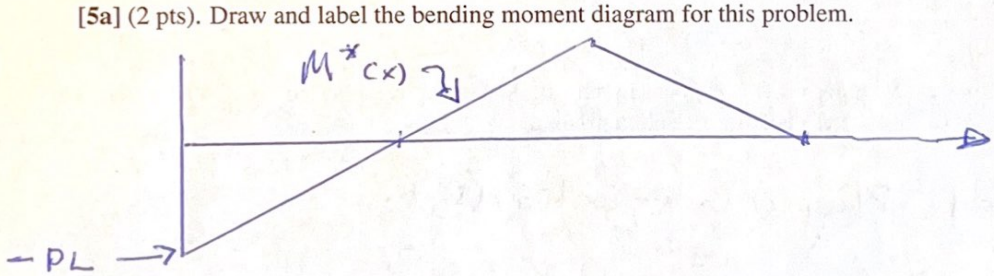


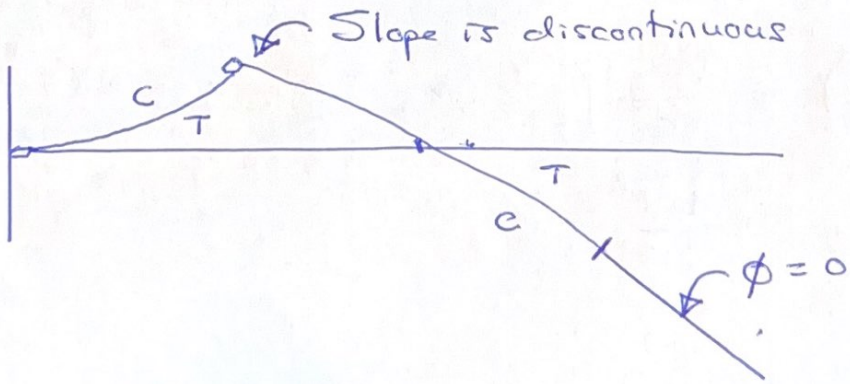
Figure 5: Elevation view of articulated cantilever beam structure.

At Point A, the cantilever is fully fixed (no movement) to a wall. Point B is a hinge. Both members have cross section properties EI. A single point load **P** (N) is applied at node D as shown in the figure.

[5a] (2 pts). Draw and label the bending moment diagram for this problem.



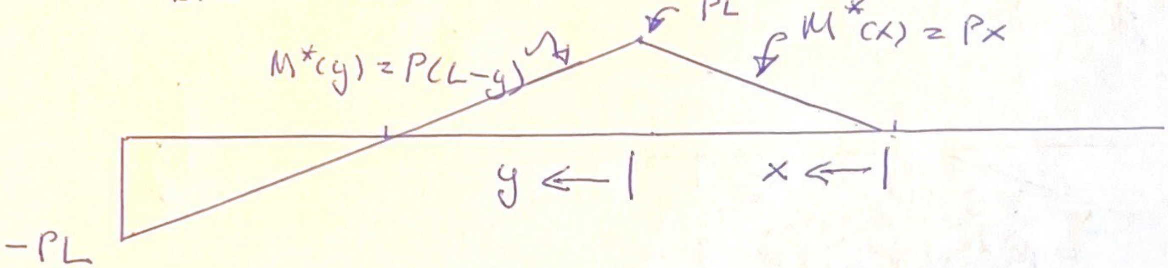
[5b] (2 pts). Qualitatively sketch the deflected shape. Indicate regions of tension/compression, and any points where slope of the beam is discontinuous.



[5c] (6 pts). Use the method of **virtual forces** to compute the **vertical displacement** and **end rotation** of the beam at E.

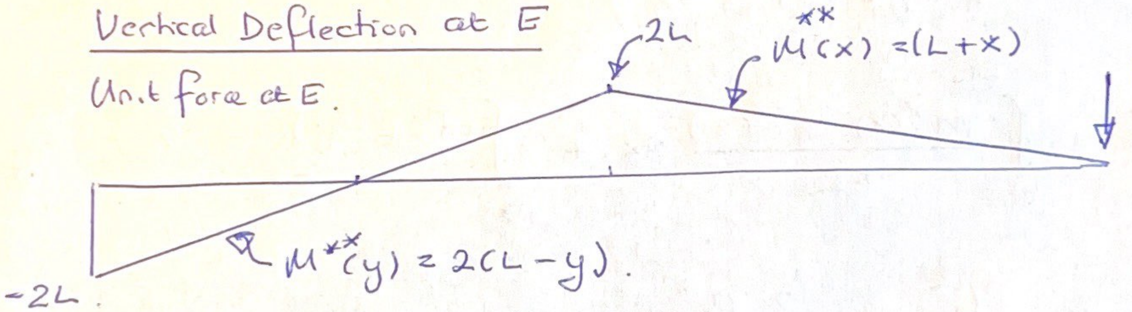
Show all of your working.

BMD due to the load at D.



Vertical Deflection at E

Unit force at E.



$$\Delta_E = \underbrace{\int_0^L \frac{M^*(x) \cdot M^{**}(x)}{EI}\,dx}_{I_1} + \underbrace{\int_0^{2L} \frac{M^*(y)\, M^{**}(y)}{EI}\,dy}_{I_2}.$$

$$I_1 = \int_0^L \frac{Px(L+x)}{EI}\,dx = \frac{5}{6}\frac{PL^3}{EI}.$$

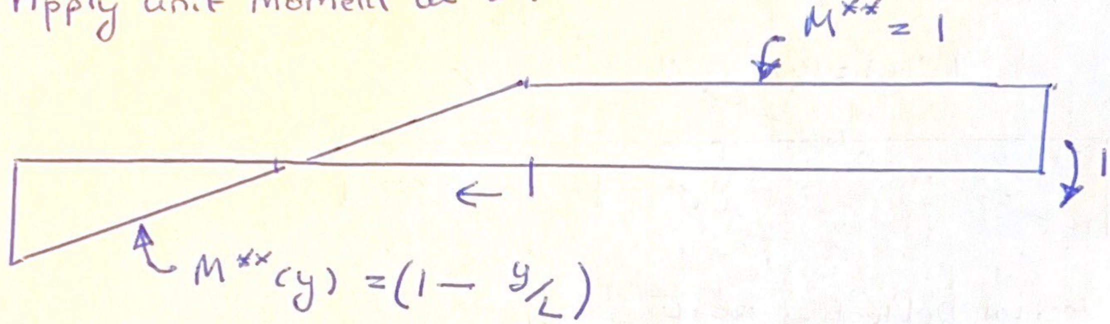
$$I_2 = \int_0^{2L} \frac{P(L-y)\,2(L-y)}{EI}\,dy = \frac{4}{3}\frac{PL^3}{EI}$$

$$\Delta_E = I_1 + I_2 = \frac{13}{6}\frac{PL^3}{EI}.$$

Question [5c] continued:

Rotation at E.

Apply unit moment at E.



$$\theta_D = \int_0^{2L} \frac{M^{*}(x)\, M^{**}(x)}{EI}\, dx + \int_0^{2L} \frac{P(L-y)(1 - y/L)}{EI}\, dy.$$

$$= \frac{PL^2}{2EI} + \frac{2}{3}\frac{PL^2}{EI}$$

$$\theta_D. = \frac{7}{6}\frac{PL^2}{EI}.$$

Question 6: 10 points

OPTIONAL: Principle of Virtual Work. The left-hand side of Figure 6 shows a simple two-bar truss that supports vertical and horizontal loads at node B. The right-hand side of Figure 6 shows the same truss with a third bar added – the latter makes the truss structure statically indeterminate to degree one. All of the truss members have cross section properties AE.

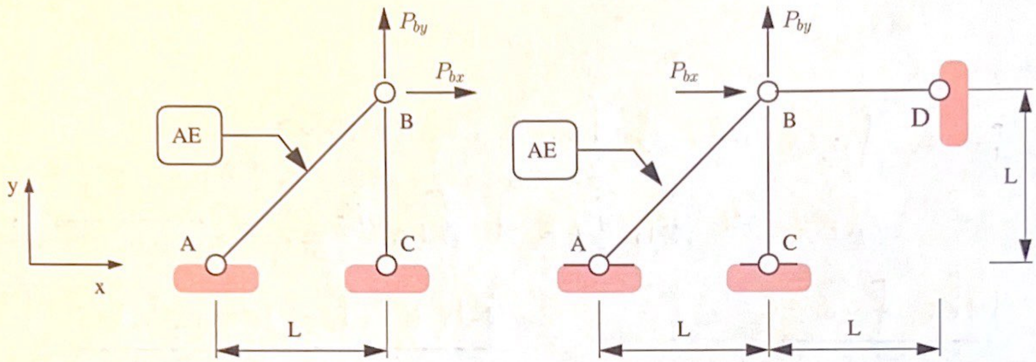
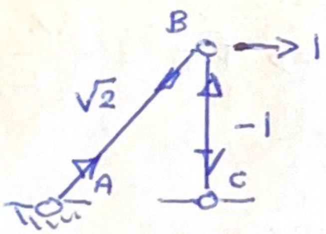


Figure 6: Front elevation view of: (left) A simple two-bar truss, and (right) a simple three-bar truss.

Let's start with the two-bar truss:

[6a] (5 pts) Use the method of **virtual forces** to compute the two-by-two flexibility matrix connecting the horizontal and vertical displacements at node B to the applied loads P_{bx} and P_{by}, L and AE.



$$\begin{bmatrix} \Delta_{bx} \\ \Delta_{by} \end{bmatrix} = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} \begin{bmatrix} P_{bx} \\ P_{by} \end{bmatrix}. \quad (6)$$

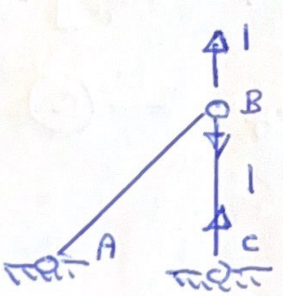
Flexibility Matrix.

#	L/AE	f_1	f_2
AB	$\sqrt{2}L/AE$	$\sqrt{2}$	0
BC	L/AE	-1	1

$$f_{11} = \sum^{1}_{1} \frac{f_1^2 L}{AE} = (1 + 2\sqrt{2}) \frac{L}{AE}$$

$$f_{22} = L/AE$$

$$f_{12} = -L/AE.$$



$$f = \frac{L}{AE} \begin{bmatrix} (1+2\sqrt{2}) & -1 \\ -1 & 1 \end{bmatrix}$$

Now let's consider the simple three-bar truss:

[6b] (5 pts). Using the method of **virtual forces** and the results of part [6a], or otherwise, derive a formula for the member force **BD** as a function of applied loads P_{bx} and P_{by}.

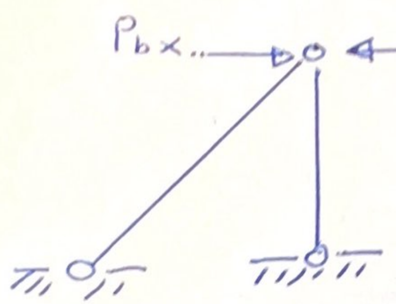
Show that if $P_{by} = 0$, then the compressive force in member BD is:

$$\text{Member force BD} = \left[\frac{1 + 2\sqrt{2}}{2 + 2\sqrt{2}} \right] P_{bx}. \tag{7}$$

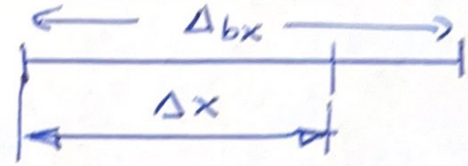
From first row of flexibility matrix:

$$\Delta_{bx} = f_{11} P_{bx} + f_{12} P_{by}. \quad \text{—— (A)}$$

<u>Net forces:</u>



<u>Net displacements:</u>



Displacement of BD.

$$\Delta x = \frac{\overline{BD}\, L}{AE} \Rightarrow \overline{BD} = \frac{\Delta x\, AE}{L}.$$

<u>Combining Results:</u>

$$\Delta X = \frac{\overline{BD} \cdot L}{AE} = \frac{L}{AE} \left[(1 + 2\sqrt{2})(P_{bx} - \overline{BD}) - P_{by} \right]$$

$$\Rightarrow \overline{BD} = (1 + 2\sqrt{2}) P_{bx} - (1 + 2\sqrt{2}) \overline{BD} - P_{by}, \quad \text{—— (C)}$$

When $P_{by} = 0$.

$$\overline{BD} = \frac{(1 + 2\sqrt{2})\, P_{bx}}{(2 + 2\sqrt{2})} \quad \checkmark\checkmark$$