Department of Civil and Environmental Engineering, Fall Semester, 2021

ENCE 353 Midterm 2, Open Notes and Open Book

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Exam Format and Grading. Attempt all three questions. Partial credit will be given for partially correct answers, so please **show all of your working**.

Question	Points	Score
1	15	
2	10	
3	15	
Total	40	

Question 1: 15 points

Elastic Curve for Beam Deflections. Figure 1 is a front elevation view of a cantilever beam carrying two external loads P. EI is constant along the cantilever beam.

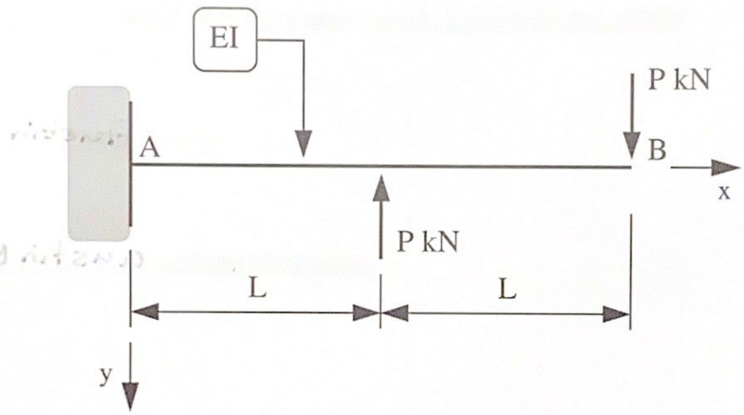
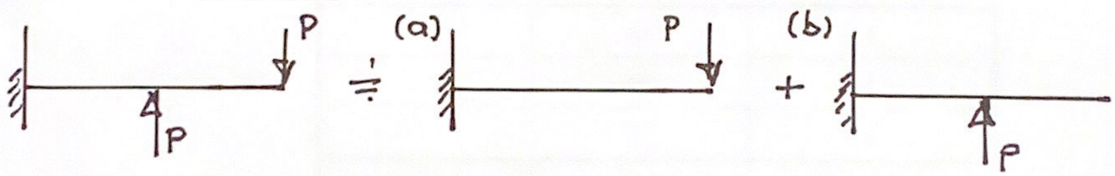


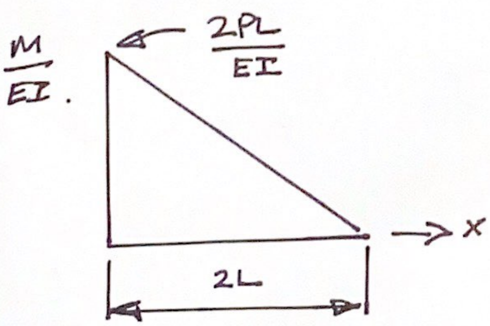
Figure 1: Cantilever beam carrying two applied loads P (kN).

[1a] (2 pts) Briefly explain how the **principle of superposition** – hint, hint, hint, hint !!! – can be applied to **this problem**.

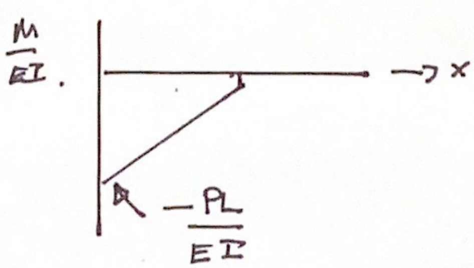


[1b] (2 pts) Draw and label the M/EI diagrams for the two subproblems.

For system (a)



For system (b)



[1c] (6 pts) Starting from the differential equation,

$$\frac{d^2y}{dx^2} = \left[\frac{M(x)}{EI}\right], \tag{1}$$

derive formulae for displacement y(x) in the two subproblems indicated in part 1a. For convenience, let's call these displacement curves $y_1(x)$ and $y_2(x)$.

Hint: Your formula should be piecewise continuous over two regions, i.e., region 1: $0 \leq x \leq L$, and region 2: $L \leq x \leq 2L$.

For system (a).

$$M(x) = P(L\cdot 2 - x)$$
$$= P(2L - x)$$

$$\frac{d^2y}{dx^2} \cdot \frac{P(2L-x)}{EI}$$

Boundary conditions: $y(0) = 0$, $\left.\frac{dy}{dx}\right|_{x=0} = 0$

Integrating twice.

$$y_1(x) = \frac{P}{6EI} x^2(6L - x), \quad 0 \leq x \leq 2L.$$

For System (b).

$$M(x) = -P(L - x)$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{-P(L-x)}{EI}, \quad 0 \leq x \leq L.$$

Boundary conditions, $y(0) = 0$, $\left.\frac{dy}{dx}\right|_{x=0} = 0$

Integrating twice:

$$y_2(x) = \frac{-P}{6EI}(x^2)(3L - x), \quad 0 \leq x \leq L.$$

Question 1c continued:

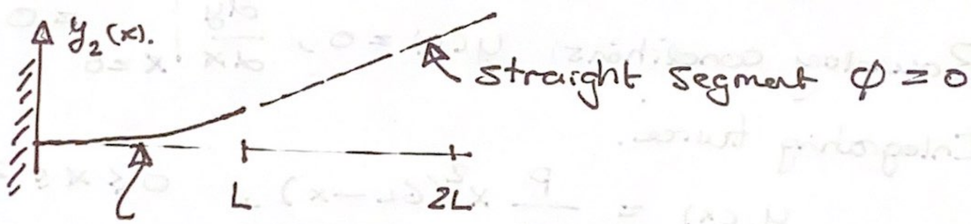
Notice that at $x = L$, $y(L) = \dfrac{-PL^3}{3EI}$ ↑

and the end rotation is $\theta_L = \dfrac{-PL^2}{2EI}$.

Hence the piecewise displacement curve is.

$$y_2(x) = \begin{cases} \dfrac{-Px^2}{6EI}(3L - x), & 0 \le x \le L \\ -\dfrac{PL^3}{3EI} - \dfrac{PL^2}{2EI}(x - L), & L \le x \le 2L. \end{cases}$$

ie.



$$\phi = \frac{M(x)}{EI}.$$

The net displacement is; $y(x) = y_1(x) + y_2(x)$.

$$y(x) = \begin{cases} \dfrac{Px^2}{6EI}(6L - x) - \dfrac{Px^2}{6EI}(3L - x), & \text{for } 0 \le x \le L \\ \dfrac{Px^2}{6EI}(6L - x) - \dfrac{PL^3}{3EI} - \dfrac{PL^2}{2EI}(x - L), & \text{for } L \le x \le 2L. \end{cases}$$

The first part simplifies to $\dfrac{Px^2}{6EI}(6L - x - 3L + x) = \dfrac{PLx^2}{2EI}$.

[1d] (3 pts) Compute the total displacement y(x) = $y_1(x) + y_2(x)$ along the beam.

$$y(x) = \begin{cases} \dfrac{PLx^2}{2EI}, & \text{for } 0 \leq x \leq L \\ \dfrac{Px^2}{6EI}(6L - x) - \dfrac{PL^3}{3EI} - \dfrac{PL^2}{2EI}(x - L), & \text{for } L \leq x \leq 2L. \end{cases}$$

[1e] (2 pts) Compute the vertical deflection of point B.

$$y(2L) = y_1(2L) + y_2(2L).$$

$$\Rightarrow y_1(2L) = \frac{P(2L)^2(4L)}{6EI} = \frac{16}{6}\frac{PL^3}{EI} \downarrow$$

$$y_2(2L) = -\frac{PL^3}{3EI} - \frac{PL^3}{2EI} = \frac{-5}{6}\frac{PL^3}{EI} \uparrow$$

$$\Rightarrow y(2L) = \left(\frac{16}{6} - \frac{5}{6}\right)\frac{PL^3}{EI} = \frac{11}{6}\frac{PL^3}{EI}.$$

Note: See Midterm II, Spring 2020, Question 1.

Tip deflection $= \dfrac{11}{6}\dfrac{PL^3}{EI}$, computed using method of moment area.

Question 2: 10 points

Analysis of a Three-Pinned Parabolic Arch. Consider the three-pinned parabolic arch shown in Figure 2

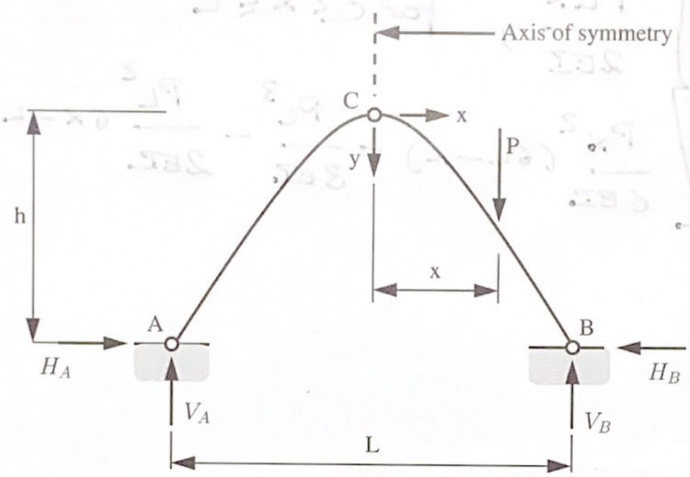


Figure 2: Three-pinned parabolic arch carrying a point load.

The height and width of the arch are h and L, respectively. A point load P is applied at a distance x, $0 \leq x \leq L/2$, from the axis of symmetry.

[2a] (5 pts) Starting from first principles of engineering (i.e., equations of equilibrium), show that the vertical and horizontal reaction forces at A and B are:

$$V_A(x) = P\left[\frac{1}{2} - \frac{x}{L}\right] \quad (2)$$

$$V_B(x) = P\left[\frac{1}{2} + \frac{x}{L}\right] \quad (3)$$

$$H_A(x) = H_B(x) = \frac{P}{2h}\left[\frac{L}{2} - x\right] \quad (4)$$

$\sum M_A = 0 \Rightarrow P(\frac{L}{2} + x) = V_B \cdot L$

$\Rightarrow V_B = P(\frac{1}{2} + \frac{x}{L})$

$\sum V = 0 \Rightarrow V_A + V_B = P$

Question 2a continued:

$$\Rightarrow V_A = P - V_B$$

$$= P - P\left(\tfrac{1}{2}\right) - \frac{Px}{L}$$

$$= P\left(\tfrac{1}{2} - \tfrac{x}{L}\right)$$

$$\overset{\curvearrowleft}{\sum} M_C = 0 \qquad Px + H_B h = V_B \cdot \left(\tfrac{L}{2}\right)$$

$$\Rightarrow H_B = \frac{P}{2h}\left(\tfrac{L}{2} - x\right)$$

Summary:

$$V_A(x) = P\left(\tfrac{1}{2} - \tfrac{x}{L}\right)$$

$$V_B(x) = P\left(\tfrac{1}{2} + \tfrac{x}{L}\right)$$

$$H_A(x) = H_B(x) = \frac{P}{2h}\left(\tfrac{L}{2} - x\right)$$

Now suppose that the point load P is replaced by a uniform load w_o (N/m) along the right-hand side of the arch, i.e.,

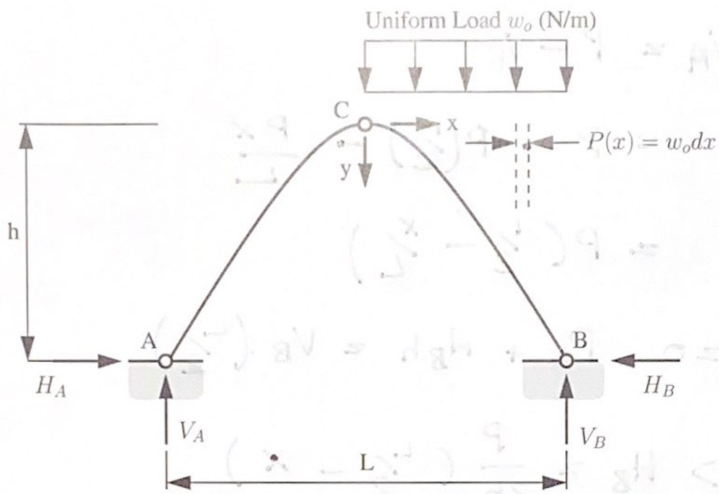


Figure 3: Three-pinned parabolic arch carrying a uniform load.

[2b] (5 pts) By using equations 2 through 4 as a starting point, and noting that a tiny increment of uniform loading can be written P(x) = w_0 dx, show that:

$$V_A = \left[\frac{w_o L}{8}\right], \quad V_B = \left[\frac{3 w_o L}{8}\right] \quad \text{and} \quad H_A = H_B = \left[\frac{w_o L^2}{16 h}\right]. \tag{5}$$

Now let $P(x) = w(x)\,dx = w_0\,dx$.

The total reaction forces will simply be the sum of the tiny increments, integrated over $0 \leq x \leq L/2$.

$$V_B = \int_0^{L/2} w_0 \left[\frac{1}{2} + \frac{x}{L} \right] dx$$

$$= \left[w_0 \left(\frac{x}{2} + \frac{x^2}{2L} \right) \right]_0^{L/2}$$

$$= \frac{3}{8} w_0 L.$$

Question 2b continued:

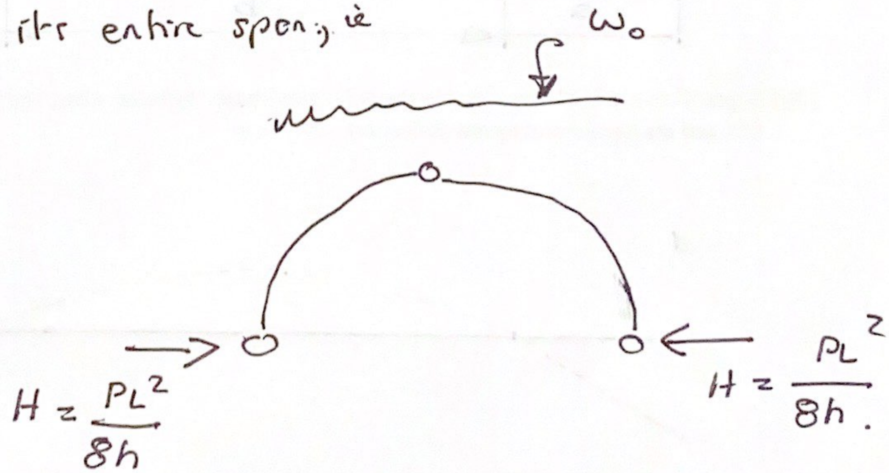
$$V_A = \int_0^{L/2} w_o\left[\frac{1}{2} - \frac{x}{L}\right]dx = \left(\frac{w_o L}{8}\right)$$

$$H_A = \int_0^{L/2} \frac{P}{2h}\left(\frac{L}{2} - x\right)dx$$

$$= \left[\frac{P}{2h}\left(\frac{Lx}{2} - \frac{x^2}{2}\right)\right]_0^{L/2}$$

$$= \frac{PL^2}{16h}.$$

Note: This is exactly one half of the horizontal reaction force caused by a parabolic arch carrying a uniform load across its entire span, i.e.



Question 3: 15 points

Analysis of a Cantilevered Beam with Moment Area. Figure 4 is a front elevation view of a cantilevered beam carrying two external loads P. EI is constant along the beam structure A-B-C-D.

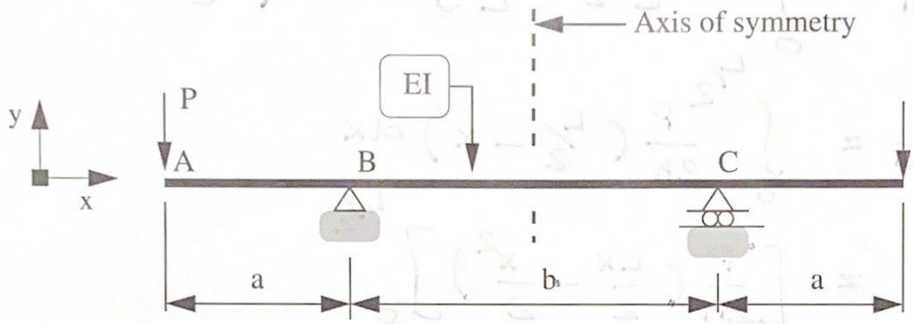
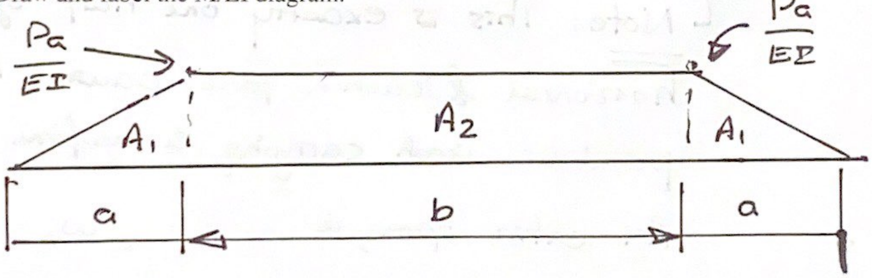


Figure 4: Cantilevered beam carrying two applied loads P.

Notice that the beam geometry and load pattern are symmetric – hint, hint, hint!! – about the beam midpoint.

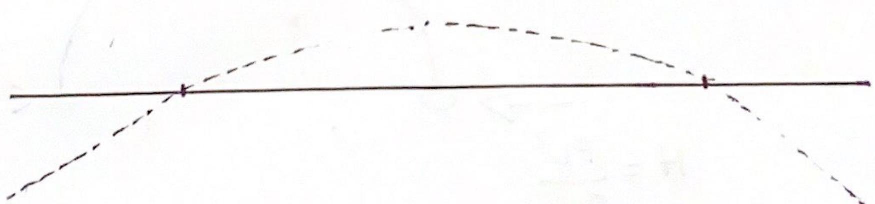
[3a] (2 pts) Draw and label the M/EI diagram.



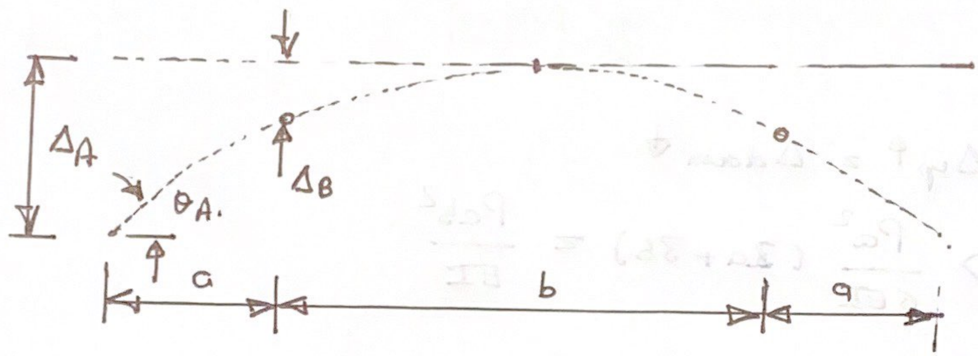
$$A_1 = \frac{Pa^2}{2EI}$$

$$A_2 = \frac{Pab}{EI}$$

[3b] (2 pts) Create a qualitative sketch the deflection shape. Indicate where the maximum upwards deflection and maximum downward deflections will occur.



[3c] (2 pts) Draw and label the **moment area diagram** (i.e., with rotations, tangents, displacements, etc) for this problem.



[3d] (2 pts) Use the **method of moment area** to show that the maximum upward deflection is:

$$y_{\text{maximum upward}} = \frac{Pab^2}{8EI}. \tag{6}$$

$$\Delta_B^{\uparrow} = \frac{b}{4} \cdot \frac{A_2}{2} = \frac{Pab^2}{8EI}. \quad \text{—} \quad A$$

[3e] (3 pts) Use the **method of moment area** to show that the maximum downward deflection is:

$$y_{\text{maximum downward}} = \frac{Pa^2}{6EI}\left[2a + 3b\right]. \tag{7}$$

$$\Delta_A = A_1\left(\tfrac{2}{3}a\right) + \frac{A_2}{2}\left(a + b/4\right)$$

$$= \frac{Pa^3}{3EI} + \frac{Pab}{8EI}(4a+b). \quad \text{—} \quad B$$

Max downwards deflection = B − A

$$\Delta_{\downarrow} = \frac{Pa^3}{3EI} + \frac{Pab}{8EI}(4a+b) - \frac{Pab^2}{8EI} = \frac{Pa^2}{6EI}(2a+3b).$$

[3f] (4 pts) Show that the maximum upward and maximum downward deflections will have equal magnitude when

$$\left[\frac{a}{b}\right] = \left[\frac{\sqrt{15}-3}{4}\right]. \tag{8}$$

If $\Delta_{up}\uparrow = \Delta_{down}\downarrow$

$$\Rightarrow \frac{Pa^2}{6EI}(3a+3b) = \frac{Pab^2}{EI}$$

$$\Rightarrow 8a^2 + 12ab - 3b^2 = 0$$

$$\Rightarrow a = \frac{-12b + \sqrt{144b^2 + 4\times 8\times 3b^2}}{16}$$

$$\Rightarrow a = \left[\frac{-3+\sqrt{15}}{4}\right]b.$$