Homework 5

Due: December 10, 2025 (No Extensions)

This homework covers integration using Trapezoid and Simpson's Rule, Gauss Quadrature and Romberg Integration.

Question 1: 20 points

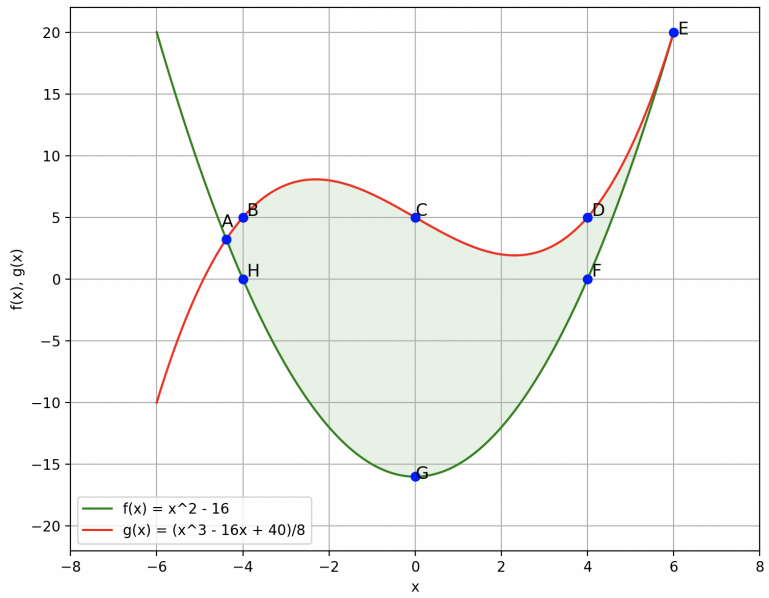

Figure 1: Filled region A-B-C-D-E-F-G-H.

Figure 1 shows an enclosed region A-B-C-D-E-F-G-H whose boundary is defined by two curves:

$$f(x) = \left[x^2 - 16\right]. \tag{1}$$

and

$$g(x) = \left[\frac{x^3}{8} - 2x + 5\right]. \tag{2}$$

From the graphic we see that curves f(x) and g(x) intersect at points A and E (i.e., in the interval $[-4, -5]$ and again $[5, 7]$).

1. Show that coordinate points A and E are defined by solutions to the cubic equation:

$$x^3 - 8x^2 - 16x + 168 = 0. \tag{3}$$

   This is a hand calculation, so show all of your working.

2. Using calculus, or otherwise, show that the area of the shaded region is $\left[781 + 145\sqrt{29}\right]/12 \approx 130.154$.

3. Demonstrate how you can use one step of Simpson's Rule to obtain a high-accuracy estimate of the area of region A-B-C-D-E-F-G-H.

4. What is the expected error with Simpson's Rule? Briefly compare and discuss the analytical and numerical results.

Question 2: 10 points.

Consider the integral

$$I = \int_0^4 3x^2 + 4x^3 + 5x^4 dx = 1,344. \tag{4}$$

Write a Python program to compute numerical approximations to equation 4 using: (1) The Trapezoid rule, (2) Simpson's rule, and (3) Two-point Gauss Quadrature. For cases 1 and 2, use only three data ordinates. Compute and print the absolute and relative errors for each numerical procedure.

Question 3: 10 points.

Write a Python program that uses Romberg Integration to show:

$$I = \int_0^2 \left[\frac{4}{1+x^2}\right] dx = 4\tan^{-1}(2) = 4.4286. \tag{5}$$

Start off by evaluating the function at 0, $\frac{1}{2}$, 1, $\frac{3}{2}$, and 2. Compute and print the absolute and relative errors.

Question 4: 10 points

Theoretical considerations indicate that:

$$\int_0^4 x^3 \left[16 - x^2\right] dx = \frac{1024}{3} \approx 341.33. \tag{6}$$

1. Use the method of Romberg integration to obtain an $O(h^6)$ accurate estimate of equation 6. Be sure to show all steps in your working.
2. Evaluate equation 6 using 3-pt Gauss Quadrature. Be sure to show all steps in your working.

Question 5: 10 points

Consider the integral:

$$I = \int_0^{2\pi} \mid \sin(x) - \cos(x) \mid dx. \tag{7}$$

and the plot shown in Figure 2:

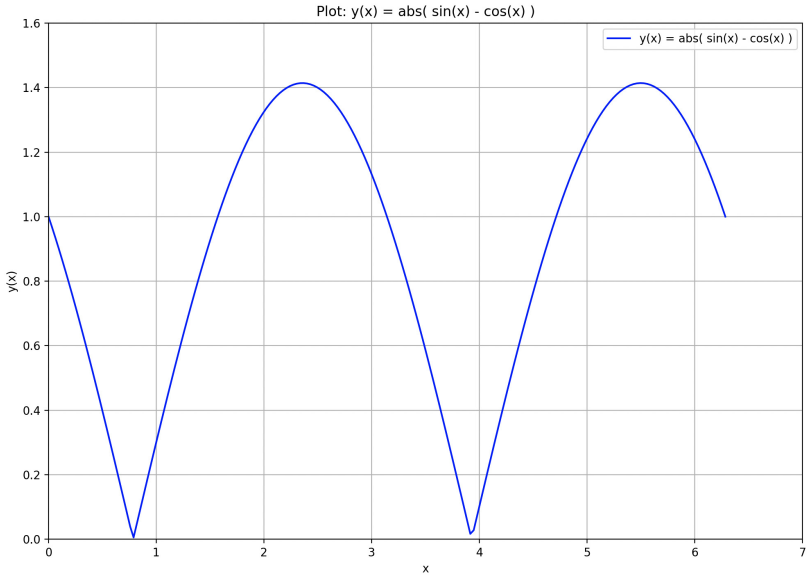

Figure 2: Graph of y(x) $\mid \sin(x) - \cos(x) \mid$ from 0 to 2π.

Notice that y(x) = $\mid \sin(x) - \cos(x) \mid$ is periodic. The values of y(x) can be summarized as follows:

x	0.0	pi/8	pi/4	3pi/8	pi/2	5pi/8
y(x)	1.0000	0.5412	0.0000	0.5412	1.0000	1.3066

x	3pi/4	7pi/8	pi	9pi/8	5pi/4	11pi/8
y(x)	1.4142	1.3066	1.0000	0.5412	0.0000	0.5412

x	6pi/4	13pi/8	7pi/4	15pi/8	2pi
y(x)	1.0000	1.3066	1.4142	1.3066	1.0000

[5a] (2 pts). Briefly explain how you can use the periodic nature of y(x) to simplify the evaluation of equation 7.

[5b] (3 pts). Compute the analytical solution to equation 7 This is a hand calculation, so show all of your working.

[5c] (5 pts). Now suppose that equation 7 is evaluated using only four steps of the Trapezoid Rule. With your answer to part [4a] in mind, what is the maximum error that will occur with this numerical approximation? Is the actual error within this bound?

Question 6: 10 points

Consider the family of integration problems:

$$I_n = \int_0^{\pi/2} \left[\frac{\sin^n(x)}{\sin^n(x) + \cos^n(x)} \right] dx \tag{8}$$

where n is an integer. When $n = 2$, for example, we have:

$$I_2 = \int_0^{\pi/2} \left[\frac{\sin^2(x)}{\sin^2(x) + \cos^2(x)} \right] dx = \int_0^{\pi/2} \sin^2(x) dx = \frac{\pi}{4}. \tag{9}$$

You can check this result via integration by parts (calculus), or Wolfram Alpha, or ChatGPT 3.5/4.

That was a little too easy – how about we set n = 5. The values for $f(x) = \sin^5(x)/(\sin^5(x) + \cos^5(x))$ can be summarized as follows:

x	0.0	pi/8	pi/4	3pi/8	pi/2
f(x)	0.0	0.012046	0.5	0.987954	1.0

[6a] (3 pts). Use the method of Romberg integration to obtain an $O(h^4)$ accurate estimate of equation 8.

[6b] (4 pts). Evaluate equation 8 using 2-pt Gauss Quadrature. Be sure to show all steps in your working.

Sometimes in calculus a simple change of variables (or adjustment in your point of view) can transform a problem from one that seems impossible into something quite simple. With that in mind:

[6c] (3 pts). Show that:

$$\int_0^a f(x)dx = \int_0^a f(a-x)dx. \tag{10}$$

Hence, or otherwise, prove:

$$I_n = \int_0^{\pi/2} \left[\frac{\sin^n(x)}{\sin^n(x) + \cos^n(x)}\right] dx = \frac{\pi}{4}. \tag{11}$$

for **any integer** n. Show all of your working.