Solutions to Homework 5

Question 1: 20 points.

Problem Statement. Figure 1 shows an enclosed region A-B-C-D-E-F-G-H

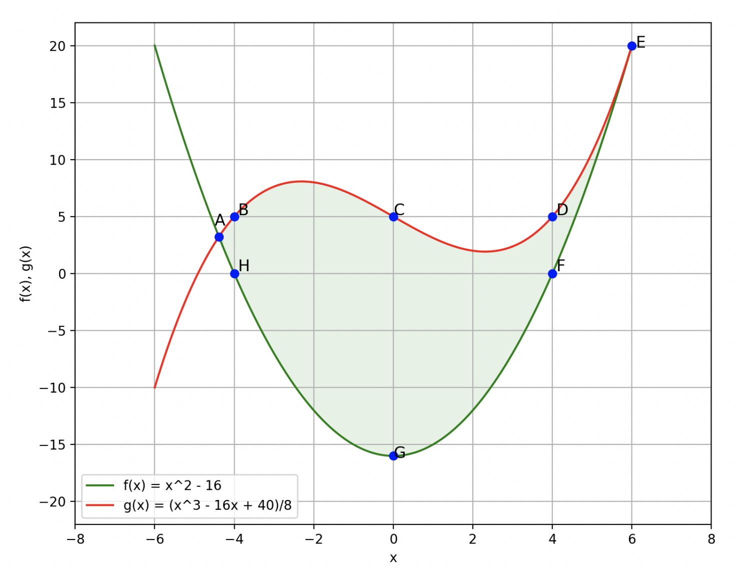

Figure 1: Filled region A-B-C-D-E-F-G-H.

whose boundary is defined by two curves:

$$f(x) = \left[x^2 - 16\right]. \tag{1}$$

and

$$g(x) = \left[\frac{x^3}{8} - 2x + 5\right]. \tag{2}$$

From the graphic we see that curves f(x) and g(x) intersect at points A and E (i.e., in the interval $[-4, -5]$ and again $[5, 7]$).

Part [1a] (6 pts). Show that coordinate points A and E are defined by solutions to the cubic equation:

$$x^3 - 8x^2 - 16x + 168 = 0. \tag{3}$$

This is a hand calculation, so show all of your working.

Solution. At the point of intersection, $f(x) = g(x)$, i.e.,

$$\left[x^2 - 16\right] = \left[\frac{x^3}{8} - 2x + 5\right] \tag{4}$$

Rearranging equation 4 gives the required result. Equation 3 can be factorized:

$$(x-6)\left(x^2 - 2x - 28\right) = 0. \tag{5}$$

Thus, Point A has coordinates: (x,y) = (1 - $\sqrt{29}$, 14-2$\sqrt{29}$). Point E has coordinates: (x,y) = (6,20).

Part [1b] (4 pts). Using calculus, or otherwise, show that the area of the shaded region is $\left[781 + 145\sqrt{29}\right]/12$ $\approx$ 130.154.

Solution. Notice that within the interval [1 - $\sqrt{29}$, 6], g(x) > f(x). If we let s(x) = g(x) - f(x), then the area of the shaded region is:

$$\text{Shaded Area} = \int_{1-\sqrt{29}}^{6} s(x)dx$$

$$= \left[\frac{x^4}{32} - \frac{x^3}{3} - x^2 + 21x\right]_{1-\sqrt{29}}^{6}$$

$$= \frac{1}{12}\left[781 + 145\sqrt{29}\right] \approx 130.154.$$

Part [1c] (6 pts). Demonstrate how you can use **one step** of Simpson's Rule, to obtain a high-accuracy estimate the area of region A-B-C-D-E-F-G-H.

Solution. Using one step of Simpson's Rule:

$$I = \int_a^b s(x)dx = \frac{h}{3}\left[s(a) + 4s(m) + s(b)\right] \tag{6}$$

where point a = (1 - $\sqrt{29}$), point b = 6, h = (b-a)/2, and m is the midpoint m = (a+b)/2 = (7 - $\sqrt{29}$)/2. Also, notice that by design, that s(a) = s(b) = 0.0. Hence, equation 6 simplfies to:

$$I = \int_a^b s(x)dx = \frac{4h}{3}s(m) = \frac{1}{12}\left[781 + 145\sqrt{29}\right] \approx 130.154. \tag{7}$$

Part [1d] (4 pts). What is the expected error with Simpson's Rule? Briefly compare and discuss the analytical and numerical results.

Solution. The error estimate for Simpson's Rule is as follows:

$$I = \int_a^b s(x)dx = S_1 - \frac{|s^{''''}(\xi)|}{180}h^4(b-a). \tag{8}$$

Given that s(x) is a cubic function, the fourth derivative will be zero. Hence, we expect one step of Simpson's Rule to provide an exact answer.

Question 2: 10 points.

Problem Statement. Consider the integral

$$I = \int_0^4 3x^2 + 4x^3 + 5x^4 dx = 1,344. \tag{9}$$

Write a Python program to compute numerical approximations to equation 9 using: (1) The Trapezoid rule, (2) Simpson's rule, and (3) Two-point Gauss Quadrature. For cases 1 and 2, use only three data ordinates. Compute and print the absolute and relative errors for each numerical procedure.

Source Code:

```
# ================================================================================
# TestIntegration03.py: Integrate test function three ways: (1) with Trapezoid
# Rule, (2) with Simpson's Rule, and (3) using Gauss Quadrature.
#
# Written By: Mark Austin                                            December 2025
# ================================================================================

import math;
import Integration;

# Define mathematical functions ...

def f1(x):
    return 3*x**2 + 4*x**3 + 5*x**4

# main method ...

def main():
    print("--- Enter TestIntegration03.main()    ... ");
    print("--- ======================================== ... ");

    analytic_i = 1344;

    print("--- ");
    print("--- Part 1: Integrate f1(x) with Trapezoid ... ");
    print("--- ");

    # Initialize problem setup ...

    a =  0.0
    b =  4.0
    nointervals = 2

    print("--- Inputs:")
    print("---   a = {:9.4f} ...".format(a) )
    print("---   b = {:9.4f} ...".format(b) )
    print("---   no intervals = {:d} ...".format(nointervals) )
```

```
# Compute numerical solution to integral ..

print("--- Execution:")
xi = Integration.trapezoid( f1, a, b, nointervals )

# Summary of computations ...

print("--- Output:")
print("---   Numerical Integral = {:14.4e} ...".format( xi ) )

# Error Analysis ...

print("--- Error Analysis:")
print("---   Absolute error = {:14.4e} ...".format( abs( xi-analytic_i )) )
print("---   Relative error = {:14.4e} ...".format( abs( xi-analytic_i )/analytic_i ) )

print("--- ");
print("--- Part 2: Integrate f1(x) with Simpson ... ");
print("--- ");

# Initialize problem setup ...

a = 0.0;
b = 4.0
nointervals = 2

print("--- Inputs:")
print("---   a = {:9.4f} ...".format(a) )
print("---   b = {:9.4f} ...".format(b) )
print("---   no intervals = {:d} ...".format(nointervals) )

# Compute numerical solution to integral ..

print("--- Execution:")
xi = Integration.simpson( f1, a, b, nointervals )

# Summary of computations ...

print("--- Output:")
print("---   Numerical Integral = {:14.4e} ...".format( xi ) )

# Error Analysis ...

print("--- Error Analysis:")
print("---   Absolute error = {:14.4e} ...".format( abs( xi-analytic_i )) )
print("---   Relative error = {:14.4e} ...".format( abs( xi-analytic_i )/analytic_i ) )

print("--- ");
print("--- Part 3: Integrate f1(x) with Two-Point Gauss Quadrature ... ");
print("--- ");

a = 0.0;
b = 4.0
```

```
    w0 = 1.0
    u0 = - 1.0/math.sqrt(3.0)
    x0 = ((b-a)/2.0)*(1 + u0)
    w1 = 1.0
    u1 =   1.0/math.sqrt(3.0)
    x1 = ((b-a)/2.0)*(1 + u1)

    xi = ((b-a)/2.0)*( w0*f1(x0) + w1*f1(x1) )

    print("---   w0 = {:14.6e}, u0 = {:14.6e}, x0 = {:14.6e} ...".format( w0, u0, x0 ) )
    print("---   w1 = {:14.6e}, u1 = {:14.6e}, x1 = {:14.6e} ...".format( w1, u1, x1 ) )
    print("---   Numerical Integral f1(x) dx = {:14.4f} ...".format( xi ) )

    # Error Analysis ...

    print("--- Error Analysis:")
    print("---   Absolute error = {:14.4e} ...".format( abs( xi-analytic_i )) )
    print("---   Relative error = {:14.4e} ...".format( abs( xi-analytic_i )/analytic_i ) )

    print("--- ========================================= ... ");
    print("--- Leave TestIntegration03.main()   ... ");

# call the main method ...

main()
```

Abbreviated Output:

```
--- Part 1: Integrate f1(x) with Trapezoid ...
--- ------------------------------------- ...
---
--- Inputs:
---   a =    0.0000 ...
---   b =    4.0000 ...
---   no intervals = 2 ...
--- Execution:
--- Output:
---   Numerical Integral =     1.8320e+03 ...
---
--- Error Analysis:
---   Absolute error =     4.8800e+02 ...
---   Relative error =     3.6310e-01 ...
---
--- Part 2: Integrate f1(x) with Simpson ...
--- ----------------------------------- ...
---
--- Inputs:
---   a =    0.0000 ...
---   b =    4.0000 ...
---   no intervals = 2 ...
--- Execution:
--- Output:
---   Numerical Integral =     1.3867e+03 ...
```

```
---
--- Error Analysis:
---   Absolute error =     4.2667e+01 ...
---   Relative error =     3.1746e-02 ...
---
--- Part 3: Integrate f1(x) with Two-Point Gauss Quadrature ...
--- ------------------------------------------------------- ...
---
---   w0 =   1.000000e+00, u0 =  -5.773503e-01, x0 =   8.452995e-01 ...
---   w1 =   1.000000e+00, u1 =   5.773503e-01, x1 =   3.154701e+00 ...
---   Numerical Integral f1(x) dx =      1315.5556 ...
---
--- Error Analysis:
---   Absolute error =     2.8444e+01 ...
---   Relative error =     2.1164e-02 ...
```

Question 3: 10 points.

Problem Statement. Write a Python program that uses Romberg Integration to show:

$$I = \int_0^2 \left[\frac{4}{1+x^2} \right] dx = 4\tan^{-1}(2) = 4.4286. \tag{10}$$

Start off by evaluating the function at 0, $\frac{1}{2}$, 1, $\frac{3}{2}$, and 2. Compute and print the absolute and relative errors.

Python Source Code: ...

```
# ==============================================================================
# TestIntegration04.py: Use rhomberg algorithm to integrate functions.
#
# Written By: Mark Austin                                          December 2025
# ==============================================================================

import math;
import Integration;

# Define mathematical functions ...

def f1(x):
    numerator   = 4
    denominator = 1 + x**2
    return numerator/denominator

# main method ...

def main():
    print("--- Enter TestIntegration04.main()    ... ");
    print("--- ======================================== ... ");

    # Analytic solution ...

    analytic_i = 4*math.atan(2.0)
    print("--- Analytic solution = {:16.10f} ...".format(analytic_i) )

    print("--- ");
    print("--- Part 1: Integrate f1(x) with Romberg Integration ... ");
    print("--- ");

    # Initialize problem setup ...

    a = 0.0;
    b = 2.0
    nointervals = 4

    print("--- Inputs:")
    print("---   a = {:9.4f} ...".format(a) )
    print("---   b = {:9.4f} ...".format(b) )
```

```
    print("---   no intervals = {:d} ...".format(nointervals) )

    # Compute numerical solution to integral ..

    print("--- Execution:")
    xi = Integration.romberg( f1, a, b, nointervals )

    # Summary of computations ...

    print("--- Output:")
    print("---   Numerical Integral = {:16.10f} ...".format( xi ) )

    # Error Analysis ...

    print("--- Error Analysis:")
    print("---   Absolute error = {:14.4e} ...".format( abs( xi-analytic_i )) )
    print("---   Relative error = {:14.4e} ...".format( abs( xi-analytic_i )/analytic_i ) )

    print("--- ========================================= ... ");
    print("--- Leave TestIntegration04.main()   ... ");

# call the main method ...

main()
```

Abbreviated Output:

```
--- Enter TestIntegration04.main()   ...
--- ========================================= ...
--- Analytic solution =     4.4285948712 ...
---
--- Part 1: Integrate f1(x) with Romberg Integration ...
---
--- Inputs:
---   a =    0.0000 ...
---   b =    2.0000 ...
---   no intervals = 4 ...
--- Execution:
---   Initialize Romberg Integration Table ...

Matrix: Romberg Integration Table (empty)
  0.00000000e+00   0.00000000e+00   0.00000000e+00   0.00000000e+00
  0.00000000e+00   0.00000000e+00   0.00000000e+00   0.00000000e+00
  0.00000000e+00   0.00000000e+00   0.00000000e+00   0.00000000e+00
  0.00000000e+00   0.00000000e+00   0.00000000e+00   0.00000000e+00

---   Compute trapezoid rule for first column ...
---   Iterate over levels of refinement ...
---     Extrapolation for column  2 ...
---     Extrapolation for column  3 ...
---     Extrapolation for column  4 ...

Matrix: Romberg Integration Table (instantiated)
```

```
  4.80000000e+00   0.00000000e+00   0.00000000e+00   0.00000000e+00
  4.40000000e+00   4.26666667e+00   0.00000000e+00   0.00000000e+00
  4.41538462e+00   4.42051282e+00   4.43076923e+00   0.00000000e+00
  4.42526653e+00   4.42856050e+00   4.42909701e+00   4.42907047e+00

--- Output:
---   Numerical Integral =     4.4290704660 ...

--- Error Analysis:
---   Absolute error =     4.7559e-04 ...
---   Relative error =     1.0739e-04 ...

--- ========================================= ...
--- Leave TestIntegration04.main()   ...
```

Question 4: 10 points.

Problem Statement. Theoretical considerations indicate that:

$$\int_0^4 x^3 \left[16 - x^2 \right] dx = \frac{1024}{3} \approx 341.33. \tag{11}$$

Part [4a]. Use the method of Romberg integration to obtain an $O(h^6)$ accurate estimate of equation 11. Be sure to show all steps in your working.

Solution: To get an $O(h^6)$ accurate estimate we need three levels of refinement with Trapezoid (i.e., h = 4, h = 2 and h = 1).

```
--- Inputs:
---   a =    0.0000 ...
---   b =    4.0000 ...
---   no intervals = 3 ...
---
--- Execution:
---   Compute trapezoid rule for first column ...
---   Extrapolation for column  2 ...
---   Extrapolation for column  3 ...

Matrix: Romberg Integration Table (instantiated)
  0.00000000e+00   0.00000000e+00   0.00000000e+00
  1.92000000e+02   2.56000000e+02   0.00000000e+00
  3.00000000e+02   3.36000000e+02   3.41333333e+02

---  Integral I = 341.3333333333 ...  <--- Numerical integration is exact ...
```

Part [4b] (5 pts). Evaluate equation 11 using 3-pt Gauss Quadrature. Be sure to show all steps in your working.

Solution: Integrating f1(x) with 3-point quadrature ...

```
---   w0 =   5.555556e-01, u0 =  -7.745967e-01, x0 =   4.508067e-01 ...
---   w1 =   8.888889e-01, u1 =   0.000000e+00, x1 =   2.000000e+00 ...
---   w2 =   5.555556e-01, u2 =   7.745967e-01, x2 =   3.549193e+00 ...
---
---   f1(x0) =   1.447236e+00 ...
---   f1(x1) =   9.600000e+01 ...
---   f1(x2) =   1.521528e+02 ...
---
---   Integral I = 3.41333333e+02 ... <--- Numerical integration is exact ...
```

Question 5: 10 points.

Problem Statement. Consider the integral:

$$I = \int_0^{2\pi} \mid \sin(x) - \cos(x) \mid dx. \tag{12}$$

and the plot shown in Figure 2:

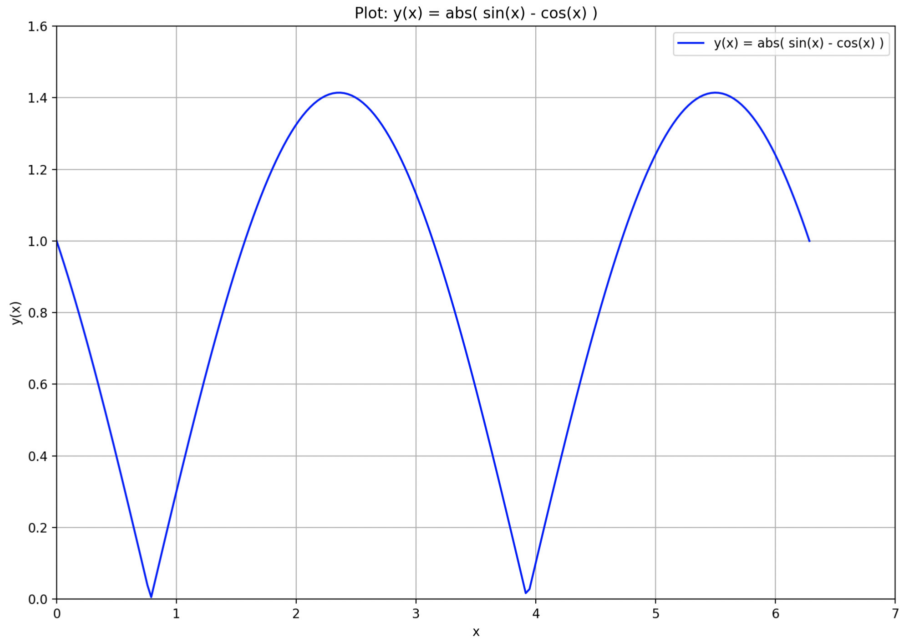

Figure 2: Graph of y(x) $\mid \sin(x) - \cos(x) \mid$ from 0 to 2π.

Notice that y(x) = $\mid \sin(x) - \cos(x) \mid$ is periodic. The values of y(x) can be summarized as follows:

x	0.0	pi/8	pi/4	3pi/8	pi/2	5pi/8
y(x)	1.0000	0.5412	0.0000	0.5412	1.0000	1.3066

x	3pi/4	7pi/8	pi	9pi/8	5pi/4	11pi/8
y(x)	1.4142	1.3066	1.0000	0.5412	0.0000	0.5412

x	6pi/4	13pi/8	7pi/4	15pi/8	2pi
y(x)	1.0000	1.3066	1.4142	1.3066	1.0000

Part [5a] (2 pts). Briefly explain how you can use the periodic nature of y(x) to simplify the evaluation of equation 12.

Solution. First note that: $|\sin(x) - \cos(x)|$ = 0 at x = $\pi/4$ and x = $5\pi/4$. The periodic nature implies:

$$\int_0^{2\pi} |sin(x) - cos(x)|dx = 2\int_{\pi/4}^{5\pi/4} |sin(x) - cos(x)|dx. \tag{13}$$

Part [5b] (3 pts). Compute the analytical solution to equation 12 This is a hand calculation, so show all of your working.

Solution. First, let's ignore the periodic nature:

$$\begin{aligned} I &= \int_0^{\pi/4} \cos(x) - \sin(x)dx + \int_{\pi/4}^{5\pi/4} \sin(x) - \cos(x)dx + \int_{5\pi/4}^{2\pi} \cos(x) - o\sin(x)dx \\ &= [\sin(x) + \cos(x)]_0^{\pi/4} + [-\cos(x) - \sin(x)]_{\pi/4}^{5\pi/4} + [\sin(x) + \cos(x)]_{5\pi/4}^{2\pi} \\ &= (\frac{2}{\sqrt{2}} - 1) + (\frac{2}{\sqrt{2}} + \frac{2}{\sqrt{2}}) + (1 + \frac{2}{\sqrt{2}}) = \frac{8}{\sqrt{2}} = 4\sqrt{2}. \end{aligned} \tag{14}$$

Accounting for the periodic nature:

$$I = 2\int_{\pi/4}^{5\pi/4} \sin(x) - \cos(x)dx = 4\sqrt{2}. \tag{15}$$

Part [5c] (5 pts). Now suppose that equation 12 is evaluated using only four steps of the Trapezoid Rule. With your answer to part [4a] in mind, what is the maximum error that will occur with this numerical approximation? Is the actual error within this bound?

Solution. By taking advantage of the periodic nature of the integral, we have:

$$I = 2\int_{\pi/4}^{5\pi/4} \sin(x) - \cos(x)dx. \tag{16}$$

Using 4 intervals of Trapezoid:

$$
\begin{aligned}
T_4 &= \frac{\pi/4}{2}\left[f(\frac{\pi}{4}) + 2(f(\frac{\pi}{2}) + f(\frac{3\pi}{4}) + f(\pi)) + f(\frac{5\pi}{4})\right] \\
&= \frac{\pi}{8}\left[0 + 2(1 + 1.4142 + 1) + 0\right] \\
&= \frac{3.4142}{4}\pi
\end{aligned} \tag{17}
$$

Hence,

$$
I_{\text{estimate}} = 2\left[\frac{3.4142\pi}{4}\right] = 5.3630. \tag{18}
$$

The analytic solution is $4\sqrt{2}$ = 5.5668, indicating a numerical error 0.2938.

The error estimate for trapezoid is:

$$
\text{error estimate} \leq \frac{|f(\epsilon)|_{max}}{12} h^2 (b-a). \tag{19}
$$

Can show that $f^{''}(x)$ = $-f(x)$, with a maximumn value 1.42 (taken from the figure). But strictly speaking $|f(x)|$ is not continuously differentiable, so the error formula does not work!

Question 6: 10 points.

Problem Statement. Consider the family of integration problems:

$$I_n = \int_0^{\pi/2} \left[\frac{\sin^n(x)}{\sin^n(x) + \cos^n(x)} \right] dx \tag{20}$$

where n is an integer. When $n = 2$, for example, we have:

$$I_2 = \int_0^{\pi/2} \left[\frac{\sin^2(x)}{\sin^2(x) + \cos^2(x)} \right] dx = \int_0^{\pi/2} \sin^2(x) dx = \frac{\pi}{4}. \tag{21}$$

You can check this result via integration by parts (calculus), or Wolfram Alpha, or ChatGPT 3.5/4.

That was a little too easy – how about we set n = 5. The values for $f(x) = \sin^5(x)/(\sin^5(x) + \cos^5(x))$ can be summarized as follows:

x	0.0	pi/8	pi/4	3pi/8	pi/2
f(x)	0.0	0.012046	0.5	0.987954	1.0

Part [6a]. (3 pts). Use the method of Romberg integration to obtain an $O(h^4)$ accurate estimate of equation 20.

Solution. Apply Trapezoid rule:

Trapezoid: h = $\frac{\pi}{2}$

$$T_1 = \frac{h}{2} \left[f(0) + f(\pi/2) \right] = \frac{\pi}{4} \left[0 + 1 \right] = \frac{\pi}{4}. \tag{22}$$

Trapezoid: h = $\frac{\pi}{4}$

$$T_2 = \frac{h}{2} \left[f(0) + 2f(\pi/4) + f(\pi/2) \right] = \frac{\pi}{8} \left[0 + 2 * .50 + 1 \right] = \frac{\pi}{4}. \tag{23}$$

Romberg Integration:

$$R_{21} = \left[frac4T_2 - T_1 4 - 1 \right] = \frac{3\pi/4}{4-1} = \frac{\pi}{4}. \tag{24}$$

Part [6b]. (4 pts). Evaluate equation 20 using 2-pt Gauss Quadrature. Be sure to show all steps in your working.

Solution. Let n = 5, i.e.,

$$I = \int_0^{\pi/2} \left[\frac{\sin^5(x)}{\sin^5(x) + \cos^5(x)} \right] dx \tag{25}$$

Map domain $\left[0, \frac{\pi}{2}\right] \longrightarrow [-1, 1]$. Next, let:

$$x = \frac{\pi}{4} [1 + u] \quad \longrightarrow \quad dx = \frac{\pi}{4} du. \tag{26}$$

Hence,

$$I = \frac{\pi}{4} \int_{-1}^{1} \left[\frac{\sin^5(\frac{\pi}{4}(1+u))}{\sin^5(\frac{\pi}{4}(1+u)) + \cos^5(\frac{\pi}{4}(1+u))} \right] du \tag{27}$$

Two point quadrature: w_o = w_1 = 1. u_o = $\frac{-1}{\sqrt{3}}$, U_1 = $\frac{1}{\sqrt{3}}$. Hence,

$$I = \frac{\pi}{4} \left[\left[\frac{\sin^5(\frac{\pi}{4}(1 - \frac{1}{\sqrt{3}}))}{\sin^5(\frac{\pi}{4}(1 - \frac{1}{\sqrt{3}})) + \cos^5(\frac{\pi}{4}(1 - \frac{1}{\sqrt{3}}))} \right] + \left[\frac{\sin^5(\frac{\pi}{4}(1 + \frac{1}{\sqrt{3}}))}{\sin^5(\frac{\pi}{4}(1 + \frac{1}{\sqrt{3}})) + \cos^5(\frac{\pi}{4}(1 + \frac{1}{\sqrt{3}}))} \right] \right] \tag{28}$$
$$= 0.785398.$$

Note: Analytic sol'n = $\frac{\pi}{4}$ = 0.785398.

Part [6c]. (3 pt) Sometimes in calculus a simple change of variables (or adjustment in your point of view) can transform a problem from one that seems impossible into something quite simple. With that in mind, show that:

$$\int_0^a f(x) dx = \int_0^a f(a - x) dx. \tag{29}$$

Hence, or otherwise, prove:

$$I_n = \int_0^{\pi/2} \left[\frac{\sin^n(x)}{\sin^n(x) + \cos^n(x)} \right] dx = \frac{\pi}{4}. \tag{30}$$

for **any integer** n. Show all of your working.

Solution. From equation 29:

$$I = \int_0^a f(a-x)dx \tag{31}$$

Let u = a-x. Then du/dx = -1. Also, when x = 0, u = a and when x = a, u=0. Hence,

$$I = \int_a^0 -f(u)du = \int_0^a f(u)du = \int_0^a f(x)dx = I. \tag{32}$$

Now consider the hint (equation 29):

$$I_n = \int_0^{\pi/2} \left[\frac{\sin^n(x)}{\sin^n(x) + \cos^n(x)}\right] dx = \int_0^{\pi/2} \left[\frac{\sin^n(\frac{\pi}{2} - x)}{\sin^n(\frac{\pi}{2} - x) + \cos^n(\frac{\pi}{2} - x)}\right] dx. \tag{33}$$

But $\sin(\frac{\pi}{2} - x)$ = $\cos(x)$ and $\cos(\frac{\pi}{2} - x)$ = $\sin(x)$. Hence,

$$I_n = \int_0^{\pi/2} \left[\frac{\cos^n(x)}{\sin^n(x) + \cos^n(x)}\right] dx. \tag{34}$$

Adding equations 30 and 34:

$$2I_n = \int_0^{\pi/2} \left[\frac{\sin^n(x) + \cos^n(x)}{\sin^n(x) + \cos^n(x)}\right] dx = \int_0^{\pi/2} 1 dx = \frac{\pi}{2}. \tag{35}$$

Hence I_n = $\frac{\pi}{4}$.