Homework 4

Due: November 21

This homework covers problems requiring interpolation and least squares.

Question 1: 20 points.

Figure 1 is a three-dimensional view of a 2 by 2 km site that is believed to overlay a thick layer of mineral deposits.

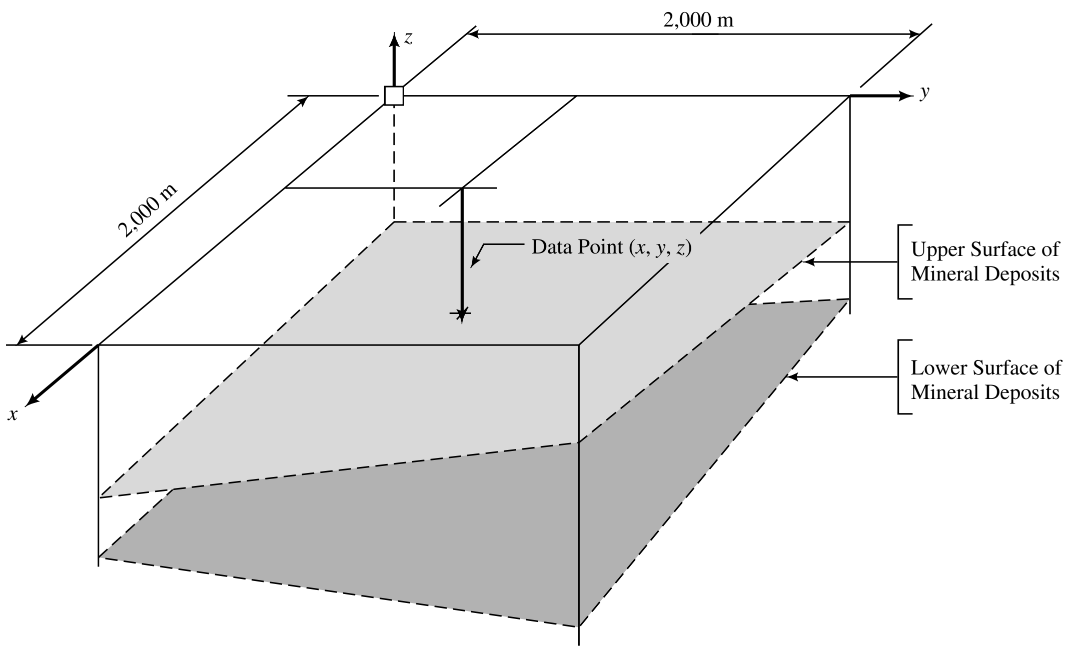

Figure 1: Three-dimensional view of mineral deposits.

To create a model of the mineral deposit profile and establish the economic viability of mining the site, a preliminary subsurface exploration consisting of 16 bore holes is conducted. Each bore hole is drilled to approximately 45 m, with the upper and lower boundaries of mineral deposits being recorded. The bore hole data is as follows:

```
Borehole    X (m)     Y(m) Upper Surface (m) Lower Surface (m)
--------------------------------------------------------------
```

```
      1,   30.0,   30.0,              -8.5,             -52.5
      2,  770.0,   30.0,              -7.0,             -51.8
      3, 1230.0,   30.0,              -6.0,             -51.3
      4, 1970.0,   30.0,              -4.6,             -50.5
      5,   30.0,  770.0,             -22.2,             -53.4
      6,  770.0,  770.0,             -10.8,             -52.6
      7, 1230.0,  770.0,              -9.8,             -52.1
      8, 1970.0,  770.0,              -8.3,             -51.4
      9,   30.0, 1230.0,             -14.7,             -54.0
     10,  770.0, 1230.0,             -13.2,             -53.2
     11, 1230.0, 1230.0,             -12.2,             -52.7
     12, 1970.0, 1230.0,             -10.8,             -52.0
     13,   30.0, 1970.0,             -18.4,             -54.8
     14,  770.0, 1970.0,             -17.0,             -54.1
     15, 1230.0, 1970.0,             -16.0,             -53.6
     16, 1970.0, 1970.0,             -14.5,             -53.9
------------------------------------------------------------------
```

With the bore hole data collected, the next step is to create a simplified three-dimensional computer model of the site and subsurface mineral deposits. The mineral deposits will be modeled as a single six-sided object. The four vertical sides are simply defined by the boundaries of the site. The upper and lower sides are to be defined by a three-dimensional plane

$$z(x,y) = a_o + a_1 \cdot x + a_2 \cdot y \tag{1}$$

where coefficients a_o, a_1, and a_2 correspond to minimum values of

$$S\left(a_o,\ a_1,\ a_2\right) = \sum_{i=1}^{N} \left[z_i - z(x_i, y_i)\right]^2 \tag{2}$$

Things to do:

1. Show that minimum value of $S(a_o, a_1, a_2)$ corresponds to the solution of the matrix equations

$$\begin{bmatrix} N & \sum_{i=1}^N x_i & \sum_{i=1}^N y_i \\ \sum_{i=1}^N x_i & \sum_{i=1}^N x_i^2 & \sum_{i=1}^N x_i \cdot y_i \\ \sum_{i=1}^N y_i & \sum_{i=1}^N x_i \cdot y_i & \sum_{i=1}^N y_i^2 \end{bmatrix} \cdot \begin{bmatrix} a_o \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^N z_i \\ \sum_{i=1}^N x_i \cdot z_i \\ \sum_{i=1}^N y_i \cdot z_i \end{bmatrix} \tag{3}$$

2. Create a comma-separated datafile (e.g., borehole-data.csv) for the geological surface data.
3. Write a Python program to read the borehole csv datafile, and then create a three-dimensional plot of the geological borehole data at the lower and upper surfaces.

4. Set up and solve the matrix equations derived in part 1 for the upper and lower mineral planes. Compute and print the average depth and volume of mineral deposits enclosed within the site.

Note. The least squares solution corresponds to the minimum value of function $S(a_o, a_1, a_2)$. At the minimum function value, we will have

$$\frac{\partial S}{\partial a_o} = \frac{\partial S}{\partial a_1} = \frac{\partial S}{\partial a_2} = 0 \tag{4}$$

Matrix Equation 3 is simply the three equations 4 written in matrix form. You should find that the equation of the upper surface is close to `z(x,y) = -10.5 + x/500 - y/200` and the lower surface close is to `z(x,y) = -52.5 + x/1000 - y/850`.

Question 2: 10 points

It is well known that first derivative of $f(x) = sin(x)$ is $cos(x)$. Given the double angle formulae,

$$sin(a + b) = sin(a)cos(b) + cos(a)sin(b) \tag{5}$$

and

$$sin(a - b) = sin(a)cos(b) - cos(a)sin(b) \tag{6}$$

write a Python program that will estimate forward and central finite difference approximations. For each approximation, plot the error in the derivative estimate versus h about the point x=0. Plots covering the interval h = -$\pi/4$ to h = $\pi/4$ might be reasonable. Repeat for x=$\pi/2$.

You should find that the central difference approximation is significantly more accurate than the forward difference approximation.

Question 3: 10 points

This question covers function interpolation with the methods of divided differences and lagrange interpolation, and curve fitting with the method of least squares. The whole question is motivated by the small dataset:

```
   x |  0.0 |   2.0 |   3.0 |   5.0
-----------------------------------
f(x) | 36.0 |  32.0 |  27.0 |  11.0
```

1. Use the method of divided differences to find a polynomial of lowest order that will fit the dataset. Be sure to show all of your working.

2. Check your answer to the previous part by computing the functional form via the method of Lagrange Interpolation. Be sure to show all of your working.