Homework 5

Due: May 9, 2025 (No Extensions)

This homework covers problems requiring interpolation, least squares, and integration using Trapezoid and Simpson's Rule, Gauss Quadrature and Romberg Integration.

Question 1: 20 points.

Figure 1 is a three-dimensional view of a 2 by 2 km site that is believed to overlay a thick layer of mineral deposits.

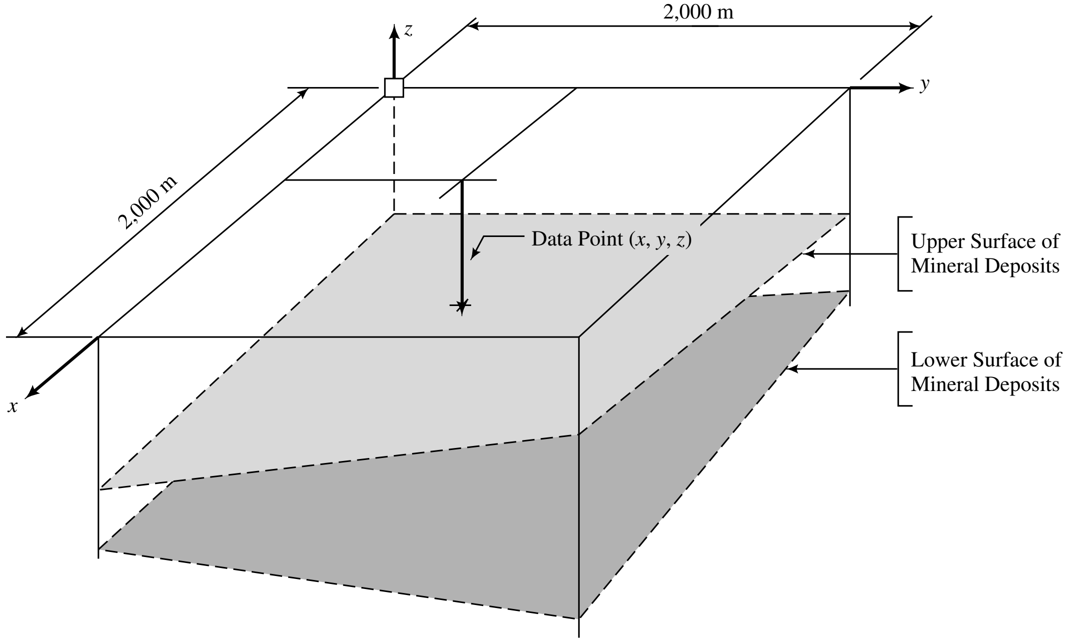

Figure 1: Three-dimensional view of mineral deposits.

To create a model of the mineral deposit profile and establish the economic viability of mining the site, a preliminary subsurface exploration consisting of 16 bore holes is conducted. Each bore hole is drilled to approximately 45 m, with the upper and lower boundaries of mineral deposits being recorded. The bore hole data is as follows:

```
Borehole    X (m)     Y(m) Upper Surface (m) Lower Surface (m)
```

```
--------------------------------------------------------------
      1,    30.0,    30.0,             -18.5,             -42.5
      2,   770.0,    30.0,             -17.0,             -41.8
      3,  1230.0,    30.0,             -16.0,             -41.3
      4,  1970.0,    30.0,             -14.6,             -40.5
      5,    30.0,   770.0,             -32.2,             -43.4
      6,   770.0,   770.0,             -20.8,             -42.6
      7,  1230.0,   770.0,             -19.8,             -42.1
      8,  1970.0,   770.0,             -18.3,             -41.4
      9,    30.0,  1230.0,             -24.7,             -44.0
     10,   770.0,  1230.0,             -23.2,             -43.2
     11,  1230.0,  1230.0,             -22.2,             -42.7
     12,  1970.0,  1230.0,             -20.8,             -42.0
     13,    30.0,  1970.0,             -28.4,             -44.8
     14,   770.0,  1970.0,             -27.0,             -44.1
     15,  1230.0,  1970.0,             -26.0,             -43.6
     16,  1970.0,  1970.0,             -24.5,             -43.9
--------------------------------------------------------------
```

With the bore hole data collected, the next step is to create a simplified three-dimensional computer model of the site and subsurface mineral deposits. The mineral deposits will be modeled as a single six-sided object. The four vertical sides are simply defined by the boundaries of the site. The upper and lower sides are to be defined by a three-dimensional plane

$$z(x,y) = a_o + a_1 \cdot x + a_2 \cdot y \tag{1}$$

where coefficients a_o, a_1, and a_2 correspond to minimum values of

$$S\left(a_o,\ a_1,\ a_2\right) = \sum_{i=1}^{N} \left[z_i - z(x_i, y_i)\right]^2 \tag{2}$$

Things to do:

1. Show that minimum value of $S(a_o, a_1, a_2)$ corresponds to the solution of the matrix equations

$$\begin{bmatrix} N & \sum_{i=1}^N x_i & \sum_{i=1}^N y_i \\ \sum_{i=1}^N x_i & \sum_{i=1}^N x_i^2 & \sum_{i=1}^N x_i \cdot y_i \\ \sum_{i=1}^N y_i & \sum_{i=1}^N x_i \cdot y_i & \sum_{i=1}^N y_i^2 \end{bmatrix} \cdot \begin{bmatrix} a_o \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^N z_i \\ \sum_{i=1}^N x_i \cdot z_i \\ \sum_{i=1}^N y_i \cdot z_i \end{bmatrix} \tag{3}$$

2. Create a comma-separated datafile (e.g., borehole-data.csv) for the geological surface data.
3. Write a Python program to read the borehole csv datafile, and then create a three-dimensional plot of the geological borehole data at the lower and upper surfaces.

4. Set up and solve the matrix equations derived in part 1 for the upper and lower mineral planes. Compute and print the average depth and volume of mineral deposits enclosed within the site.

Note. The least squares solution corresponds to the minimum value of function $S(a_o, a_1, a_2)$. At the minimum function value, we will have

$$\frac{\partial S}{\partial a_o} = \frac{\partial S}{\partial a_1} = \frac{\partial S}{\partial a_2} = 0 \tag{4}$$

Matrix Equation 3 is simply the three equations 4 written in matrix form. You should find that the equation of the upper surface is close to `z(x,y) = -20.5 + x/500 - y/200` and the lower surface close is to `z(x,y) = -42.5 + x/1000 - y/850`.

Question 2: 20 points

This question covers function interpolation with the methods of divided differences and lagrange interpolation, and curve fitting with the method of least squares. The whole question is motivated by the small dataset:

x	0.0	2.0	3.0	5.0
f(x)	36.0	32.0	27.0	11.0

1. Use the method of divided differences to find a polynomial of lowest order that will fit the dataset. Be sure to show all of your working.
2. Check your answer to the previous part by computing the functional form via the method of Lagrange Interpolatio. Be sure to show all of your working.

Question 3: 20 points

Figure 2 shows an enclosed region A-B-C-D-E-F-G-H whose boundary is defined by two curves:

$$f(x) = \left[x^2 - 16 \right]. \tag{5}$$

and

$$g(x) = \left[\frac{x^3}{8} - 2x + 5 \right]. \tag{6}$$

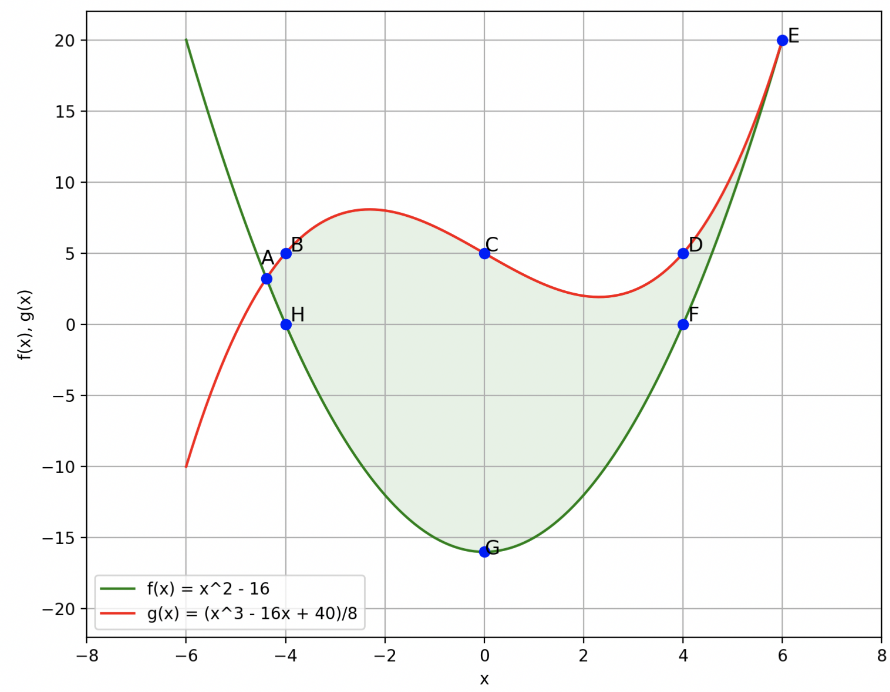

Figure 2: Filled region A-B-C-D-E-F-G-H.

From the graphic we see that curves f(x) and g(x) intersect at points A and E (i.e., in the interval $[-4, -5]$ and again $[5, 7]$).

1. Show that coordinate points A and E are defined by solutions to the cubic equation:

$$x^3 - 8x^2 - 16x + 168 = 0. \tag{7}$$

   This is a hand calculation, so show all of your working.

2. Using calculus, or otherwise, show that the area of the shaded region is $\left[781 + 145\sqrt{29}\right]/12 \approx 130.154$.

3. Demonstrate how you can use one step of Simpson's Rule to obtain a high-accuracy estimate of the area of region A-B-C-D-E-F-G-H.

4. What is the expected error with Simpson's Rule?

   Briefly compare and discuss the analytical and numerical results.

Question 4: 10 points.

Consider the integral

$$I = \int_0^2 3x^2 + 4x^3 + 5x^4 dx = 56. \tag{8}$$

Write a Python program to compute numerical approximations to equation 8 using:

1. The Trapezoid rule,
2. Simpson's rule, and
3. Two-point Gauss Quadrature.

For cases 1 and 2, use only three data ordinates. Compute and print the absolute and relative errors for each numerical procedure.

Question 5: 10 points.

Write a Python program that uses Romberg Integration to show:

$$I = \int_0^1 \left[\frac{4}{1+x^2} \right] dx = \pi. \tag{9}$$

Start off by evaluating the function at 0, $\frac{1}{4}$, $\frac{1}{2}$, $\frac{3}{4}$, and 1. Compute and print the absolute and relative errors.

Question 6: 10 points

Theoretical considerations indicate that:

$$\int_0^4 x^3 \left[16 - x^2 \right] dx = \frac{1024}{3} \approx 341.33. \tag{10}$$

1. Use the method of Romberg integration to obtain an $O(h^6)$ accurate estimate of equation 10. Be sure to show all steps in your working.
2. Evaluate equation 10 using 3-pt Gauss Quadrature. Be sure to show all steps in your working.