Function Approximation

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Overview

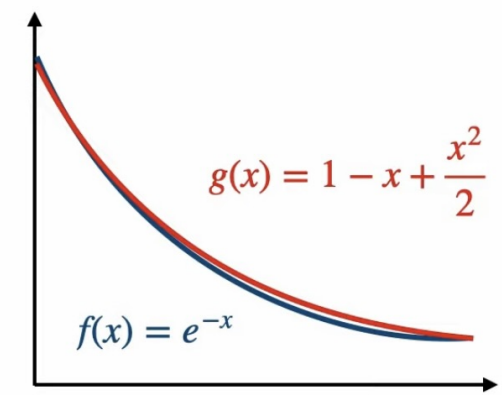
Function Approximation

Motivating Ideas

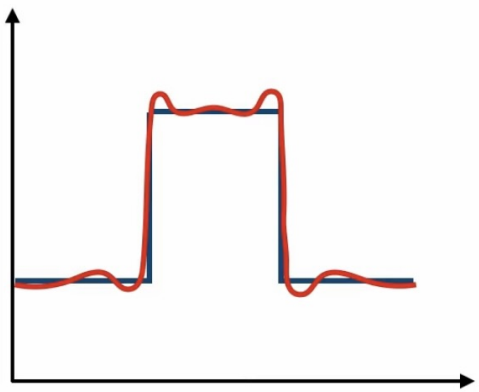
Function Approximation

A function approximation asks us to select a function, g(x), among a well-defined set of options that approximates – closely matches – a second function, f(x), in a task-specific way.

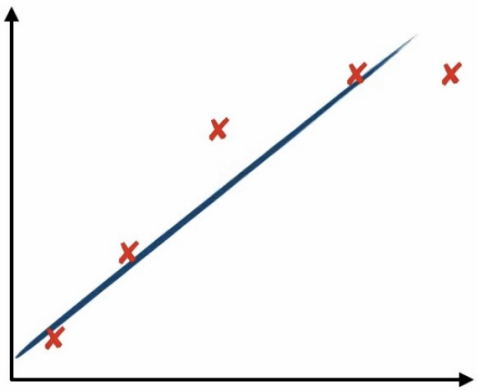
Approximation Examples: Many approaches ...



Polynomial Approximation



Fourier Series



Machine Learning

Strategy 1: Polynomial Approximation

Polynomial Approximation

Replace function f(x) by a simplier polynomial approximation g(x). Then, use g(x) in computations instead of f(x).

Example 1: Replace $y = f(x) = e^{-x}$ by a quadratic approximation:

$$f(x) = e^{-x} \quad \longrightarrow \quad g(x) = 1 - x + \frac{x^2}{2}. \tag{1}$$

Example 2: Replace $y = sin(x)$ on $x \in [0, \pi]$ by a quadratic approximation:

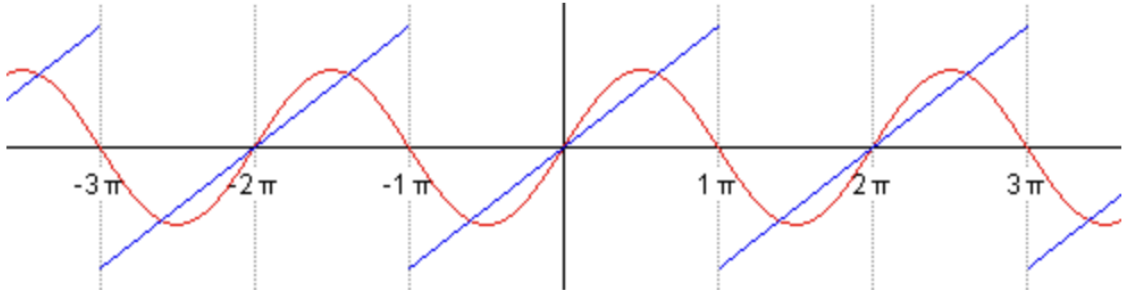
$$f(x) = sin(x) \quad \longrightarrow \quad g(x) = \frac{4x}{\pi^2}\left[\pi - x\right]. \tag{2}$$

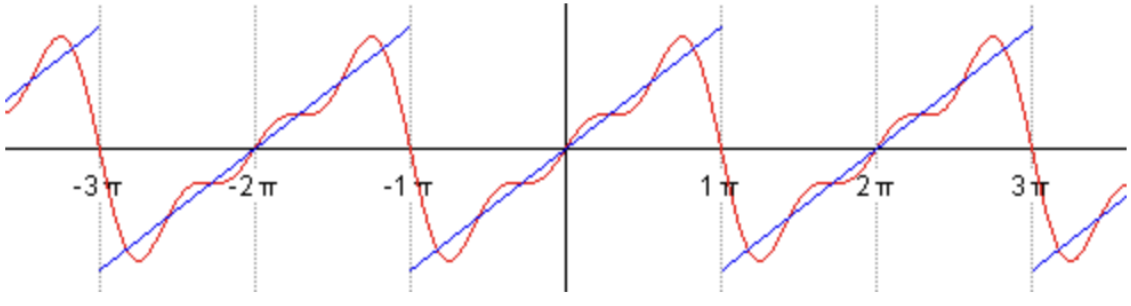
Strategy 2: Fourier Series

Fourier Series

A Fourier series is an expansion of a periodic function f(x) in terms of an infinite sum of trigonometric (i.e., sines and cosines) and/or exponential functions.

Example 1: Progressive refinement of sawtooth function:



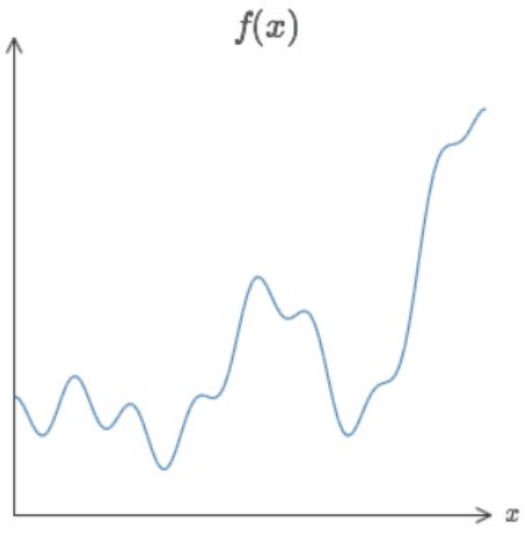


Strategy 3: Machine Learning

Neural Network

Use observable (or experimentally measured) data to train a neural network to capture (or estimate) input-to-output functionality.

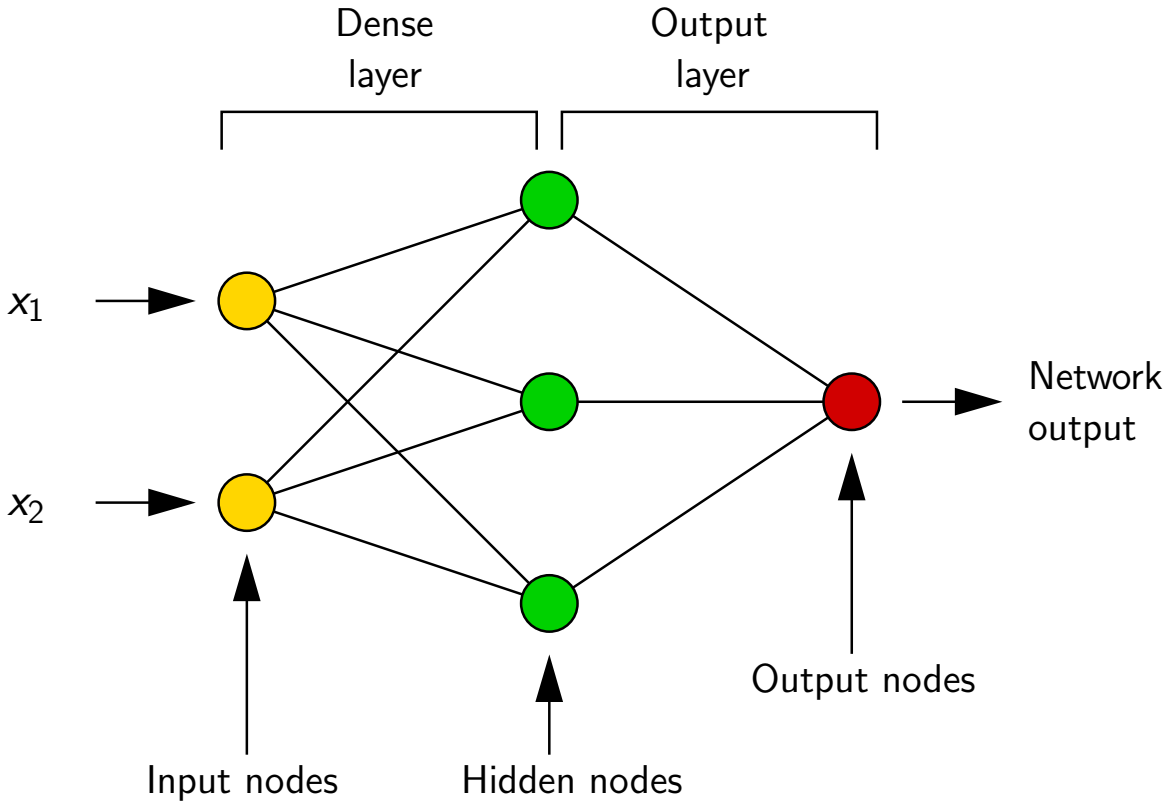
- Neural networks are universal function approximators, no matter how complex:
- Neural network architectures are highly scalable and flexible.



Caveat: Very large neural networks may be close to impossible to train and generalize correctly $\rightarrow$ AI chips.

Strategy 3: Machine Learning

Example 1: Neural Network with One Hidden Layer:



Data inputs x_1 and x_2 are transformed into a prediction $f(x_1, x_2)$.

Strategy 3: Machine Learning

Example 2: Learn how to classify univariate time series as belonging to one of six categories:

Data

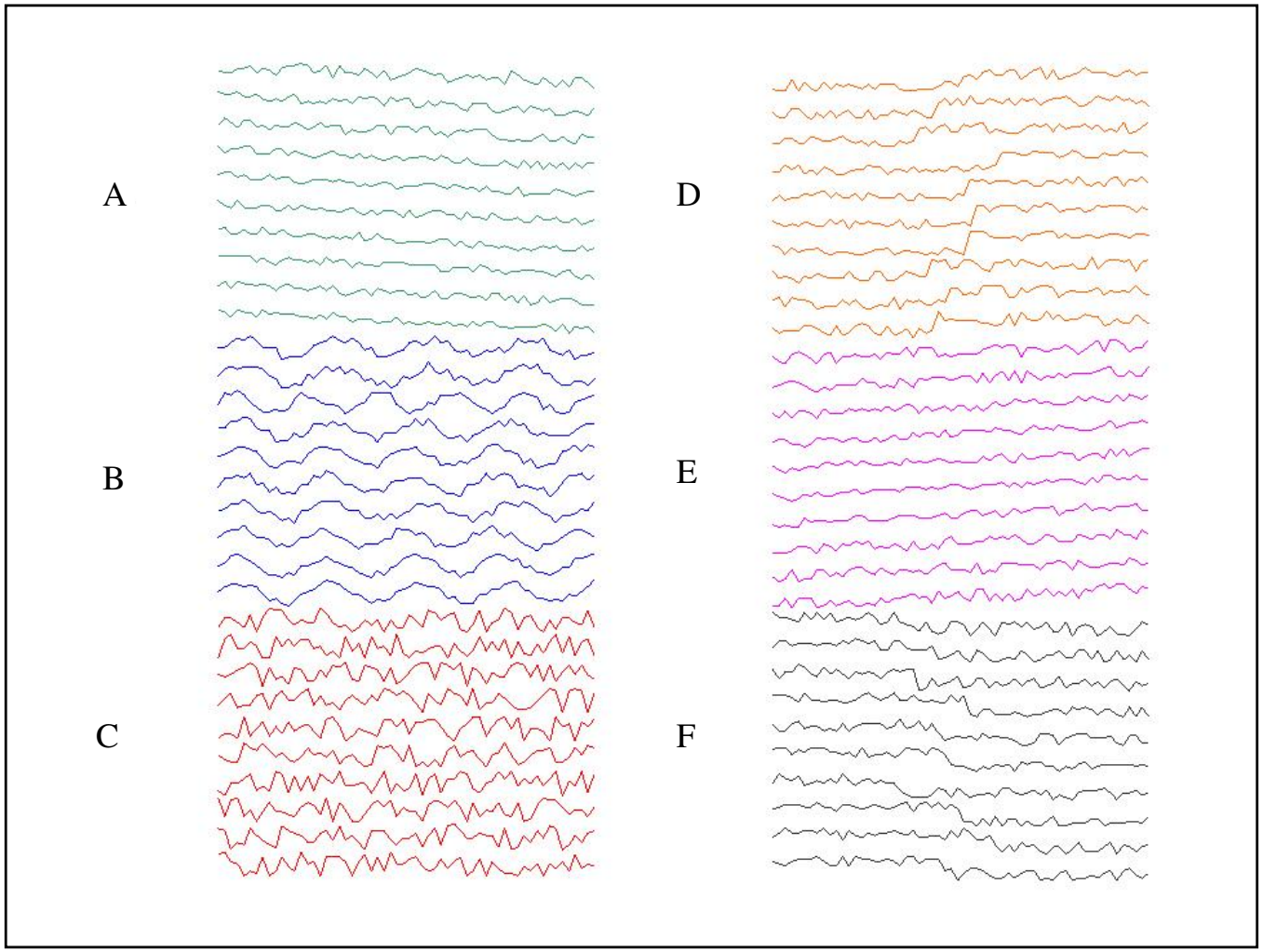
- UCI Synthetic Control Chart Time Series Data Set contains 600 sequences of data.
- Partition data: 450 items for training; 150 items for testing.

Six Categories of Datastream

- **A and E:** Decreasing and Increasing Trend.
- **B:** Cyclic.
- **C:** Normal.
- **D and F:** Upward and Downward Shift.

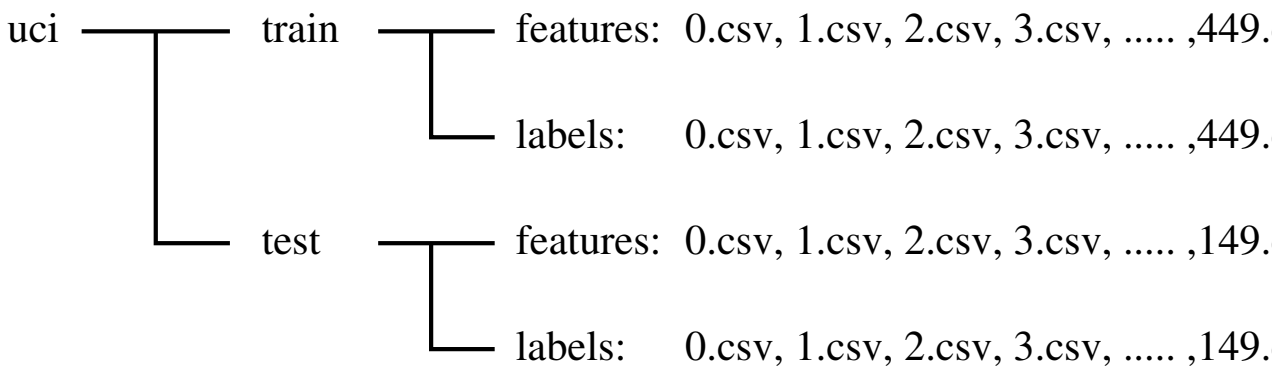
Strategy 3: Machine Learning

Representative Data Streams:

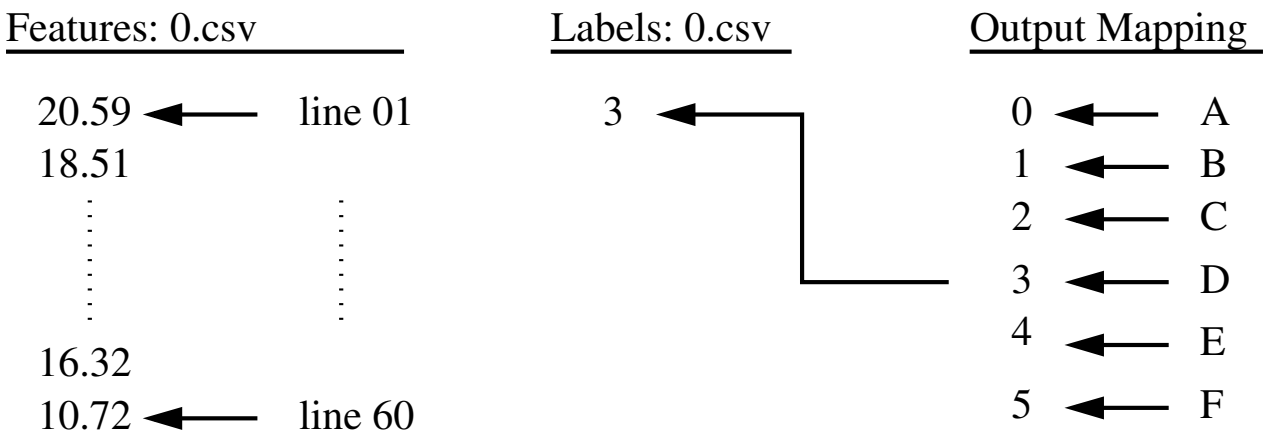


Strategy 3: Machine Learning

Training and Testing Corpus

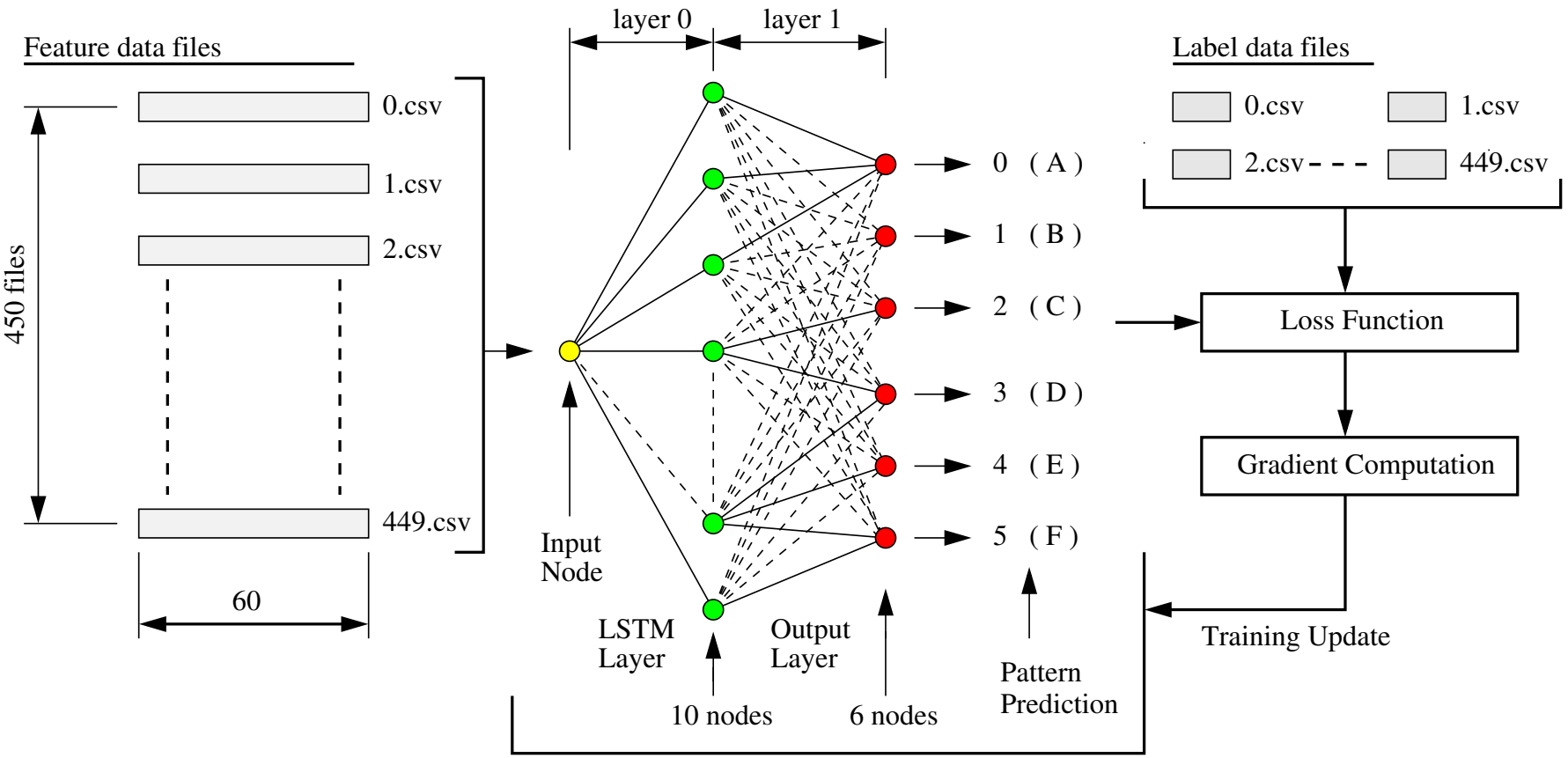


CSV Data File Format and Label Mappings



Strategy 3: Machine Learning

RNN Architecture + Sequences of Feature and Label vectors



Strategy 3: Machine Learning

Training the Model for 40 Epochs:

```
int nEpochs = 40;
net.fit(trainData, nEpochs);
```

Evaluation Metrics and Confusion Matrix:

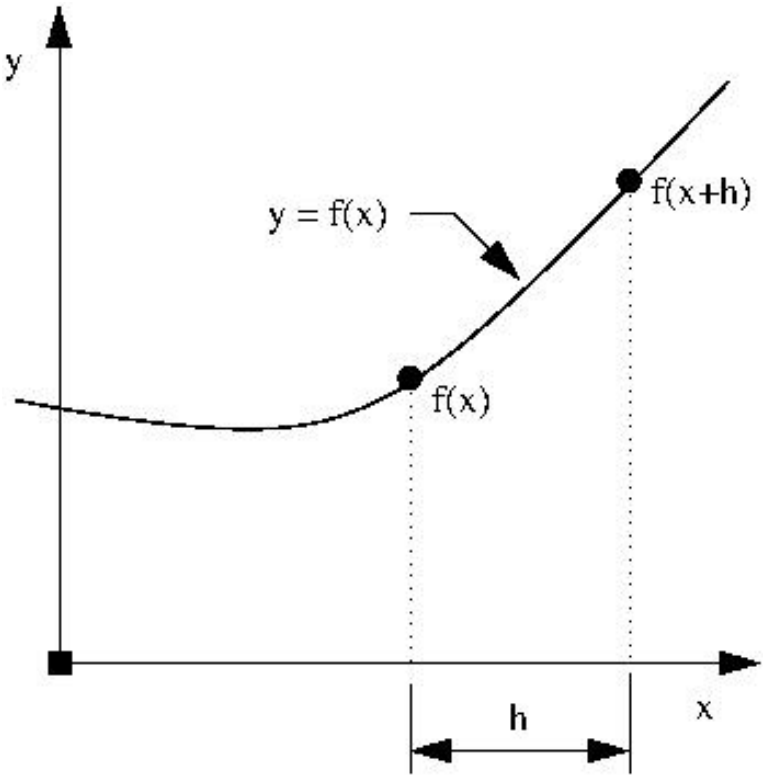
```
Accuracy:  0.8867   Recall:   0.8890   <--- It works!
Precision: 0.8886   F1 Score: 0.8883

   0  1  2  3  4  5
-------------------
  26  0  0  0  0  0 |  0 = 0
   0 29  0  0  0  0 |  1 = 1
   0  0 15  0  7  0 |  2 = 2
   0  0  0 20  0  1 |  3 = 3
   0  0  9  0 21  0 |  4 = 4
   0  0  0  0  0 22 |  5 = 5
```

Taylor Series

(Brook Taylor, 1715)

Motivating Idea

Let $y = f(x)$ be a smooth differentiable function.



Given $f(x)$ and derivatives $f^{'}(a)$, $f^{''}(a)$, $f^{'''}(a)$, etc, the purpose of Taylor's series is to estimate $f(x+h)$ at some distance h from x.

Taylor Series Expansion

Mathematical Expansion.

$$f(x+h) = \sum_{i=0}^{\infty} \frac{f^i(x)}{(i)!} h^i = f(x) + f'(x)h + \frac{f''(x)}{2!}h^2 + \frac{f'''(x)}{3!}h^3 + \cdots \tag{3}$$

For a Taylor series approximation containing $(n+1)$ terms

$$f(x+h) = \sum_{i=0}^{i=n} \frac{f^i(x)}{(i)!} h^n + R_n(x) \tag{4}$$

The remainder, $R_n(x)$, after truncation is:

$$R_n(x) = \frac{f^{(n+1)}(c(x+h))}{(n+1)!} h^{(n+1)} \quad \text{with} \quad [x \leq c \leq x+h]\,. \tag{5}$$

Taylor Series Expansion

We can also write:

$$f(x+h) = \sum_{i=0}^{i=n} \frac{f^i(x)}{(i)!} h^n + O(h^{(n+1)}) \tag{6}$$

The big-O notation $O(h^n)$ indicates how quickly the error will change as a function of h.

- O(0) $\rightarrow$ Magnitude of error is constant, regardless of h.
- O(h) $\rightarrow$ Magnitude of error proportional to h.
- O(h^2) $\rightarrow$ Magnitude of error proportional to h squared.

Maclaurin Series Expansion

Maclaurin Series: A Maclaurin Series is nothing more than a Taylor series expansion about a = 0, i.e.,

$$f(h) = f(0) + \frac{f'(0)}{1!}h + \frac{f''(0)}{2!}h^2 + \frac{f'''(0)}{3!}h^3 + \cdots \tag{7}$$

Trigonometric Series

$$sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}x^{2n} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

Surprisingly, these series converge for all values of x.

Remainder Function

Remainder Function. The formula for $R_n(x)$,

$$R_n(x) = \frac{f^{(n+1)}(c(x+h))}{(n+1)!}h^{(n+1)} \quad \text{with} \quad [x \leq c \leq x+h]\,. \quad (8)$$

This formula is derived via the "mean value theorem" (see extra slides for details) and simply states there exists a point $c(x+h)$ such that:

$$\frac{f(x+h)-f(x)}{(x+h-x)} = f^1(c) \rightarrow f(x+h) = f(x) + f^1(c)\,(h)\,. \quad (9)$$

We can estimate $R_n(x)$ without knowing $c(x+h)$ explicitly.

Ratio Test and Interval of Convergence

For the power series centered about x = a,

$$P(a+h) = C_0(a)+C_1(a)h+C_2(a)h^2+C_3(a)h^3+C_4(a)h^4+\cdots \quad (10)$$

suppose that:

$$\lim_{n\to\infty}\left[\frac{|C_n(a)|}{|C_{n+1}(a)}\right] = R. \quad (11)$$

then:

- If R = ∞, then the series converges for all values.
- If $0 < R < \infty$, then the series converges for all $h < R$.
- If R = 0, then the series converges only for h = 0.

We call R the **radius of convergence.**

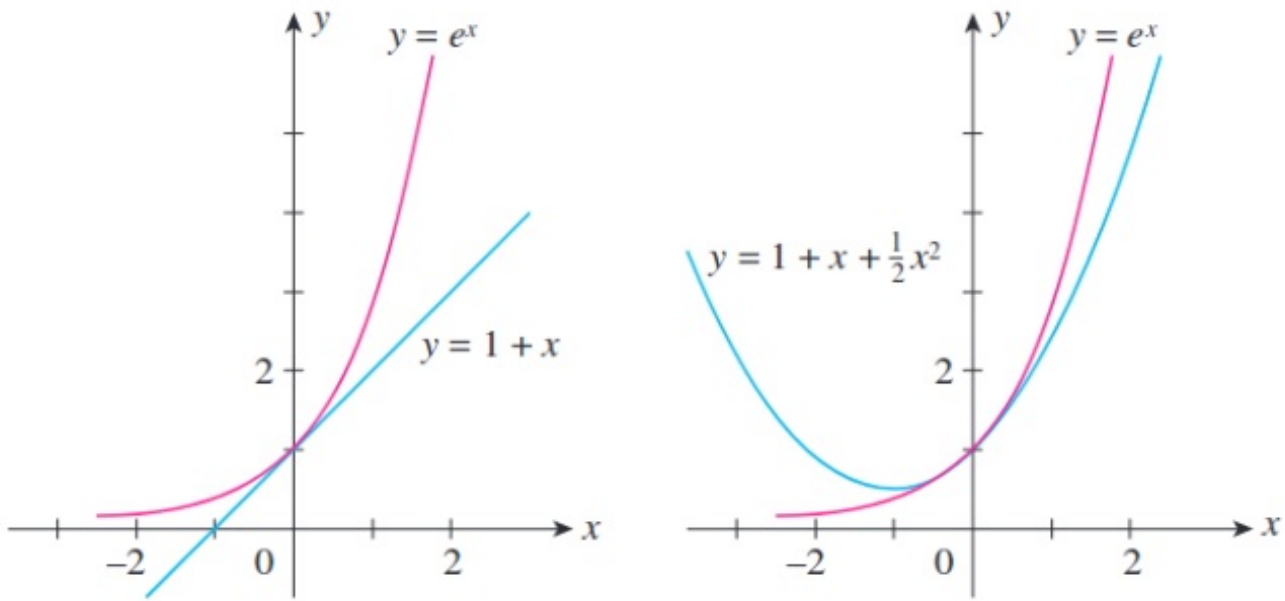
Solved Problems

Example 1: Taylor Series Expansion for e^x

Problem 1. Approximating e^x about $x = 0$.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots \tag{12}$$

Linear and quadratic approximations:



Example 1: Taylor Series Expansion for e^x

Now lets predict: $e^2 = 2.71828 * 2.71828 = 7.38904$.

No Terms	Numerical Estimate
1	1.0000
2	$1 + 2 \rightarrow 3.0000$
3	$1 + 2 + 4/2! \rightarrow 5.00000$
4	$1 + 2 + 4/2! + 8/3! \rightarrow 6.33333$
5	$1 + 2 + 4/2! + 8/3! + 16/4! \rightarrow 6.99999$

Estimate of Maximum Error: After five terms:

$$R_5(2) \leq \frac{e^{c(2)}}{6!}2^6 = \frac{e^2}{6!}2^6 = \frac{7.38904 * 64}{720} = 0.657. \qquad (13)$$

The actual error is: 0.389.

Example 1: Taylor Series Expansion for e^x

Test for Convergence. We have:

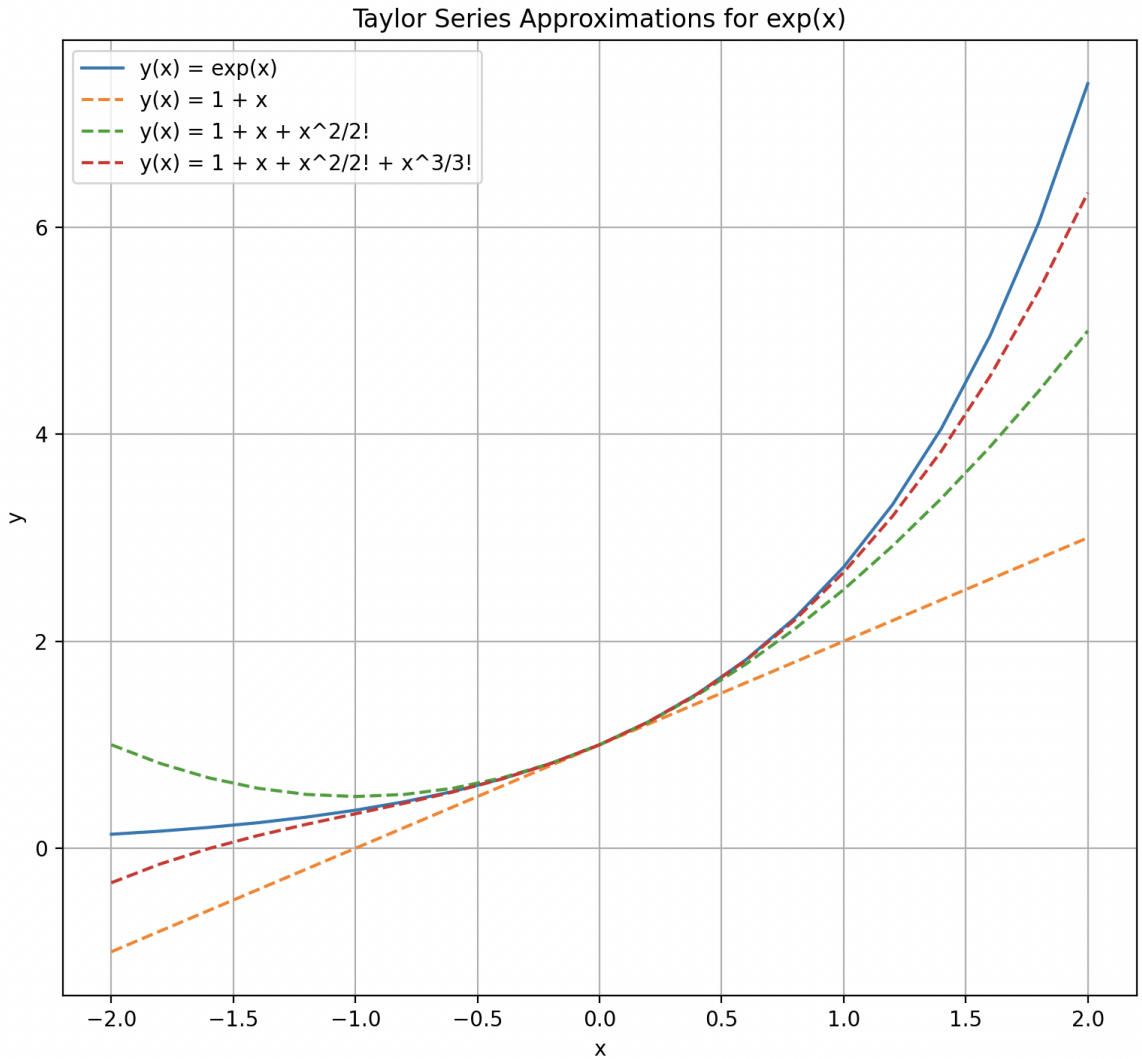
$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \tag{14}$$

The limit

$$\lim_{n\to\infty} \left[\frac{|C_n(a)|}{|C_{n+1}(a)} \right] = \lim_{n\to\infty} \left[\frac{\frac{1}{n!}}{\frac{1}{(n+1)!}} \right] = \lim_{n\to\infty} (n+1) = \infty. \tag{15}$$

gives $\mathrm{R} = \infty$ and the series converges for all values of x.

Example 1: Taylor Series Expansion for $exp(x)$



Example 2: Taylor Series Expansion for $cos(x)$

Example 2. Taylor Series expansion for $cos(x)$ about x = 0.

We have: f(0) = cos(0), $f^1(0) = -sin(0)$, and so forth. Hence,

$$cos(x) = cos(0) - \frac{sin(0)}{1!}x + \frac{-cos(0)}{2!}x^2 + \frac{sin(0)}{3!}x^3 + \frac{cos(0)}{4!}x^4 + \cdots$$
$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$$

Can also use the same procedure to show:

$$sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots \tag{16}$$

Example 2: Taylor Series Expansion for $cos(x)$

Test for Convergence. We have:

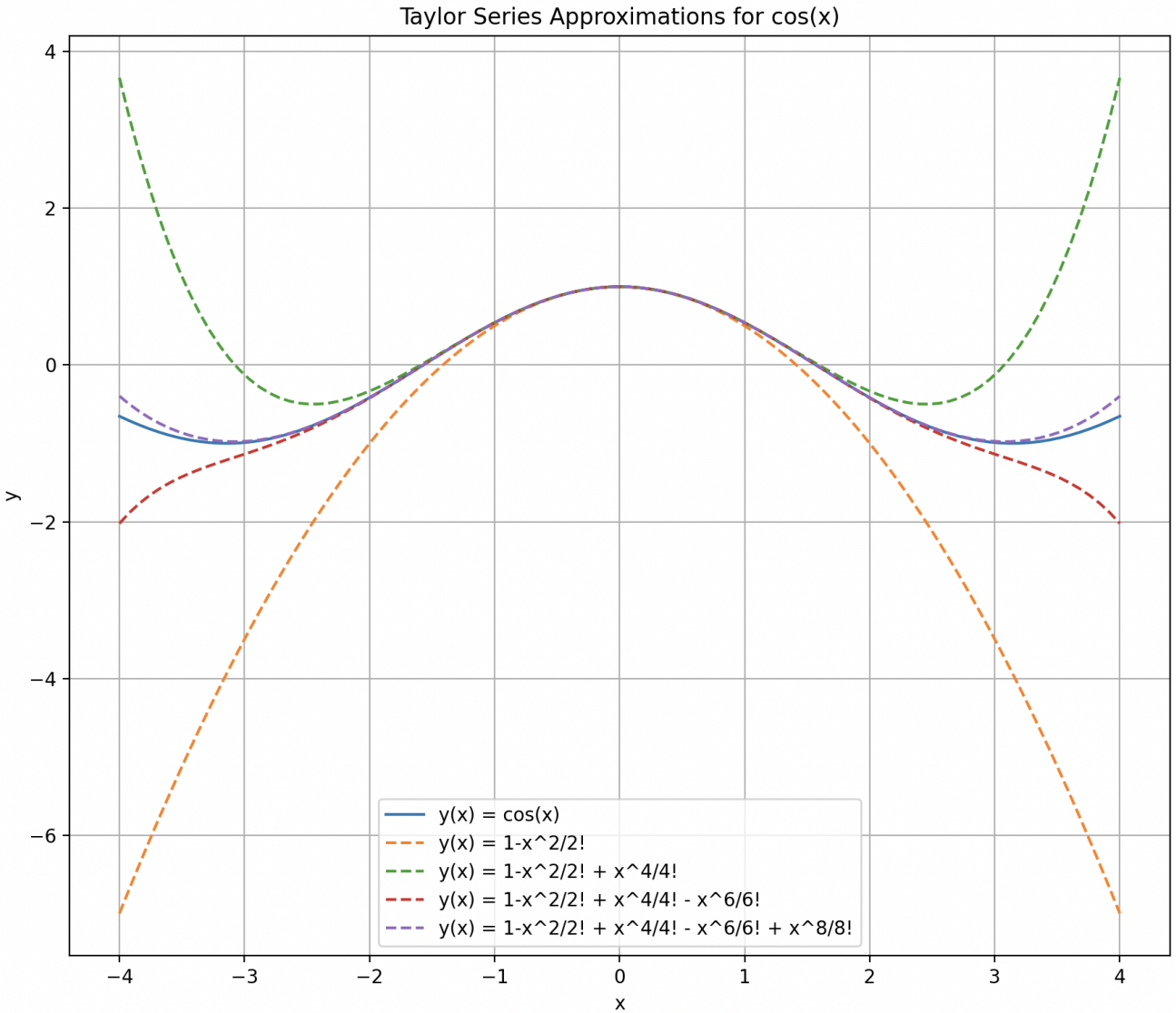
$$cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}. \tag{17}$$

Hence, $C_n = (-1)^n/(2n)!$ and:

$$\lim_{n\to\infty} \left[\frac{|C_n(a)|}{|C_{n+1}(a)|} \right] = \lim_{n\to\infty} \left[\frac{\frac{(-1)^n}{2n!}}{\frac{(-1)^{(n+1)}}{(2n+2)!}} \right] = \lim_{n\to\infty} (2n+1)(2n+2) = \infty. \tag{18}$$

The series converges for all real x.

Example 2: Taylor Series Expansion for $cos(x)$



Summary: Taylor Series Expansion

Strengths:

- When they are used to approximate complex functions with polynomials, the latter are much easier to differentiate and integrate.
- Can be easily manipulated to provide finite difference approximations to function derivatives.
- Can be used to get theoretical error bounds.

Weaknesses:

- Taylor series approximations require underlying function to be continuously differentiable.
- Series convergence for a wide range of values may only occur with an unreasonably large number of terms, a task that may be computationally infeasible.

Fourier Series

and

Fourier Integral

Fourier Series (Motivating Idea)

Fourier Series

A Fourier series is an expansion of a periodic function f(x) in terms of an infinite sum of trigonometric (i.e., sines and cosines) and/or exponential functions.

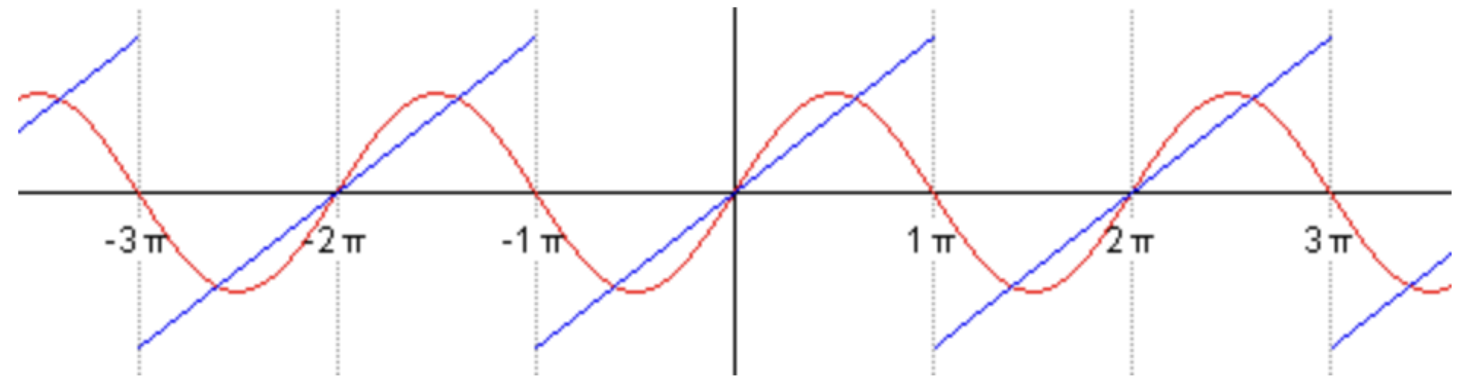
Periodic Function

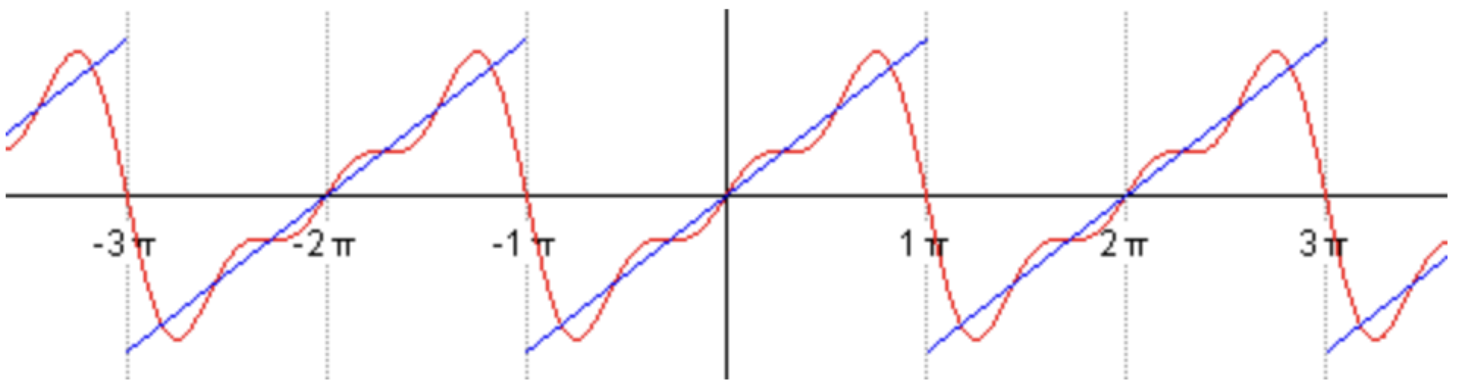
A function f(x) is periodic if and only if there exists a positive number 2p such that for every x in the domain of f, $f(x + 2p) = f(x)$. The number 2p is called the period of f.

Applications: Modeling of waveforms (e.g., surface waves on ocean; ocean tides; acoustics, musical tones; weather phenomena), analysis of resonant frequencies in a structure.

Fourier Series

Example 1: Progressive refinement of sawtooth function:





Fourier Series (Mathemtics)

Mathematics: (triginometry version) For x $\in [0, 2P]$:

$$f(x) \approx A_o + \sum_{n=1}^{\infty} \left[A_n \cos(\frac{n\pi x}{P}) + B_n \sin(\frac{n\pi x}{P}) \right]. \qquad (19)$$

The coefficients are given by:

$$A_o = \frac{1}{P} \int_0^{2P} f(x) dx, \qquad (20)$$

and $$A_n = \frac{1}{P} \int_0^{2P} f(x) \cos(\frac{n\pi x}{L}) dx, \quad \text{for n} \geq 1, \qquad (21)$$

$$B_n = \frac{1}{P} \int_0^{2P} f(x) \sin(\frac{nx}{L}) dx, \quad \text{for n} \geq 1. \qquad (22)$$

Fourier Series (Mathematics)

Example 1: Fourier Series for Sawtooth Function ...

The sawtooth shape can be written:

$$f(x) = \begin{cases} x & 0.0 \leq x \leq \pi, \\ x - 2\pi & \pi < x \leq 2\pi \end{cases} \tag{23}$$

Substituting 23 into 20 – 22 gives:

$$A_n = \frac{1}{\pi} \int_0^{2\pi} f(x) cos(nx) dx = 0.0. \tag{24}$$

$$B_n = \frac{1}{\pi} \int_0^{2\pi} f(x) sin(nx) dx = -\frac{2}{n}(-1)^n. \tag{25}$$

Fourier Series (Mathematics)

Example 1: Sawtooth function continued ...

The Fourier exapansion is:

$$f(x) = 2\left[sin(x) - \frac{sin(2x)}{2} + \frac{sin(3x)}{3} - \cdots + (-1)^{n+1}\frac{sin(nx)}{n}\cdots\right]$$
$$= 2\sum(-1)^{n+1}\frac{sin(nx)}{n}, \quad \text{for} \quad -\pi \leq x \leq \pi. \tag{26}$$

Note 1: The Fourier series coverges only conditionally because of the discontinuity of f(x) at x $\pm$ π.

Note 2: At the discontinuity itself, the Fourier series will coverge to the arithmetic mean of the end values.

Fourier Series (Mathemtics)

Mathematics: (exponential series). Recall Euler's formula:

$$e^{inx} = \cos(nx) + i\sin(nx). \tag{27}$$

Rearranging equation 27 gives:

$$\cos(nx) = \left[\frac{e^{inx} + e^{-inx}}{2}\right], \quad \sin(nx) = \left[\frac{e^{inx} - e^{-inx}}{2i}\right]. \tag{28}$$

Substituting 27 into 19, and rearranging terms gives:

$$f(x) = \frac{A_o}{2} + \sum_{n=1}^{\infty}\left[A_n\left[\frac{e^{inx} + e^{-inx}}{2}\right] + B_n\left[\frac{e^{inx} - e^{-inx}}{2i}\right]\right]. \tag{29}$$

Fourier Series

Simplifying:

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{-i\pi nx/L} \tag{30}$$

where:

$$c_n = \frac{1}{2}\left[A_n - iB_n\right], c_{-n} = \frac{1}{2}\left[A_n + iB_n\right], c_o = \frac{A_o}{2}. \tag{31}$$

Fourier Integral Analysis

Motivation:

- In many practical problems, the function involved is non-periodic (i.e., Fourier series are not possible).

Solution:

- Consider limiting form of Fourier Series when $p \longrightarrow \infty$.
- Fourier Series becomes Fourier Integral Analysis

Fourier Transform Pair: Waveforms in time/frequency domains:

$$f(t) = \int_{-\infty}^{\infty} g(w) e^{iwt} dw.$$
$$g(w) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\tau) e^{iw\tau} d\tau. \tag{32}$$

Discrete Fourier Transform

Discrete Fourier Transform (DFT)

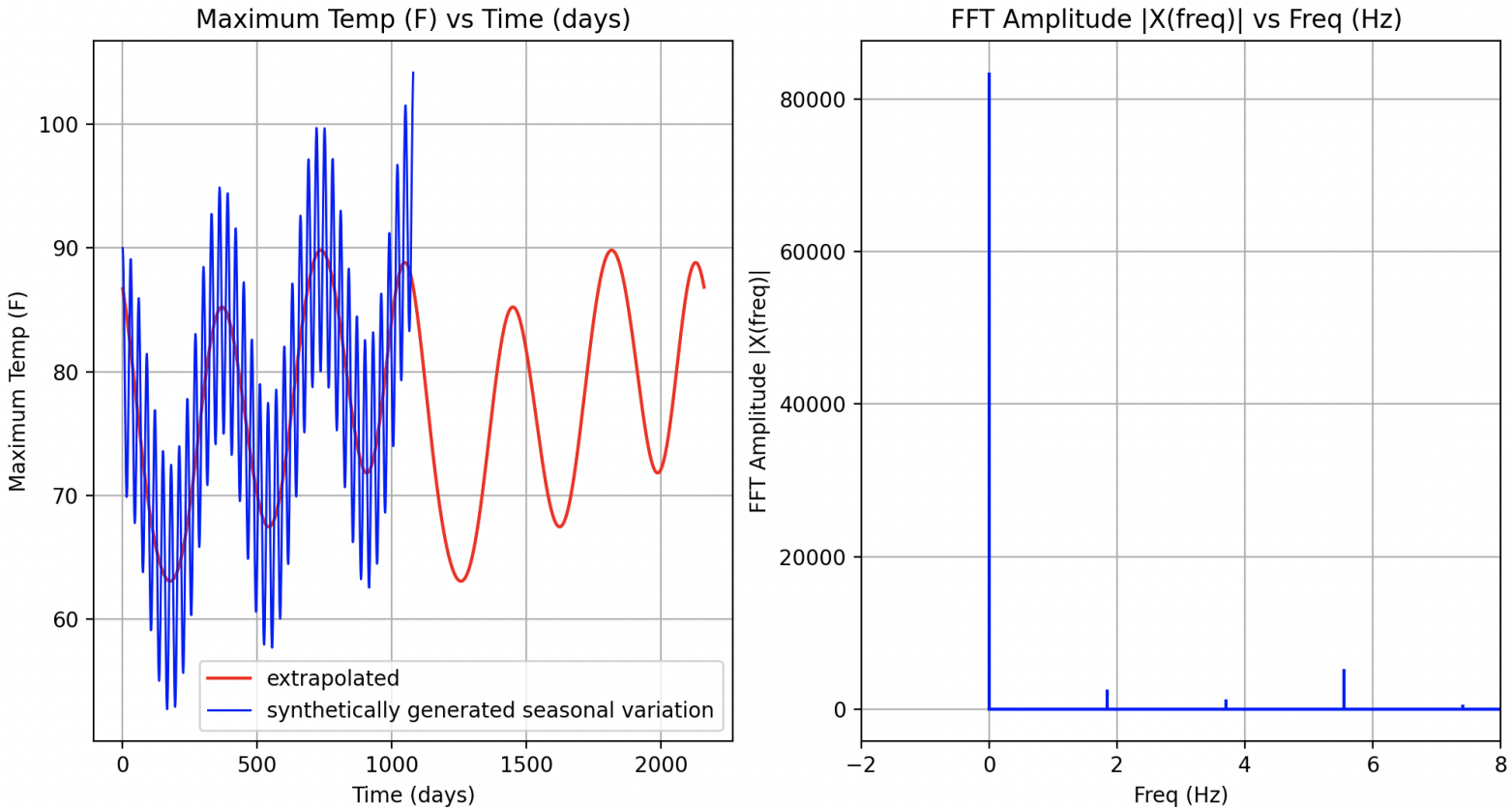
- Represents waveforms in both the time and frequency domains.
- Standard implementation of DFT requires $O(n^2)$ computational work, where n is the size of the data.

Fast Fourier Transform (FFT)

- An algorithm that computes the DFT of a sequence, most often in the time domain to a representation in the frequency domain.
- The inverse FFT transforms the frequency domain representation back into the time domain.
- FFT requires $O(nlog(n))$ computational work, so is very efficient.

Fast Fourier Transform Analysis

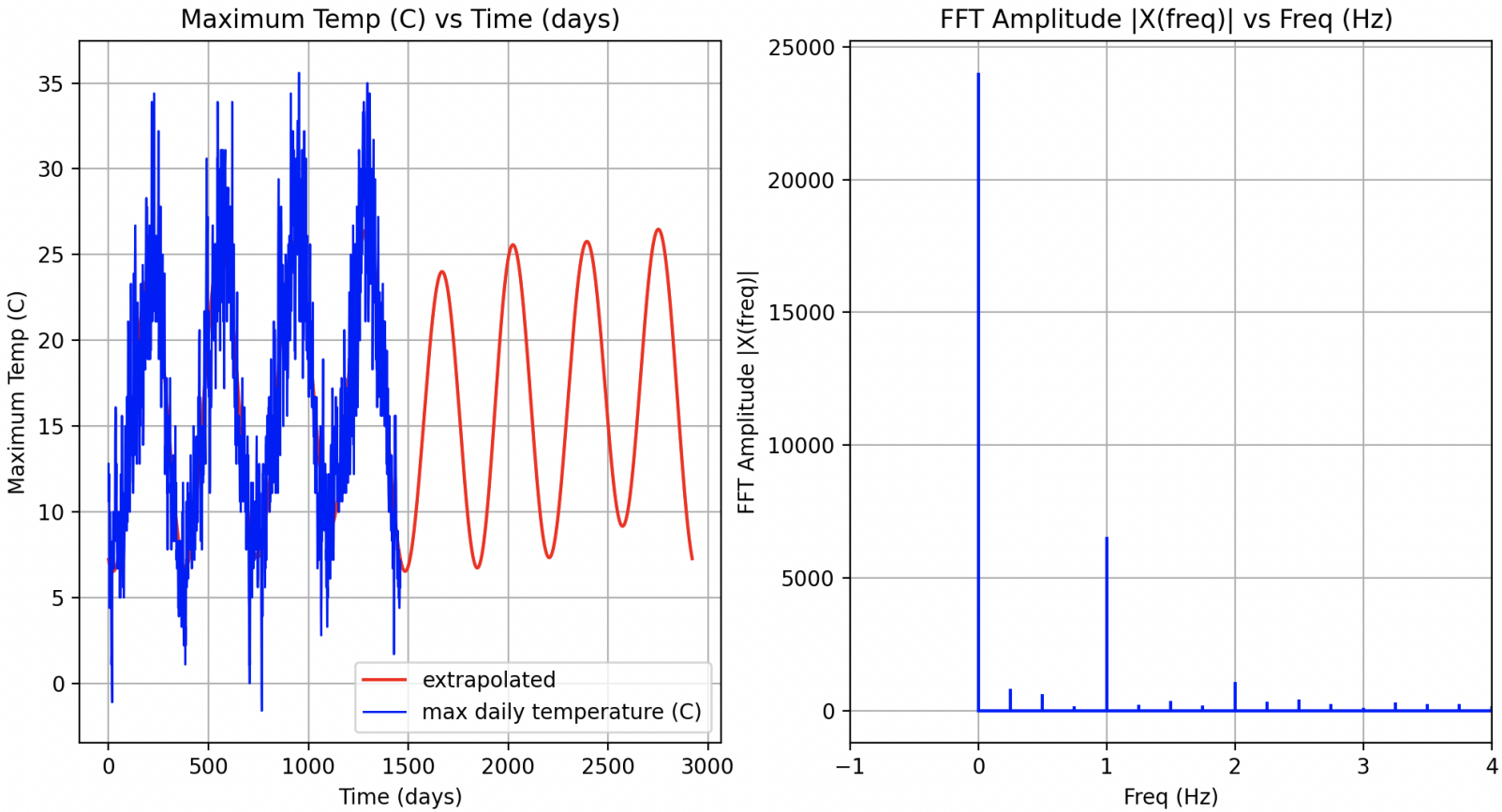
Example 1: FFT analysis of synthetically generated time series ...



Source Code: See python-code.d/math/

Fast Fourier Transform Analysis

Example 2: FFT analysis of max daily temperature in Seattle.



Source Code: See python-code.d/math/

Extra Slides

Extra Slides

Mean Value Theorem

Let function f be continuous on $[a, b]$

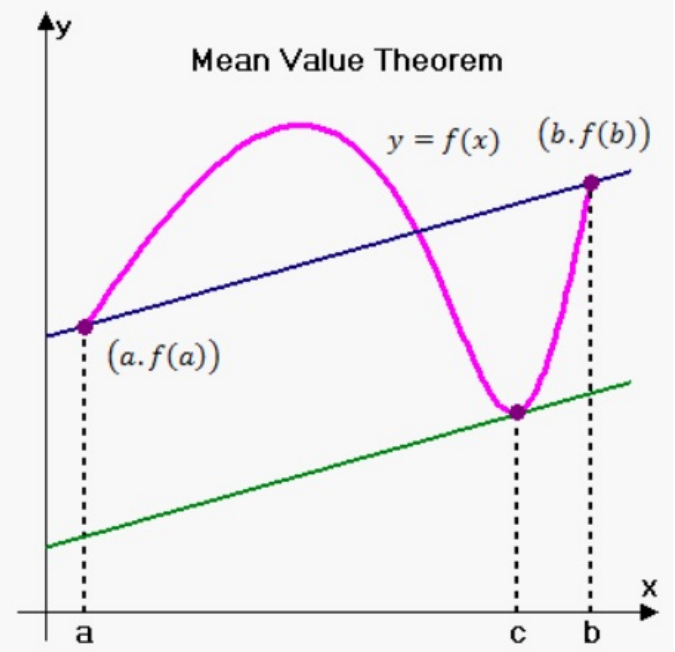
and differentiable on $[a, b]$.

There exists a point c in $[a, b]$ such that:

$$f^1(c) = \left[\frac{f(b) - f(a)}{b - a)}\right]. \qquad (33)$$

The right-hand side of the equation is the secant line connecting end points.

The green line is the tangent slope at point c.



Python Code Listings

Code 1: Taylor Series Approximations for cos(x)

```
1  # ============================================================================
2  # TestTaylorSeries02.py: Compute and plot Taylor Series approximations for
3  # cos(x).
4  #
5  # Written By: Mark Austin                                          August 2023
6  # ============================================================================
7
8  import math;
9  import numpy as np;
10 import matplotlib.pyplot as plt;
11
12 # Taylor series approximation for cos(x) ...
13
14 def func_cos(x,n):
15     cos_approx = 0
16     for i in range(n):
17         coeff = (-1)**i
18         num   = x**(2*i)
19         denom = math.factorial(2*i)
20         cos_approx = cos_approx + (coeff)*(num/denom)
21
22     return cos_approx
23
24 # main method ...
25
26 def main():
27     print("--- Enter TestTaylorSeries02.main()                ... ");
28     print("--- ========================================== ... ");
```

Code 1: Taylor Series Approximations for cos(x)

```
29
30      # Part 1: Compute x values for numerical computation ...
31
32      x = np.linspace( -4.0, 4.0, num = 101)
33      y = np.cos(x)
34
35      print(x)
36      print(y)
37
38      # Part 2: Two-term polynomial approximation: y(x) = 1 - x^2/2! ...
39
40      yapprox2 = []
41      for xi in x:
42         yapprox2.append( func_cos(xi,2) );
43
44      # Part 3: Three-term polynomial approximation: y(x) = 1 - x^2/2! + x^4/4! ...
45
46      yapprox3 = []
47      for xi in x:
48         yapprox3.append( func_cos(xi,3) );
49
50      # Part 4: Four-term polynomial approximation: y(x) = 1 - x^2/2! + x^4/4! - x^6/6! ..
51
52      yapprox4 = []
53      for xi in x:
54         yapprox4.append( func_cos(xi,4) );
55
56      # Part 5: Five-term polynomial approximation: y(x) = 1 - x^2/2! + x^4/4! - x^6/6! +
57
58      yapprox5 = []
```

Code 1: Taylor Series Approximations for cos(x)

```
59        for xi in x:
60           yapprox5.append( func_cos(xi,5) );
61
62        # Part 6: Plot cos(x) and various polynomial approximations ...
63
64        plt.plot(x,         y, label="y(x) = cos(x)",             linestyle="-")
65        plt.plot(x, yapprox2, label="y(x) = 1-x^2/2!",            linestyle="--")
66        plt.plot(x, yapprox3, label="y(x) = 1-x^2/2! + x^4/4!", linestyle="--")
67        plt.plot(x, yapprox4, label="y(x) = 1-x^2/2! + x^4/4! - x^6/6!", linestyle="--")
68        plt.plot(x, yapprox5, label="y(x) = 1-x^2/2! + x^4/4! - x^6/6! + x^8/8!", linestyle=
69        plt.title("Taylor Series Approximations for cos(x)")
70        plt.xlabel('x')
71        plt.ylabel('y')
72        plt.grid()
73        plt.legend()
74        plt.show()
75
76        print("--- ============================================== ... ");
77        print("--- Leave TestTaylorSeries02.main()                ... ");
78
79    # call the main method ...
80
81    main()
```

Code 2: FFT of Max Temperature in Seattle

```
# ============================================================================
# TestFastFourierTransform03.py: Compute FFT extrapolation and dominant
# frequencies in Temperature Measurements, Seattle, WA.
#
# Modified by: Mark Austin                                    September 2023
# ============================================================================

from math import cos,pi
import numpy as np
import pandas as pd
import matplotlib.pyplot as plt

# ============================================================================
# main method ...
# ============================================================================

def main():
    print("--- Enter TestFastFourierTransform03.main()      ...");
    print("--- ========================================== ...");

    # Part 1: Set parameters ...

    print("--- Part 1: Set parameters ...");

    sr = 365 # <-- sampling rate (no samples per year) ...

    # Part 2: Read Seattle weather ...
```

Code 2: FFT of Max Temperature in Seattle

```
29      print("--- Part 2: Read datafile for weather in Seattle ...");
30
31      rainfall = pd.read_csv('../data/seattle-weather.csv')
32      x = np.array( rainfall['temp_max'].values)
33
34      # Part 3: Compute fast fourier transform for maximum temperature ...
35
36      print("--- Part 3: Compute FFT for max daily temperature in Seattle ...");
37
38      X = np.fft.fft(x)
39      N = len(X)
40      n = np.arange(N)
41      T = N/sr
42      freq = n/T
43
44      print("---          No samples per year = {:10.2f} ...".format(sr) );
45      print("---          No data points: N = {:10.2f} ...".format(N) );
46      print("---          No periods:     T = {:10.2f} ...".format(T) );
47      print(n)
48      print("--- Fourier transform result ...");
49
50      # print(X)
51
52      # Part 4: Plot data + extrapolated curve ...
53
54      print("--- Part 4: Plot data + extrapolated curve ...");
55
56      plt.figure(figsize = (12, 6))
57      plt.subplot(121)
```

Code 2: FFT of Max Temperature in Seattle

```
59        for num_ in [6]:
60            fft_list              = np.copy(X)
61            fft_list[num_:-num_] = 0
62
63            # Inverse Fast Fourier transform
64
65            t = np.fft.ifft(fft_list)
66
67            # Plot general trend ...
68
69            plt.plot(np.concatenate([t,t]), color = 'red', label='extrapolated' )
70
71        plt.plot( np.arange(0, x.size), x, 'b', label = 'max daily temperature (C)', linewid
72        plt.title("Maximum Temp (C) vs Time (days)")
73        plt.ylabel("Maximum Temp (C)", fontsize=10, rotation=90)
74        plt.xlabel("Time (days)", fontsize=10, rotation=0)
75        plt.grid()
76        plt.legend()
77
78        # Part 5: Plot data in frequency domain ...
79
80        print("--- Part 5: Plot data in frequency domain ...");
81
82        plt.subplot(122)
83        plt.stem( freq, np.abs(X), 'b', markerfmt=" ", basefmt="-b")
84        plt.title("FFT Amplitude |X(freq)| vs Freq (Hz) ")
85        plt.xlabel('Freq (Hz)')
```

Code 2: FFT of Max Temperature in Seattle

```
86          plt.ylabel('FFT Amplitude |X(freq)|')
87          plt.grid()
88          plt.xlim(-1, 4)
89
90          plt.show()
91
92          print("--- ======================================== ... ");
93          print("--- Leave TestFastFourierTransform03.main()  ...");
94
95      # call the main method ...
96
97      main()
```