

Solutions to Midterm II Exam

Question 1: 20 points

Problem Statement. Recall from our class lectures that if $[A]$ is an $(n \times n)$ matrix then, in general, it can be factored into a product of lower and upper triangular matrices, i.e., $[A] = [L][U]$ where $[L]$ and $[U]$ are also $(n \times n)$ matrices. Our case study programs with Python and sympy assume that the lower diagonal elements are unity (i.e., $L_{ii} = 1$), but this is only one way of enabling the factorization. A second possibility is to set the upper diagonal elements to unity (i.e., $U_{ii} = 1$). The key point here is that any set of constraints that reduces the total number of unknowns from $(n^2 + n)$ to n^2 might work.

Part [1a]. (10 pts). Calculate the LU decomposition for the matrix

$$[A] = \begin{bmatrix} 1 & 1 & 0 \\ a & 8 & 4 \\ 0 & 1 & 2 \end{bmatrix}. \quad (1)$$

by assuming that $U_{ii} = 1$. Show all of your working ...

Solution: With $U_{ii} = 1$, the LU decomposition can be written:

$$\begin{bmatrix} L_{11} & 0 & 0 \\ L_{21} & L_{22} & 0 \\ L_{31} & L_{32} & L_{33} \end{bmatrix} \begin{bmatrix} 1 & U_{12} & U_{13} \\ 0 & 1 & U_{23} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ a & 8 & 4 \\ 0 & 1 & 2 \end{bmatrix}. \quad (2)$$

We have 9 matrix terms and 9 unknowns.

Matching terms in the first row of L and columns of U gives:

$$1 \cdot L_{11} = 1, \longrightarrow L_{11} = 1. \quad (3)$$

$$L_{11}U_{12} = 1, \longrightarrow U_{12} = 1. \quad (4)$$

$$L_{11}U_{13} = 0, \longrightarrow U_{13} = 0. \quad (5)$$

Matching terms in the second row of L and columns of U gives:

$$L_{21}U_{11} = a, \longrightarrow L_{21} = a. \quad (6)$$

$$L_{21} + L_{22} = 8, \longrightarrow L_{22} = (8 - a). \quad (7)$$

$$L_{22}U_{23} = 4, \longrightarrow U_{23} = \frac{4}{8 - a}. \quad (8)$$

Matching terms in the third row of L and U gives:

$$L_{31}U_{11} = 0, \longrightarrow L_{31} = 0. \quad (9)$$

$$L_{31} + L_{32} = 1, \longrightarrow L_{32} = 1. \quad (10)$$

$$L_{31}U_{13} + L_{32}U_{23} + L_{33}U_{33} = 2, \longrightarrow L_{33} = 2 - \frac{4}{8 - a}. \quad (11)$$

Collecting terms gives:

$$\begin{bmatrix} L_{11} & 0 & 0 \\ L_{21} & L_{22} & 0 \\ L_{31} & L_{32} & L_{33} \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ a & 8 - a & 0 \\ 0 & 1 & 2 - \frac{4}{8 - a} \end{bmatrix} \quad (12)$$

$$\begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & \frac{4}{8 - a} \\ 0 & 0 & 1 \end{bmatrix} \quad (13)$$

Part [1b]. (3 pts). Hence, write down the $\det[A]$?. Note: Do not calculate the determinate by the method of cofactors - there is a much faster way that is a one line calculation!

Solution: $\det(A) = \det(L) \cdot \det(U) = (8 - a) \left(2 - \frac{4}{8-a}\right) = 12 - 2a.$

Part [1c]. (7 pts). Use forward and backward substitution to show that the general solution to $[L][U][x] = [1, 2, b]^T$ can be written:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} (-4 - 2b)/(a - 6) \\ (a + 2b - 2)/(a - 6) \\ (ab - a - 8b + 2)/(2a - 12) \end{bmatrix} \quad \text{.label} \text{eqn : family - matrix - equations02} \quad (14)$$

This is a hand calculation, so show all of your working.

Solution: Two steps:

Forward Substitution: Solve $Lz = B$, i.e.,

$$\begin{bmatrix} 1 & 0 & 0 \\ a & 8 - a & 0 \\ 0 & 1 & 2 - \frac{4}{8-a} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ b \end{bmatrix} \longrightarrow \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 1 \\ (a - 2)/(a - 8) \\ (b(a - 8) - a + 2)/(2a - 12) \end{bmatrix}. \quad (15)$$

Backward Substitution: Solve $Ux = Z$, i.e.,

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & \frac{4}{8-a} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \longrightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} (-4 - 2a)/(a - 6) \\ (a + 2b - 2)/(a - 6) \\ ((a - 8)b + 2 - a)/(2a - 12) \end{bmatrix}. \quad (16)$$

Part [1d]. (3 pts). For what values (a,b) will the system have: (a) a unique solution, (b) zero solutions, (c) infinite solutions?

Solution: Unique solution: $a \neq 6$. Infinite solutions: When $a = 6$, $b = -2$, $\text{rank}(A) = \text{rank}(A | b) = 2$. Otherwise, when $a = 6$, $b \neq -2$, zero solutions.

Question 2: 10 points

Problem Statement. This question covers numerical computation of roots to the test equation:

$$f(x) = x^4 - 12x^3 + 47x^2 - 66x + 18 = 0. \quad (17)$$

Preliminary work indicates that $f(x)$ has multiple roots at a and, thus, can be factored:

$$f(x) = (x - a)^m h(x). \quad (18)$$

Here $m \geq 2$.

Part [2a] (4 pts). Show that the Newton Raphson update formula for equation 18 can be written:

$$x_{n+1} = x_n - \left[\frac{(x_n - a)h(x_n)}{mh(x_n) + (x_n - a)h'(x_n)} \right]. \quad (19)$$

Solution: First, we differentiate equation 18:

$$f'(x) = m(x - a)^{m-1}h(x) + (x - a)^m h'(x). \quad (20)$$

Substituting equations 18 and 20 into the standard N-R update formula and cancelling common terms gives:

$$x_{n+1} = x_n - \left[\frac{f(x_n)}{f'(x_n)} \right] = x_n - \left[\frac{(x_n - a)h(x_n)}{mh(x_n) + (x_n - a)h'(x_n)} \right]. \quad (21)$$

Part [2b] (3 pts). Determine appropriate values of a , m and $h(x)$ for equation 17,

Solution: Equation 17 factors into:

$$f(x) = (x - 3)^2 (x^2 - 6x + 2). \quad (22)$$

Hence, $a = 3$, $m = 2$, and $h(x) = (x^2 - 6x + 2)$. The roots of $h(x) = 0$ are $3 \pm \sqrt{7}$.

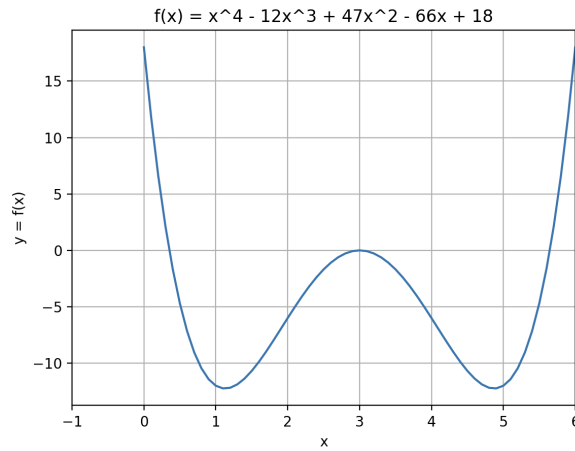


Figure 1. Plot $y = f(x)$ vs x .

Part [2c] (3 pts). Starting from an initial value, $x_0 = 0.0$, compute no more than three iterations of approximation to the lowest root of equation 17. I suggest you organize your computations into a table that shows iteration no, x , $(x - a)^m$, $h(x)$ and $f(x)$.

Solution:

Iteration:	x :	$(x-3)^2$:	$h(x)$:	$f(x)$:
0	0.000	9.000	2.000	18.000
1	0.273	7.438	0.438	3.258
2	0.349	7.030	0.030	0.212
3	0.354	7.000	0.000	0.001

Question 3: 10 points

Problem Statement. This question covers function interpolation with the methods of divided differences and lagrange interpolation, and curve fitting with the method of least squares.

The whole question is motivated by the small dataset:

x	0.0	2.0	3.0	5.0
f(x)	0.0	8.0	27.0	125.0

Part [3a] (5 pts). Use the method of divided differences to find a polynomial of lowest order that will fit the dataset.

Solution: We seek a third order polynomial $p(x)$:

$$p(x) = f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1) + f[x_0, x_1, x_2, x_3](x - x_0)(x - x_1)(x - x_2). \quad (23)$$

where

x_i	$f[x_i] = f(x_i)$	$f[,]$	$f[, ,]$	$f[, , ,]$
0.0	0	$f[x_0, x_1]$	$f[x_0, x_1, x_2]$	$f[x_0, x_1, x_2, x_3]$
2.0	8	$f[x_1, x_2]$	$f[x_1, x_2, x_3]$	
3.0	27	$f[x_2, x_3]$		
5.0	125			

Elements in the divided difference table are as follows:

$$f[x_0, x_1] = \left[\frac{f(x_1) - f(x_0)}{x_1 - x_0} \right] = \left[\frac{8 - 0}{2 - 0} \right] = 4. \quad (24)$$

$$f[x_1, x_2] = \left[\frac{f(x_2) - f(x_1)}{x_2 - x_1} \right] = \left[\frac{27 - 8}{3 - 2} \right] = 19. \quad (25)$$

$$f[x_2, x_3] = \left[\frac{f(x_3) - f(x_2)}{x_3 - x_2} \right] = \left[\frac{125 - 27}{5 - 3} \right] = 49. \quad (26)$$

The second column of computations:

$$f[x_0, x_1, x_2] = \left[\frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} \right] = \left[\frac{19 - 4}{3 - 0} \right] = 5.0. \quad (27)$$

$$f[x_1, x_2, x_3] = \left[\frac{f[x_2, x_3] - f[x_1, x_2]}{x_3 - x_1} \right] = \left[\frac{49 - 19}{5 - 2} \right] = 10.0. \quad (28)$$

The third column computations:

$$f[x_0, x_1, x_2, x_3] = \left[\frac{f[x_1, x_2, x_3] - f[x_0, x_1, x_2]}{x_3 - x_0} \right] = \left[\frac{10 - 5}{5 - 0} \right] = 1.0 \quad (29)$$

Thus, the interpolated polynomial is:

$$\begin{aligned} f(x) &= f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1) \\ &= 0 + 4(x - 0) + 5(x - 0)(x - 2) + 1(x - 0)(x - 2)(x - 3) \\ &= x^3 \end{aligned}$$

Part [3b] (5 pts). Check your answer in Part 3a by computing the functional form via the method of Lagrange Interpolation.

Solution: For the given dataset,

$$f(x) = f(x_0)p_0(x) + f(x_1)p_1(x) + f(x_2)p_2(x) + f(x_3)p_3(x) \quad (30)$$

where

$$p_0(x) = \frac{(x - 2)(x - 3)(x - 5)}{(0 - 2)(0 - 3)(0 - 5)} = \left[\frac{-x^3 + 10x^2 - 31x + 30}{30} \right]. \quad (31)$$

$$p_1(x) = \frac{(x - 0)(x - 3)(x - 5)}{(2 - 0)(2 - 3)(2 - 5)} = \left[\frac{x^3 - 8x^2 + 15x}{6} \right]. \quad (32)$$

$$p_2(x) = \frac{x(x-2)(x-5)}{3(3-2)(3-5)} = \left[\frac{-x^3 + 7x^2 - 10x}{6} \right]. \quad (33)$$

$$p_3(x) = \frac{x(x-2)(x-3)}{5(5-2)(5-3)} = \left[\frac{x^3 - 5x^2 + 6x}{30} \right]. \quad (34)$$

Also note: $f(x_0) = 0$, $f(x_1) = 8$, $f(x_2) = 27$ and $f(x_3) = 125$.

Substitute equations 31 through 34 into equation 30. Notice that the constant term in equation 31 is one, and the linear and quadratic terms cancel out completely. This leaves:

$$f(x) = x^3. \quad (35)$$