Solutions to Midterm 1 Exam

Question 1: 20 points.

Problem Statement. This question covers use of Python to model and visualize the difficult equation:

$$f(x) = \frac{1}{x^5} - \frac{2}{x^4} + \frac{4}{x^3} - \frac{8}{x^2} + \frac{16}{x} - 32 \tag{1}$$

Figure 1 plots f(x) over the range $[-4, 4]$.

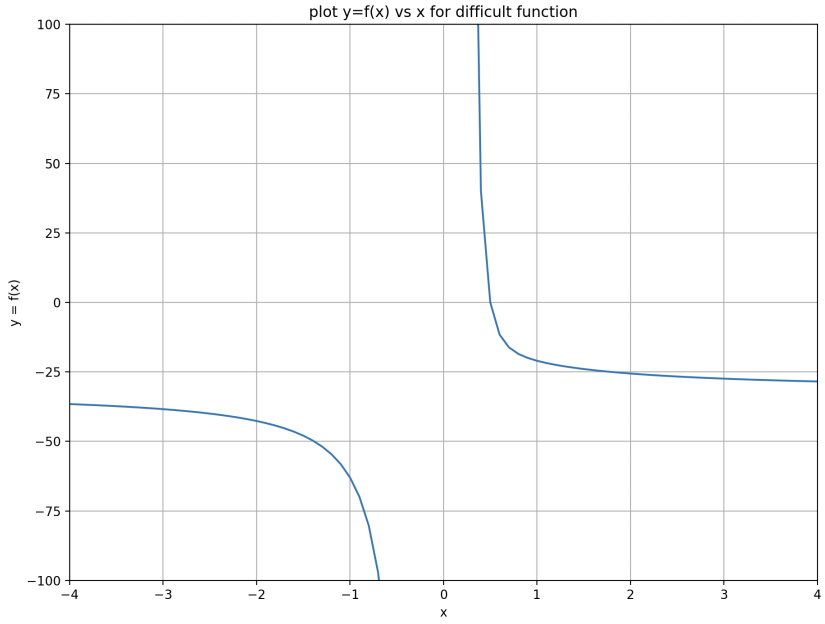


Figure 1. Plot y = f(x) vs x.

From the plot we can see that there is one solution to f(x) = 0 within the interval $[0, 1]$. Also notice that when x = 0, evaluation of $f(x)$ will result in positive/negative infinity, a run-time error.

The Python code to model equation 1 and produce Figure 1 is as follows:

```
# ============================================================
```

```
# TestDifficultFunction04.py: Explore roots to difficult equation.
#
# Written by: Mark Austin                                 October 2025
# ================================================================

import math
import numpy as np
import matplotlib.pyplot as plt

# Difficult function ...

def difficultFunction(x):
    result = 1/x**5 - 2/x**4 + 4/x**3 - 8/x**2 + 16.0/x - 32.0;
    return result

# main method ...

def main():
    print("================================");
    print("          Coord          Value");
    print("            (x)            (y)");
    print("================================");

    # Define problem parameters.

    xcoord = np.linspace( -4, 4, 81)

    # Create list for y coordinates ...

    ycoord = [];

    # Traverse xcoord array and compute y values ...

    for x in xcoord:
        result = difficultFunction(x)
        print("        {:7.2f}   {:12.3e}".format(x,result) );
        ycoord.append(result)

    print("================================");
    print("");

    # Plot y=f(x) vs x for test function ...

    plt.plot(xcoord,ycoord)
    plt.title('plot y=f(x) vs x for difficult function')
    plt.ylabel('y = f(x)')
    plt.xlabel('x')
    plt.xlim(   -4.0,   4.0 )
    plt.ylim( -100.0, 100.0 )
    plt.grid(True)
    plt.show()

# call the main method ...

if __name__ == "__main__":
```

```
main()
```

The abbreviated textual output is:

```
==============================
          Coord          Value
            (x)            (y)
==============================
          -4.00     -3.657e+01
          -3.90     -3.671e+01
          -3.80     -3.685e+01

... lines of code removed ...

          -0.10     -1.250e+05
           0.00            nan
           0.10      8.333e+04
           0.20      2.223e+03
           0.30      2.452e+02
           0.40      4.003e+01
           0.50      0.000e+00
           0.60     -1.161e+01

... lines of code removed ...

           3.80     -2.828e+01
           3.90     -2.836e+01
           4.00     -2.844e+01
==============================
```

Let's start with a theoretical evaluation of the roots to equation 1.

Part [1a] (7 pts) Use mathematics to show that equation 1 has **only one real root**.

Solution: Multplying equation 1 by x^5 gives:

$$g(x) = x^5 f(x) = 32x^5 - 16x^4 + 8x^3 - 4x^2 + 2x - 1 = 0. \tag{2}$$

Equation 2 factors into:

$$g(x) = \left(4x^2 - 2x + 1\right)(2x - 1)\left(4x^2 + 2x + 1\right) = 0 \tag{3}$$

The left-most and right-most terms have complex roots $\frac{1}{4}\left(1 \pm \sqrt{3}i\right)$ and $\frac{-1}{4}\left(1 \pm \sqrt{3}i\right)$, respectively. The center term has one real root: x = $\frac{1}{2}$.

Part [1b] (2 pts) Explain the purpose of the statements:

```
import math
import numpy as np
import matplotlib.pyplot as plt
```

and how they are used in the program.

Solution: In Python, the keyword import provides access to library modules and their functions and constants therein. Here we have three statements:

- `import math` accesses the math library, and it's constants (e.g., math.pi) and methods (e.g., math.cos(x)) therein. It is not used/necessary in this code.
- `import numpy as np` accesses the numpy library and gives is a shortened name `np`. Numpy provides support for the creation of 0-d, 1-d, 2-d, and 3-d arrays, along with computational support for array operations. Use of the shortened name means that instead of creating an array object with something like numpy.array(2,2), we can simply write np.array(2,2).

  This program uses numpy in the line xcoord = np.linspace(-4, 4, 81) to create an array of evenly-spaced values.
- `import matplotlib.pyplot as plt` accesses the `pyplot` module of the `matplotlib` library and gives it the shortened name `plt`. This module provides support for the creation of plots/figures, and in this program, is used to plot y = f(x) (see Figure 1) label the axes, and provide a title.

Part [1c] (2 pts) The fragment of textual output:

```
-0.10     -1.250e+05
 0.00            nan
 0.10      8.333e+04
```

indicates an error in the evaluation for f(x) when x = 0. What is the nature of this error, and what is the standard that prevents Python from crashing?

Solution: When x = 0, f(x) encounters a divide by zero in the expressions $1/x^5$, $2/x^4$, and so forth. Python records the divide-by-zero as a `-Inf` run-time error. Support for handling of run-time errors, such as divide by zero and NaNs, is defined by the IEEE 754 Standard.

Part [1d] (3 pts) Draw and label a figure for the coordinate values generated by:

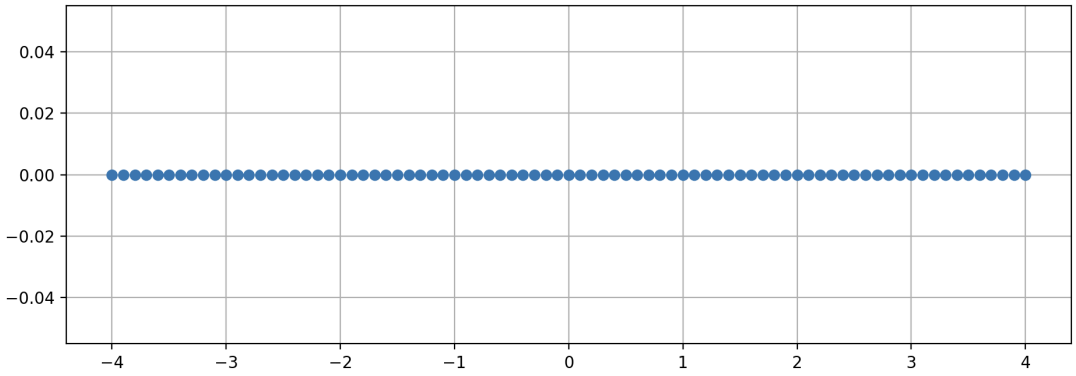
```
xcoord = np.linspace( -4, 4, 81)
```

Solution: Any hand-drawn figure that shows a sequence of 81 intervals of equal length between -4 and 4 (there are 81 nodes) is acceptable. Alternatively, the short Python script:

```
import numpy as np
import matplotlib.pyplot as plt;
   xcoord = np.linspace( -4, 4, 81);
   ycoord = np.zeros(181);

   plt.plot(xcoord, ycoord, 'o' );
   plt.grid();
   plt.show();
```

generates:



Part [1e] (2 pts) Briefly explain how the formatting specification for x and $result$

```
print("        {:7.2f}   {:12.3e}".format( x, result) );
```

can be re-written to take advantage of Python f-strings.

Solution: How about:

```
 print(f"        {x:7.2f}    {result:12.3e}" );
```

Part [1f] (4 pts) Now consider the block of code:

```
ycoord = [];
for x in xcoord:
    result = difficultFunction(x)
    ycoord.append(result)
```

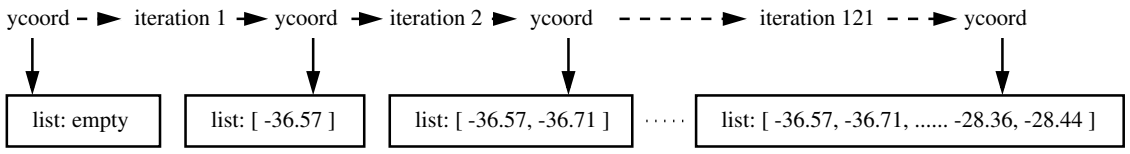
Draw and label the layout of memory for ycoord as it works its way through this looping structure. To keep things manageable, just show an abbreviation of what happens, focusing on the beginning and end of the list assembly. All reasonable answers will be accepted.

Solution: This process can be summarized in a table:

```
Loop   x value   y value                          ycoord list []
-----------------------------------------------------------------
   0       ---       ---                        empty list --> [ ]
   1      -4.0     36.57                                 [ 36.57 ]
   2      -3.9     36.71                           [ 36.57, 36.71 ]

   ... lines of output removed ...

  79       3.9    -28.36             [ 36.57, 36.71, .... -28.36 ]
  80       4.0    -28.44     [ 36.57, 36.71, .... -28.36, -28.44 ]
-----------------------------------------------------------------
```



Question 2: 20 points.

Problem Statement. This question covers use of Python to systematically assemble and draw the box girder bridge cross section shown in Figure 2, and then compute and print the polygon area.

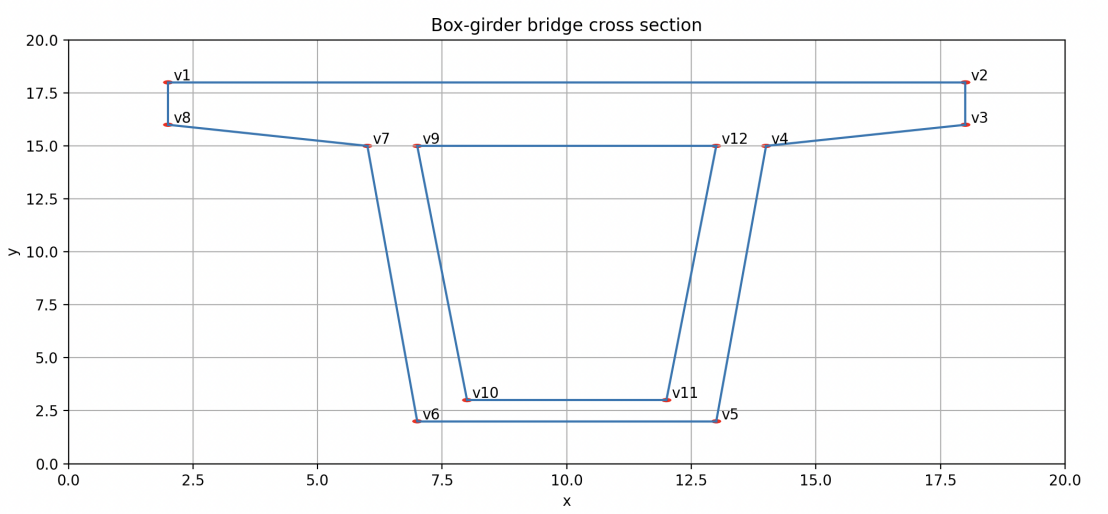


Figure 2. Cross section of box-girder bridge.

In geometry, a polygon with holes is an area-connected planar polygon with one exterior boundary and one or more interior boundaries (holes). An appropriate arrangement of Python scripts and classes for the test program and polygon modeling is shown in Figure 3.

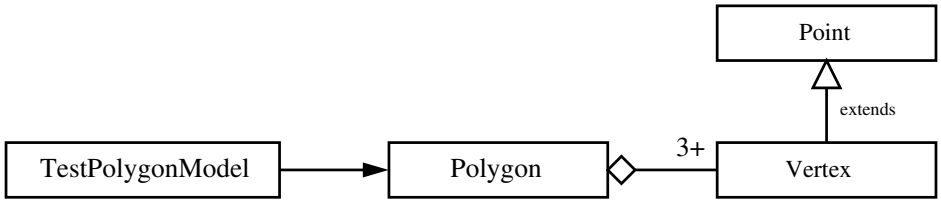


Figure 3. Test program script and classes in a polygon-with-holes system.

Our specification for polygons with holes will be assembled from lists of vertex objects, one list for the exterior boundary, and individual lists for the boundary of each hole. Each vertex will be modeled as a point (data attributes: x,y) with a label (data attribute: name). The individual lists for the boundary of each hole will be stored in a dictionary.

Finally, notice that verticies of the exterior boundary (i.e., *v1*, *v2*, *v3* $\cdots$ *v8*) are organized clockwise, and the verticies of the holes (i.e., *v9*, *v10* $\cdots$ *v12*) are organized anti-clockwise. This modeling convention simplifies the design of algorithms to compute engineering properties, such as the polygon area.

Python Code: The program source code is comprised of four Python files. Look it over carefully and answer the questions that follow:

Object Source Code: Point01.py

```
# ========================================================
# Point01.py: Bare-bones implementation of a Point class ...
#
# Written by: Mark Austin                          October 2025
# ========================================================

class Point:

    def __init__(self, xCoord=0, yCoord=0):
        self.xCoord = xCoord
        self.yCoord = yCoord

    # Get/set X coordinate

    def getX(self):
        return self.xCoord

    def setX(self, xCoord):
        self.xCoord = xCoord

    # Get/set Y coordinate

    def getY(self):
        return self.yCoord

    def setY(self, yCoord):
        self.yCoord = yCoord

    # Return string represention of object ...

    def __str__(self):
        return "( %6.2f, %6.2f )" % ( self.xCoord, self.yCoord )
```

Object Source Code: Vertex01.py

```
# ================================================
# Vertex01.py: Bare-bones vertex ...
# ================================================

from Point01 import Point

class Vertex(Point):
   label = ""

   # Constructor method ...
```

```
  def __init__(self, x, y) :
    Point.__init__(self, x, y)

  # Set/get label ...

  def setLabel(self, label ):
    self.label = label

  def getLabel(self):
    return self.label

  # Assemble string represention of Vertex ...

  def __str__(self):
     vertexinfo = [];
     vertexinfo.append("\n");
     vertexinfo.append("--- Vertex: {:s} ... \n".format( self.getLabel()));
     vertexinfo.append("--- ------------------------------------------------ \n");
     vertexinfo.append("---   Coordinate: (x,y) = {:s} ... \n".format( Point.__self__()));
     vertexinfo.append("--- ------------------------------------------------ ");
     return "".join(vertexinfo);
```

Object Source Code: PolygonModel01.py

```
# ====================================================================
# PolygonModel01.py: Bare-bones implementation of a polygon containing
# holes. Rules of implementation:
#
# -- The polygon exterior will be modeled by a sequence of nodes
#    organized into a clockwise orientation.
# -- Holes will be modeled as loop having nodes organized into an
#    anti-clockwise orientation.
#
# Written by: Mark Austin                                  October 2025
# ====================================================================

from Vertex01 import Vertex

from matplotlib.patches import Circle
from matplotlib.lines import Line2D

class Polygon:
  area       = 0
  perimeter  = 0
  name       = ""
  boundary = []     # <-- coords for polygon exterior ...
  holes    = {}     # <-- dictionary of holes (key = name, value = list of coords) ...

  # Constructor method ...

  def __init__(self, vertexlist):
    self.boundary = vertexlist;          # <--- Assign vertex list to coords ...
```

```
# Set/get name ...

def setName(self, name):
  self.name = name

def getName(self):
  return self.name

# Add hole to polygon model ...

def addHole(self, name, hole ):
  self.holes[ name ] = hole;

# ---------------------------------------
# Compute area of a loop ...
# ---------------------------------------

def getLoopArea( self, coord ):
  norows = len(coord);

  darea = 0.0;
  for i in range(1,norows):
      dx = coord[i].getX() - coord[i-1].getX();
      dy = coord[i].getY() + coord[i-1].getY();
      darea = darea + dx*dy/2.0;

  dx = coord[0].getX() - coord[norows-1].getX();
  dy = coord[0].getY() + coord[norows-1].getY();
  darea = darea + dx*dy/2.0;

  return darea;

# ---------------------------------------
# Polygon Area ...
# ---------------------------------------

def getPolygonArea(self):

    # Compute area of outer boundary polygon ...

    area = self.getLoopArea( self.boundary );

    # Compute area of holes inside polygon ...

    holeKeyList = list( self.holes.keys() )
    for hole in holeKeyList:
        area = area + self.getLoopArea( self.holes.get(hole) );

    return area;

# ---------------------------------------
# Draw loop of edges ...
# ---------------------------------------

def drawLoop(self, ax, coords ):
```

```
    # Draw polygon edges ...

    for i in range( len(coords)-1):
        xcoords = [ coords[i].getX(), coords[i+1].getX() ];
        ycoords = [ coords[i].getY(), coords[i+1].getY() ];
        ax.add_line( Line2D(xcoords, ycoords) )

    lastnode = len( coords) - 1
    xcoords = [ coords[0].getX(), coords[ lastnode ].getX() ];
    ycoords = [ coords[0].getY(), coords[ lastnode ].getY() ];
    ax.add_line( Line2D(xcoords, ycoords) )

    # Draw polygon vertices as small circles ...

    width = 0.1;
    for i in range(len( coords )):
        xcoord = coords[i].getX();
        ycoord = coords[i].getY();
        ax.add_patch( Circle( (xcoord, ycoord), width, facecolor='red') )

    # Draw node labels ...

    dx = 0.1; dy = 0.1
    for i in range(len( coords )):
        xcoord = coords[i].getX();
        ycoord = coords[i].getY();
        label  = coords[i].getLabel();
        ax.text( xcoord + dx, ycoord + dy, label )

# ----------------------------------------
# Draw polygon model ...
# ----------------------------------------

def draw(self, ax):

    # Draw exterior of polygon ...

    self.drawLoop( ax, self.boundary );

    # Draw holes inside polygon ...

    holeKeyList = list( self.holes.keys() )
    for hole in holeKeyList:
        self.drawLoop( ax, self.holes.get(hole) );

# ----------------------------------------
# String represention of simple polygon ...
# ----------------------------------------

def __str__(self):
    polygoninfo = [];
    polygoninfo.append("\n");
    polygoninfo.append("--- PolygonModel: {:s} ...  \n".format( self.name ));
    polygoninfo.append("--- ------------------------------------------------ \n");
```

```
        # String represention of polygon boundary ...

        polygoninfo.append("--- Polygon Boundary \n");
        for i in range(len( self.boundary )):
            xc    = self.boundary[i].getX();
            yc    = self.boundary[i].getY();
            label = self.boundary[i].getLabel();
            polygoninfo.append("---   Vertex {:2d}: (x,y) = ({:6.2f}, {:6.2f}): label = \"{:s}\" .
                               format( i+1, xc, yc, label));

        # String represention of holes ...

        holeKeyList = list( self.holes.keys() )
        for hole in holeKeyList:
            polygoninfo.append("--- Polygon Hole: {:s} ... \n".format( hole ));
            loop   = self.holes.get(hole);
            for i in range(len( loop )):
                xc     = loop[i].getX();
                yc     = loop[i].getY();
                label  = loop[i].getLabel();
                polygoninfo.append("---   Vertex {:2d}: (x,y) = ({:6.2f}, {:6.2f}): label = \"{:s}\
                                   format( i+1, xc, yc, label));

        polygoninfo.append("\n");
        polygoninfo.append("--- Polygon Area = {:f} ...  \n".format( self.getPolygonArea() ));
        polygoninfo.append("--- ------------------------------------------------- ");
        return "".join(polygoninfo);
```

Test Program Source Code: TestBoxGirderPolygonModel01.py

```
# ============================================================================
# TestBoxGirderPolygonModel01.py: Analysis of a box-girder bridge cross section.
# ============================================================================

from Vertex01 import Vertex
from PolygonModel01 import Polygon

import matplotlib.pyplot as plt

# main method ...

def main():
    print("--- Enter TestBoxGirderPolygonModel02.main() ... ");
    print("--- ======================================== ... ");

    print("--- Part 1: List of vertices for polygon exterior ... ");

    v01 = Vertex (  2.0, 18.0 ); v01.setLabel("v1");
    v02 = Vertex ( 18.0, 18.0 ); v02.setLabel("v2");
    v03 = Vertex ( 18.0, 16.0 ); v03.setLabel("v3");
    v04 = Vertex ( 14.0, 15.0 ); v04.setLabel("v4");
    v05 = Vertex ( 13.0,  2.0 ); v05.setLabel("v5");
```

```
    v06 = Vertex (  7.0,  2.0 ); v06.setLabel("v6");
    v07 = Vertex (  6.0, 15.0 ); v07.setLabel("v7");
    v08 = Vertex (  2.0, 16.0 ); v08.setLabel("v8");

    exteriorLoop = [ v01, v02, v03, v04, v05, v06, v07, v08 ];

    print("--- Part 2: Lists of vertices for interior hole ... ");

    v09 = Vertex (  7.0, 15.0 ); v09.setLabel("v9");
    v10 = Vertex (  8.0,  3.0 ); v10.setLabel("v10");
    v11 = Vertex ( 12.0,  3.0 ); v11.setLabel("v11");
    v12 = Vertex ( 13.0, 15.0 ); v12.setLabel("v12");

    hole01 = [ v09, v10, v11, v12 ];

    print("--- Part 3: Assemble and print box-girder polygon ... ");

    polygon01 = Polygon( exteriorLoop );
    polygon01.setName("Box-girder cross section");
    polygon01.addHole( "hole01", hole01 );

    print( polygon01 )

    print("--- Part 4: Draw box-girder cross section ... \n");

    # Define Matplotlib figure and axis

    fig, ax = plt.subplots()

    polygon01.draw(ax)

    plt.title('Box-girder bridge cross section')
    plt.ylabel('y')
    plt.xlabel('x')
    plt.ylim( 0,  20 )
    plt.xlim( 0,  20 )
    plt.grid(True)
    plt.show()

    print("--- ========================================== ... ");
    print("--- Finished TestBoxGirderPolygonModel02.main() ... ");

# call the main method ...

main()
```

Program Output: The abbreviated program output is:

```
--- Part 1: List of vertices for polygon exterior ...
--- Part 2: Lists of vertices for interior hole ...
--- Part 3: Assemble and print box-girder polygon ...

--- PolygonModel: Box-girder cross section ...
```

```
--- ------------------------------------------------
--- Polygon Boundary
---   Vertex  1: (x,y) = (  2.00,  18.00): label = "v1" ...
---   Vertex  2: (x,y) = ( 18.00,  18.00): label = "v2" ...
---   Vertex  3: (x,y) = ( 18.00,  16.00): label = "v3" ...
---   Vertex  4: (x,y) = ( 14.00,  15.00): label = "v4" ...
---   Vertex  5: (x,y) = ( 13.00,   2.00): label = "v5" ...
---   Vertex  6: (x,y) = (  7.00,   2.00): label = "v6" ...
---   Vertex  7: (x,y) = (  6.00,  15.00): label = "v7" ...
---   Vertex  8: (x,y) = (  2.00,  16.00): label = "v8" ...
--- Polygon Hole: hole01 ...
---   Vertex  1: (x,y) = (  7.00,  15.00): label = "v9" ...
---   Vertex  2: (x,y) = (  8.00,   3.00): label = "v10" ...
---   Vertex  3: (x,y) = ( 12.00,   3.00): label = "v11" ...
---   Vertex  4: (x,y) = ( 13.00,  15.00): label = "v12" ...

--- Polygon Area = 75.000000 ...
--- ------------------------------------------------
--- Part 4: Draw box-girder cross section ...
```

Questions: Let's start with Vertex01.py.

Part [2a] (2 pts) What does the line of code:

```
from Point01 import Point
```

do, and why is it needed in this program?

Solution: This line imports the class `Point` from the file `Point01.py`. With this class definition in place, other classes (e.g., Vertex) can customize its implementation through class extension (i.e., build a class hierarchy). This program extends `Point` to create `Vertex`, which in turn, is used by both the `TestBoxGirderPolygonModel01` script and the `Polygon` object model.

Part [2b] (2 pts) Briefly explain how the class hierarchy relationship between Point and Vertex is established (see right-hand side of Figure 3).

Solution: With the single statement: `class Vertex(Point):`

Part [2c] (2 pts) List the methods available to the class Vertex via inheritance from class Point.

Solution:

```
--- getX (self)
--- setX (self, xCoord )
```

```
--- getY (self)
--- setY (self, yCoord )
--- Point.__str__(self)
--- Point.__init__(self, xCoord = 0, yCoord = 0)
```

Part [2d] (2 pts) Briefly explain what the line:

```
vertexinfo.append("---   Coordinate: (x,y) = {:s} ... \n".format( Point.__str__() ));
```

is doing?

Solution: This line adds a new element to the vertexinfo list, a string representation for Point, from which Vertex is derived.

Part [2e] (4 pts) Draw and label a diagram showing the layout of memory generated by the block of code:

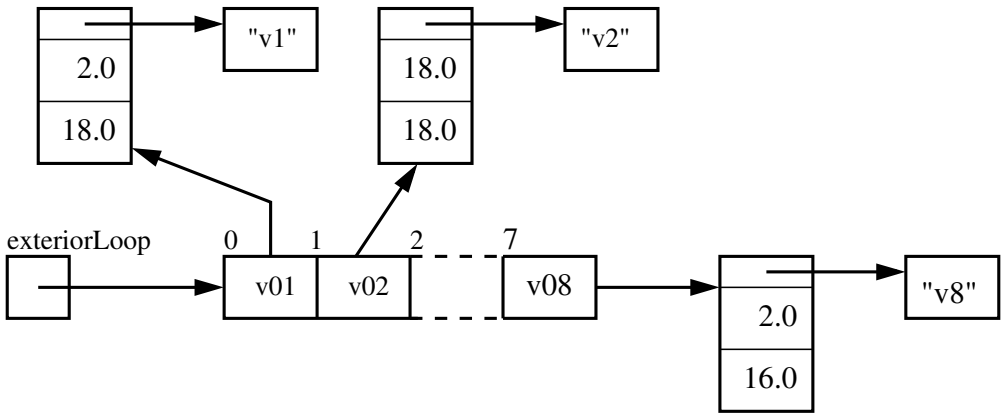
```
v01 = Vertex (  2.0, 18.0 ); v01.setLabel("v1");
v02 = Vertex ( 18.0, 18.0 ); v02.setLabel("v2");

... lines of source code removed ...

v08 = Vertex (  2.0, 16.0 ); v08.setLabel("v8");

exteriorLoop = [ v01, v02, v03, v04, v05, v06, v07, v08 ];
```

Solution:

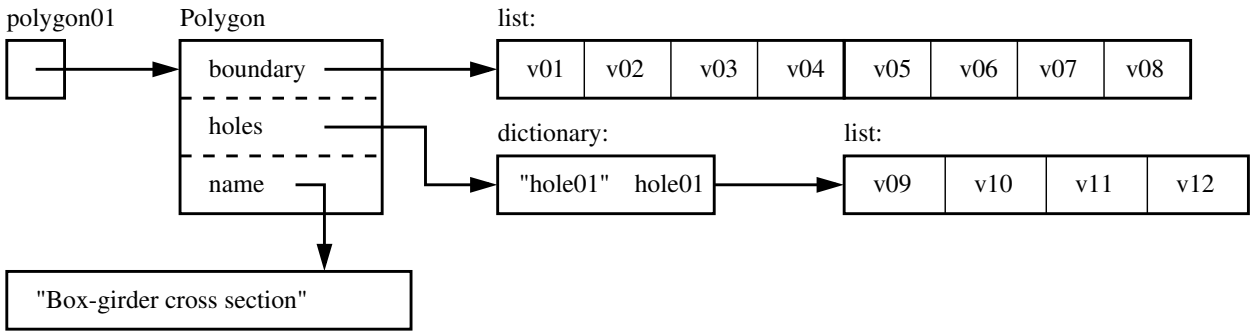


Part [2f] (4 pts) Draw and label a diagram of the layout of memory created by the block of code:

```
polygon01 = Polygon( exteriorLoop );
polygon01.setName("Box-girder cross section");
polygon01.addHole( "hole01", hole01 );
```

Clearly indicate how the boundary list (i.e., boundary = []) and holes dictionary (i.e., holes = {}) are used within the Polygon object.

Solution:



Finally, let's consider the area computation for the box-girder cross section polygon (see Figure 2).

Part [2g] (4 pts) The method `getPolygonArea()`:

```
def getPolygonArea(self):

    # Compute area of outer boundary polygon ...

    area = self.getLoopArea( self.boundary );

    # Compute area of holes inside polygon ...

    holeKeyList = list( self.holes.keys() )
    for hole in holeKeyList:
        area = area + self.getLoopArea( self.holes.get(hole) );

    return area;
```

systematically computes the cross section area by first computing the area of the outer boundary, and then adjusting the area for the inner hole.

Create a table that shows the key stages of the area computation. A reasonable table would include values for *area*, *self.boundary*, *hole*, *self.holes.get(hole)* and results from *self.getLoopArea()*.

Solution:

Task 1: Compute area of exterior boundary:

self.boundary	hole	self.holes.get(hole)	self.getLoopArea()	area
[v1, v2, ..., v8]	---	---	+135	135

Task 2. Subtract hole01:

	hole01	[v9, v10, v11, v12]	-60	75

Result:

			Final Area = 75