Department of Civil and Environmental Engineering, Fall Semester, 2025

ENCE 201 Final Exam, Open Notes and Open Book

Name : ________________________

E-mail (print neatly!): ________________________

Exam Format and Grading. This take home final exam is open notes and open book. You need to comply with the university regulations for academic integrity.

There are five questions. Partial credit will be given for partially correct answers, so please show all your working.

IMPORTANT: Please see the **class web page for instructions on how to submit your exam paper**. Also, before submitting your exam, check that **every page has been scanned correctly**.

Question	Points	Score
1	15	
2	15	
3	10	
4	10	
5	10	
Total	60	

Question 1: 15 points

This question covers solution of matrix equations using Gauss Elimination. Consider the matrix equations Ax = b, where:

$$\begin{bmatrix} 0 & 7 & 3 \\ 3 & 0 & 7 \\ 7 & 3 & 0 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 13 \\ 17 \end{bmatrix}. \tag{1}$$

[1a] (3 pts). Compute the det(A)?

[1b] (3 pts). Based on your solution to part `1a`, what can you say about: (1) the rank of the system of equations, and (2) the number of solutions to the matrix equations?

[1c] (2 pts). In terms of “complexity of matrix structure” and “row operations,” what are the goals of the method of Gauss Elimination? Why does the method work?

[1d] (7 pts). Use the method of Gauss Elimination with pivoting to compute the solution to equation 1. This is a hand calculation, so show all of your working.

Question 2: 15 points

This question covers numerical integration of quadratic and cubic functions with the Trapezoid and Simpson Rules, and Gauss Quadrature.

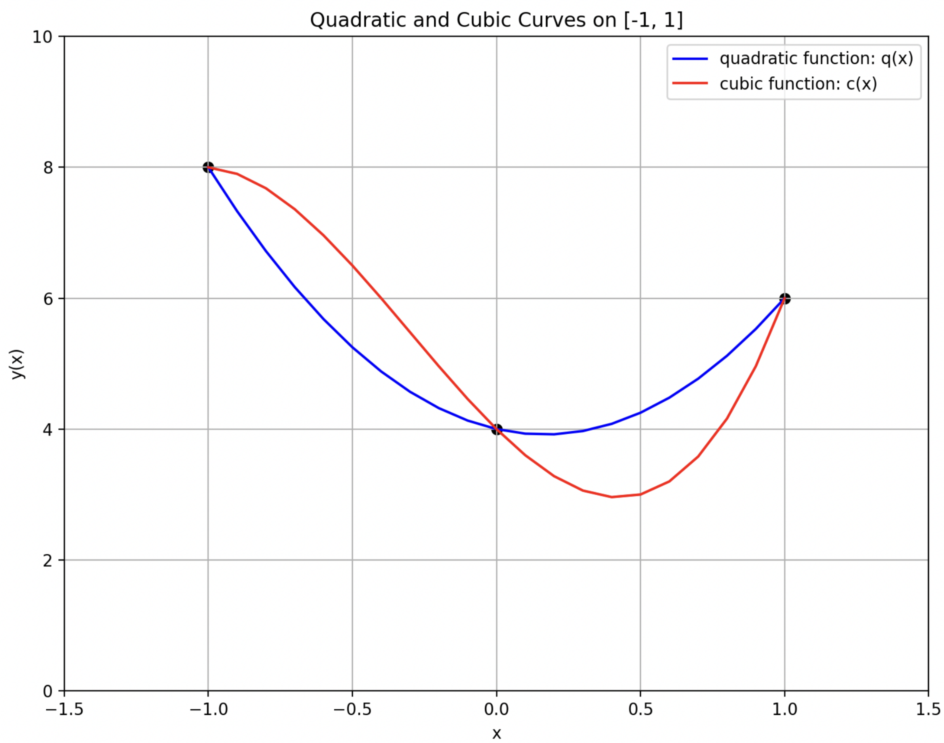

Figure 1. Interpolation of three data points with quadratic and cubic functions.

To motivate the problem setup, Figure 1 shows a scenario where quadratic (i.e., q(x) = $q_o + q_1 x + q_2 x^2$) and cubic (i.e., c(x) = $c_o + c_1 x + c_2 x^2 + c_3 x^3$) curves interpolate three equally spaced data points on the interval $[-1, 1]$.

[2a] (3 pts). Derive an expression for the integral:

$$I_q = \int_{-1}^{1} q(x) dx \qquad (2)$$

in terms of q_o, q_1, and q_2.

[2b] (3 pts). Derive an expression for the integral:

$$I_c = \int_{-1}^{1} c(x)dx \tag{3}$$

in terms of c_o, c_1, c_2 and c_3.

[2c] (3 pts). Show that one step of Simpson's Rule integrates the cubic equation exactly.

[2d] (6 pts). Now suppose that equation 3 is evaluated with **two-point** Gauss Quadrature. Develop a symbolic expression for the integral and actual error.

Question 3: 10 points

This question covers least squares analysis. Carefully read the problem statement and then answer the question that follows.

Problem Statement. Most structural engineering systems have metrics of performance (e.g., maximum allowable deformation) that are relatively insensitive to small variations in material strength, external loads, temperature, and details of construction. An important exception is structural systems required to support large mirrors in telescopes.

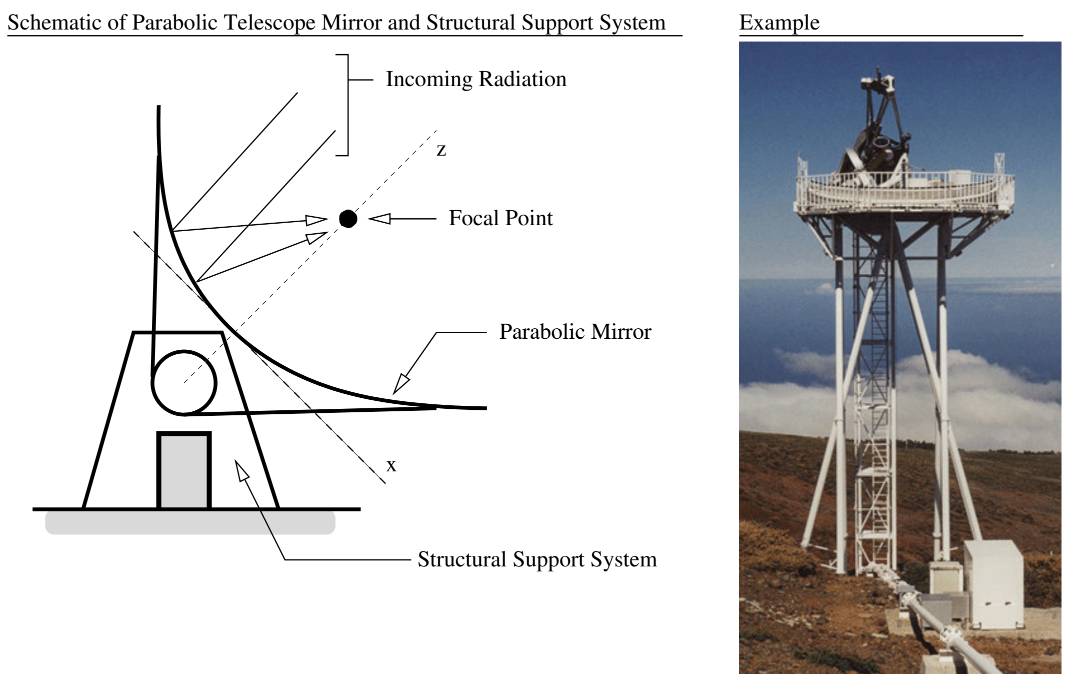

Figure 2. Schematic of telescope and structural support system.

Figure 2 shows that in order for the mirror to reflect incoming radiation on the focal point, the mirror surface must be precisely parabolic, i.e.,

$$z(x,y) = k\left[x^2 + y^2\right] \tag{4}$$

where k is a suitable design constant. But unfortunately, even a small variation in the structural support can cause the mirror surface to distort, thereby rendering the telescope inoperable. Instead of following equation 4 the as-built surface might be:

$$z(x,y) = ax + by + cx^2 + dy^2 \tag{5}$$

The terms $ax + by$ account for shear-like distortions in the mirror surface (ideally, a=b=0). The terms $cx^2 + dy^2$ account for mirror distortions in the axial/radial directions (ideally, c=d=k).

Suppose that an experiment is conducted to measure the profile of an as-built mirror surface, resulting in the data format:

```
---------------------------------
z (mm)      x (mm)          y (mm)
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... details of data omitted ...

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Least squares procedures can be used to detect the presence of distortions, which in turn, can guide adjustments to the support system.

Question. (10 pts.) Starting from first principles of least squares analysis, show that the least squares fit of equation 5 to the data set is given by the matrix equation:

$$\begin{bmatrix} \sum_{i=1}^{N} x_i^2 & \sum_{i=1}^{N} x_i \cdot y_i & \sum_{i=1}^{N} x_i^3 & \sum_{i=1}^{N} x_i \cdot y_i^2 \\ \sum_{i=1}^{N} x_i \cdot y_i & \sum_{i=1}^{N} y_i^2 & \sum_{i=1}^{N} x_i^2 \cdot y_i & \sum_{i=1}^{N} y_i^3 \\ \sum_{i=1}^{N} x_i^3 & \sum_{i=1}^{N} x_i^2 \cdot y_i & \sum_{i=1}^{N} x_i^4 & \sum_{i=1}^{N} x_i^2 \cdot y_i^2 \\ \sum_{i=1}^{N} x_i \cdot y_i^2 & \sum_{i=1}^{N} y_i^3 & \sum_{i=1}^{N} x_i^2 \cdot y_i^2 & \sum_{i=1}^{N} y_i^4 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{N} x_i \cdot z_i \\ \sum_{i=1}^{N} y_i \cdot z_i \\ \sum_{i=1}^{N} x_i^2 \cdot z_i \\ \sum_{i=1}^{N} y_i^2 \cdot z_i \end{bmatrix} \quad (6)$$

where N is the total number of data points. Be sure to show all of your working.

Question 3 cont'd ....

Question 4: 10 points

This question covers function interpolation with the methods of divided differences and Lagrange Interpolation. The whole question is motivated by the small dataset:

x	0.0	2.0	3.0	5.0
f(x)	25.0	25.0	13.0	-65.0

[4a] (5 pts). Use the method of **divided differences** to find a polynomial of lowest order that will fit the dataset.

Be sure to show all of your working.

[4b] (5 pts). Check your answer in Part 4a by computing the functional form via the method of **Lagrange Interpolation**.

Be sure to show all of your working.

Question 5: 10 points

Theoretical considerations indicate that:

$$I = \int_0^{\pi} x \sin^2(x) dx = \frac{\pi^2}{4}. \quad (7)$$

[5a] (4 pts). Use the method of Romberg integration to obtain an $O(h^6)$ accurate estimate of equation 7. Be sure to show all steps in your working.

[5b] (3 pts). Evaluate equation 7 using 2-pt Gauss Quadrature. Be sure to show all steps in your working.

[5c] (3 pts). Evaluate equation 7 using 3-pt Gauss Quadrature. Be sure to show all steps in your working.