Department of Civil and Environmental Engineering, Spring Semester, 2025

ENCE 201 Midterm 2, Open Notes and Open Book

Name : ________________________

E-mail (print neatly!): ________________________

Exam Format and Grading. This take home midterm exam is open notes and open book. You need to comply with the university regulations for academic integrity.

There are three questions. Partial credit will be given for partially correct answers, so please show all your working.

Note: Please see the **class web page for instructions on how to submit your exam paper**.

Question	Points	Score
1	15	
2	15	
3	10	
Total	40	

Question 1: 15 points.

Recall from our class lectures that if $[A]$ is an $(n \times n)$ matrix then, in general, it can be factored into a product of lower and upper triangular matrices, i.e., $[A] = [L]\,[U]$ where $[L]$ and $[U]$ are also $(n \times n)$ matrices. Our examples in class assumed that upper diagonal elements would be unity (i.e., $U_{ii} = 1$), but this is only one way of enabling the factorization. A second possibility is to set the lower diagonal elements to unity (i.e., $L_{ii} = 1$). The key point here is that any set of constraints that reduces the total number of unknowns from $\left(n^2 + n\right)$ to n^2 might work.

[1a] (6 pts). Calculate the LU decomposition for the matrix

$$[A] = \begin{bmatrix} 3 & -1 & 5 \\ 1 & 2 & -3 \\ 4 & 1 & a^2 - 14 \end{bmatrix}. \tag{1}$$

by assuming that $L_{ii} = 1$. Show all of your working ...

[1a] continued ...

[1b] (3 pts). Hence, write down the det[A]?. Note: Do not calculate the determinate by the method of cofactors - there is a much faster way that is a one line calculation!

[1c] (6 pts). Use forward and backward substitution to show that the general solution to $[L]\,[U]\,[x] = [2, 4, a+2]^T$. can be written:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} (8a+25)/(7a+28) \\ (10a+54)/(7a+28) \\ 1/(a+4) \end{bmatrix}. \tag{2}$$

This is a hand calculation, so show all of your working.

[1c] continued ...

Question 2: 15 points.

This question covers numerical solutions to roots of the 4th order equation:

$$f(x) = x^2(x-2)^2 - 1 = 0. \tag{3}$$

at x = 1. Figure 1 plots f(x) over the range $[-1, 3]$.

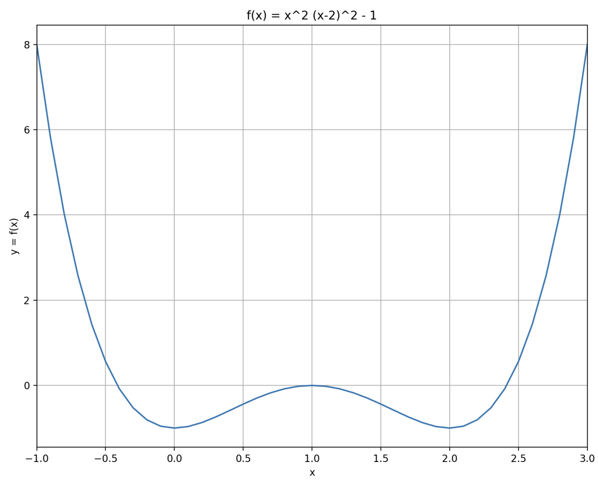


Figure 1. Plot y = f(x) vs x.

[2a] (3 pts). Show that equation 3 has roots at $1 \pm \sqrt{2}$, and a double root at 1. (Note: do not simply substitute the roots into equation equation 3):

Now lets consider numerical solutions to equation 3:

[2b] (3 pts). Show that the Newton-Raphson update formula for solutions to equation 3 can be written:

$$x_{n+1} = \left[\frac{3x_n^3 - 5x_n^2 - x_n - 1}{4x_n\left(x_n - 2\right)}\right]. \tag{4}$$

State all of your assumptions and show all of your working.

[2c] (3 pts). Briefly explain why iterations of equation 4 will struggle to converge to the root at x = 1. Be specific.

[2d] (3 pts). Derive a formula for numerical solutions to equation 3 using Modified Newton-Raphson. **Note:** The answer is a bit long, so I suggest you simply state formulae for the various pieces of the update and how they fit together.

[2e] (3 pts). Use a starting value $x_o = 1.5$ and the Modified Newton Raphson Formula to find an improved estimate of the root of the polynomial. Do no more than 1 iteration !!.

Question 3: 10 points.

This question covers linear algebra.

[3a] (5 pts) A Pythagorean triple is a set of three integers (a,b,c) satisfying constraint $a^2 + b^2 = c^2$. The right triangle with side lengths (3,4,5) is perhaps the simplest and most well known example.

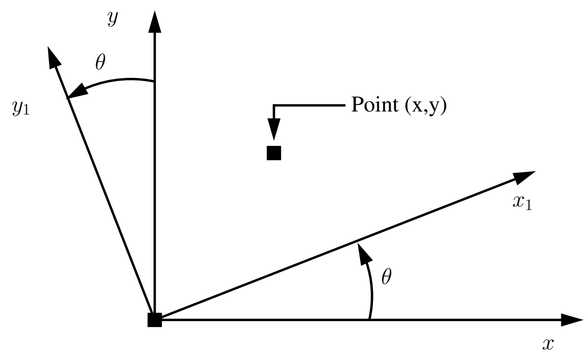
Now suppose that a Pythagorean triple (a,b,c) is written as a (3×1) column vector $v = (a, b, c)^T$, and A is the (3×3) matrix transformation:

$$A = \begin{bmatrix} 1 & -2 & 2 \\ 2 & -1 & 2 \\ 2 & -2 & 3 \end{bmatrix}, \tag{5}$$

Determine whether or not the matrix-vector product Av is also a Pythagorean triple? Show all of your working.

[3a] continued ...

[3b] (5 pts) Suppose that a x-y coordinate system is rotated anticlockwise by an angle θ to create a new coordinate system x_1-y_1.



The matrix product:

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}. \tag{6}$$

describes how points in the x-y coordinate system are transformed into the x_1-y_1 coordinate system. Let us denote the 2-by-2 coordinate transformation matrix A(θ).

For two rotations θ_1 and θ_2 verify that:

$$A(\theta_1)A(\theta_2) = A(\theta_1 + \theta_2). \tag{7}$$

Hence, derive a formula for cos(2θ) in terms of cos(θ) alone. Show all of your working.

[3b] continued ...