Solutions to Midterm 2 Exam

Question 1: 15 points.

Problem Statement. Recall from our class lectures that if $[A]$ is an $(n \times n)$ matrix then, in general, it can be factored into a product of lower and upper triangular matrices, i.e., $[A] = [L]\,[U]$ where $[L]$ and $[U]$ are also $(n \times n)$ matrices. Our examples in class assumed that upper diagonal elements would be unity (i.e., $U_{ii} = 1$), but this is only one way of enabling the factorization. A second possibility is to set the lower diagonal elements to unity (i.e., $L_{ii} = 1$). The key point here is that any set of constraints that reduces the total number of unknowns from $\left(n^2 + n\right)$ to n^2 might work.

Part [1a] (6 pts) Calculate the LU decomposition for the matrix

$$[A] = \begin{bmatrix} 3 & -1 & 5 \\ 1 & 2 & -3 \\ 4 & 1 & a^2 - 14 \end{bmatrix}. \tag{1}$$

by assuming that $L_{ii} = 1$. Show all of your working ...

Solution: With $L_{ii} = 1$, the LU decomposition can be written:

$$\begin{bmatrix} 1 & 0 & 0 \\ L_{21} & 1 & 0 \\ L_{31} & L_{32} & 1 \end{bmatrix} \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix} = \begin{bmatrix} 3 & -1 & 5 \\ 1 & 2 & -3 \\ 4 & 1 & a^2 - 14 \end{bmatrix}. \tag{2}$$

We have 9 matrix terms and 9 unknowns.

Matching terms in the first row of L and U gives:

$$1U_{11} + 0U_{21} + 0U_{31} = 3, \longrightarrow U_{11} = 3. \tag{3}$$

$$1U_{12} + 0U_{22} + 0U_{32} = -1, \longrightarrow U_{12} = -1. \tag{4}$$

$$1U_{13} + 0U_{23} + 0U_{33} = 5, \longrightarrow U_{13} = 5. \tag{5}$$

Matching terms in the second row of L and U gives:

$$L_{21}U_{11} + L_{22}U_{21} + L_{23}U_{31} = 1, \longrightarrow L_{21} = 1/3. \tag{6}$$

$$L_{21}U_{12} + L_{22}U_{22} + L_{23}U_{32} = 2, \longrightarrow U_{22} = 7/3. \tag{7}$$

$$L_{21}U_{13} + L_{22}U_{23} + L_{23}U_{33} = -3, \longrightarrow U_{23} = -14/3. \tag{8}$$

Matching terms in the third row of L and U gives:

$$L_{31}U_{11} + L_{32}U_{21} + L_{33}U_{31} = 4, \longrightarrow L_{31} = 4/3. \tag{9}$$

$$L_{31}U_{12} + L_{32}U_{22} + L_{33}U_{32} = 1, \longrightarrow L_{32} = 1. \tag{10}$$

$$L_{31}U_{13} + L_{32}U_{23} + L_{33}U_{33} = a^2 - 14, \longrightarrow U_{33} = a^2 - 16. \tag{11}$$

Collecting terms gives:

$$\begin{bmatrix} 1 & 0 & 0 \\ L_{21} & 1 & 0 \\ L_{31} & L_{32} & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 1/3 & 1 & 0 \\ 4/3 & 1 & 1 \end{bmatrix} \tag{12}$$

$$\begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix} \longrightarrow \begin{bmatrix} 3 & -1 & 5 \\ 0 & 7/3 & -14/3 \\ 0 & 0 & a^2 - 16 \end{bmatrix} \tag{13}$$

Part [1b] (3 pts) Hence, write down the det[A]?. Note: Do not calculate the determinate by the method of cofactors - there is a much faster way that is a one line calculation!

Solution: det(A) = det(L).det(U) = 7(a^2-16) = 0.

Part [1c] (6 pts). Use forward and backward substitution to show that the general solution to $[L]\,[U]\,[x] = [2, 4, a+2]^T$. can be written:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} (8a+25)/(7a+28) \\ (10a+54)/(7a+28) \\ 1/(a+4) \end{bmatrix}. \tag{14}$$

This is a hand calculation, so show all of your working.

Solution: Two steps:

Forward Substitution: Solve Lz = B, i.e.,

$$\begin{bmatrix} 1 & 0 & 0 \\ 1/3 & 1 & 0 \\ 4/3 & 1 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ a+2 \end{bmatrix} \longrightarrow \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 10/3 \\ a-4 \end{bmatrix}. \tag{15}$$

Backward Substitution: Solve Ux = Z, i.e.,

$$\begin{bmatrix} 3 & -1 & 5 \\ 0 & 7/3 & -14/3 \\ 0 & 0 & a^2-16 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \longrightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} (8a+25)/(7a+28) \\ (10a+54)/(7a+28) \\ 1/(a+4) \end{bmatrix}. \tag{16}$$

Question 2: 15 points.

Problem Statement. This question covers numerical solutions to roots of the 4th order equation:

$$f(x) = x^2(x-2)^2 - 1 = 0. \tag{17}$$

at x = 1. Figure 1 plots f(x) over the range $[-1, 3]$.

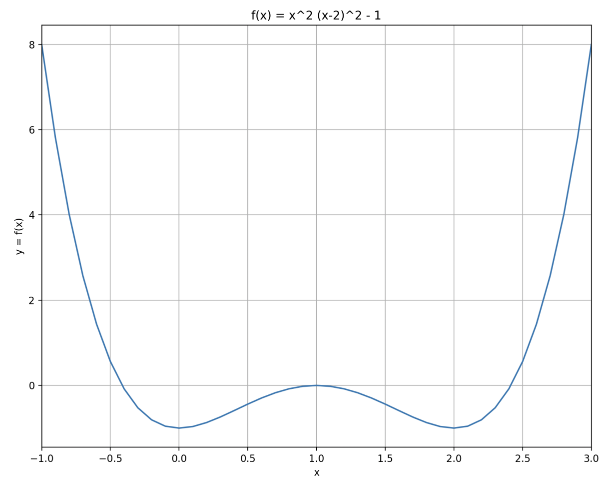


Figure 1. Plot y = f(x) vs x.

Part [2a] (3 pts) Show that equation 17 has roots at $1 \pm \sqrt{2}$, and a double root at 1. (Note: do not simply substitute the roots into equation equation 17):

Solution: Notice that equation 17 is the difference of squares – hence, it can be factored:

$$f(x) = x^2(x-2)^2 - 1 = \big(x^2 - 2x - 1\big)\big(x^2 - 2x + 1\big) = 0. \tag{18}$$

which, in turn, can be written:

$$f(x) = \big(x^2 - 2x - 1\big)(x-1)^2 = 0. \tag{19}$$

The first term leads to roots: $1 \pm \sqrt{2}$. The second term indicates a double root at 1.

Part [2b] (3 pts) Show that the Newton-Raphson update formula for solutions to equation 17 can be written:

$$x_{n+1} = \left[\frac{3x_n^3 - 5x_n^2 - x_n - 1}{4x_n\left(x_n - 2\right)}\right]. \tag{20}$$

State all of your assumptions and show all of your working.

Solution: The standard form for Newton-Raphson update is:

$$x_{n+1} = x_n - \left[\frac{f(x_n)}{f'(x_n)}\right] \tag{21}$$

where,

$$f(x_n) = \left(x_n^2 - 2x_n - 1\right)\left(x_n - 1\right)^2 \tag{22}$$

and

$$f'(x_n) = 4x_n(x_n - 1)(x_n - 2) \tag{23}$$

Plugging equations 22 and 23 into 21:

$$x_{n+1} = x_n - \left[\frac{\left(x_n^2 - 2x_n - 1\right)\left(x_n - 1\right)^2}{4x_n(x_n - 1)(x_n - 2)}\right] \tag{24}$$

Assuming $x_n \neq 1$, then we can cancel the term $(x_n - 1)$ in both the numerator and denominator. Rearranging the remaining terms gives the required result.

Part [2c] (3 pts) (3 pts). Briefly explain why iterations of equation 20 will struggle to converge to the root at x = 1. Be specific.

Solution: As x_n tends toward 1, equation 24 approaches a 0/0 situation.

Part [2d] (3 pts) Derive a formula for numerical solutions to equation 17 using Modified Newton-Raphson. **Note:** The answer is a bit long, so I suggest you simply state formulae for the various pieces of the update and how they fit together.

Solution: The update formula for Modified Newton-Raphson is:

$$x_{n+1} = x_n - \left[\frac{f(x_n)f'(x_n)}{f'(x_n)f'(x_n) - f(x_n)f''(x_n)}\right] \tag{25}$$

The second derivative of f is:

$$f''(x_n) = 4\big(3x_n^2 - 6x_n + 2\big). \tag{26}$$

Equations 22, 23 and 26 are plugged into equation 25.

Part [2e] (3 pts) Use a starting value $x_o = 1.5$ and the Modified Newton Raphson Formula to find an improved estimate of the root of the polynomial. Do no more than 1 iteration !!.

Solution: With x_o = 1.5, equation 22 evaluates to:

$$f(1.5) = \big(1.5^2 - 2 \cdot 1.5 - 1\big)(1.5 - 1)^2 = -0.438. \tag{27}$$

Equation 23 evaluates to:

$$f'(1.5) = 4 \cdot 1.5(1.5 - 1)(1.5 - 2) = -1.5. \tag{28}$$

And equation 26 evaluates to:

$$f''(1.5) = 4\big(3 \cdot 1.5^2 - 6 \cdot 1.5 + 2\big) = -1.0. \tag{29}$$

Substituting equations 27 – 29 into 25 gives:

$$x_1 = 1.5 - \left[\frac{0.438 \cdot 1.5}{1.5 \cdot 1.5 - -0.438 \cdot -1.0}\right] = 1.137. \tag{30}$$

Question 3: 10 points.

Problem Statement. This question covers linear algebra.

Part [3a] (5 pts) A Pythagorean triple is a set of three integers (a,b,c) satisfying constraint $a^2 + b^2 = c^2$. The right triangle with side lengths (3,4,5) is perhaps the simplest and most well known example.

Now suppose that a Pythagorean triple (a,b,c) is written as a (3×1) column vector $v = (a, b, c)^T$, and A is the (3×3) matrix transformation:

$$A = \begin{bmatrix} 1 & -2 & 2 \\ 2 & -1 & 2 \\ 2 & -2 & 3 \end{bmatrix}, \tag{31}$$

Determine whether or not the matrix-vector product Av is also a Pythagorean triple? Show all of your working.

Solution: First, notice that if $v = (3, 4, 5)^T$, then $Av = (5, 12, 13)^T$, which is also Pythagorean triple. So, it seems like the matrix-vector product Av generating Pythagorean triples might be true? To test this hypothesis, let:

$$\begin{bmatrix} 1 & -2 & 2 \\ 2 & -1 & 2 \\ 2 & -2 & 3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} e \\ f \\ g \end{bmatrix}. \tag{32}$$

The matrix-vector transformation Av will be a Pythagorean triple if and only if $e^2 + f^2 = g^2$. From matrix equations 32,

$$\begin{aligned} e^2 &= (a - 2b + 2c)\,(a - 2b + 2c) \\ &= a^2 + 4b^2 + 4c^2 - 4ab + 4ac - 8bc. \end{aligned} \tag{33}$$

and

$$\begin{aligned} f^2 &= (2a - b + 2c)\,(2a - b + 2c) \\ &= 4a^2 + b^2 + 4c^2 - 4ab + 8ac - 4bc. \end{aligned} \tag{34}$$

Adding equations 33 and 34 and noting $a^2 + b^2 = c^2$:

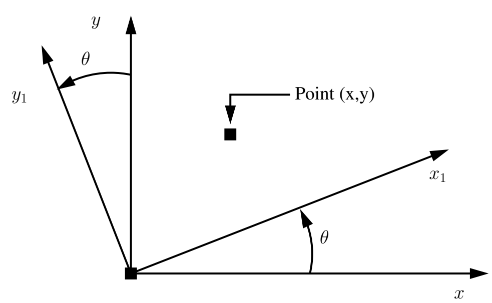
$$
\begin{aligned}
e^2 + f^2 &= 5a^2 + 5b^2 + 8c^2 - 8ab + 12ac - 12bc. \\
&= 13c^2 - 8ab + 12ac - 12bc.
\end{aligned} \tag{35}
$$

Finally,

$$
\begin{aligned}
g^2 &= (2a - 2b + 3c)\,(2a - 2b + 3c) \\
&= 4a^2 + 4b^2 + 9c^2 - 8ab + 12ac - 12bc, \\
&= 13c^2 - 8ab + 12ac - 12bc.
\end{aligned} \tag{36}
$$

Equations 35 and 36 are identical – hence, (e, f, g) is also a Pythagorean triple.

Part [3b] (5 pts) Suppose that a x-y coordinate system is rotated anticlockwise by an angle θ to create a new coordinate system x_1-y_1.



The matrix product:

$$
\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}. \tag{37}
$$

describes how points in the x-y coordinate system are transformed into the x_1-y_1 coordinate system. Let us denote the 2-by-2 coordinate transformation matrix A(θ). For two rotations θ_1 and θ_2 verify that:

$$
A(\theta_1)A(\theta_2) = A(\theta_1 + \theta_2). \tag{38}
$$

Hence, derive a formula for cos(2θ) in terms of cos(θ) alone. Show all of your working.

Solution: Let's start with the right-hand side:

$$
A(\theta_1 + \theta_2) = \begin{bmatrix} \cos(\theta_1 + \theta_2) & \sin(\theta_1 + \theta_2) \\ -\sin(\theta_1 + \theta_2) & \cos(\theta_1 + \theta_2) \end{bmatrix}. \tag{39}
$$

The matrix product is:

$$
\begin{aligned}
A(\theta_1)A(\theta_2) &= \begin{bmatrix} \cos(\theta_1) & \sin(\theta_1) \\ -\sin(\theta_1) & \cos(\theta_1) \end{bmatrix} \cdot \begin{bmatrix} \cos(\theta_2) & \sin(\theta_2) \\ -\sin(\theta_2) & \cos(\theta_2) \end{bmatrix} \\
&= \begin{bmatrix} \cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2) & \cos(\theta_1)\sin(\theta_2) + \sin(\theta_1)\cos(\theta_2) \\ -\sin(\theta_1)\cos(\theta_2) - \cos(\theta_1)\sin(\theta_2) & -\sin(\theta_1)\sin(\theta_2) + \cos(\theta_1)\cos(\theta_2) \end{bmatrix} \\
&= \begin{bmatrix} \cos(\theta_1+\theta_2) & \sin(\theta_1+\theta_2) \\ -\sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) \end{bmatrix} = A(\theta_1+\theta_2).
\end{aligned} \tag{40}
$$

Matching terms in equations 39 and 40:

$$\cos(\theta_1 + \theta_2) = \cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2) \tag{41}$$

Finally, set $\theta_1 = \theta_2 = \theta$:

$$\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta) = 2\cos^2(\theta) - 1. \tag{42}$$