Department of Civil and Environmental Engineering, Spring Semester, 2025

ENCE 201 Final Exam, Open Notes and Open Book

Name : ________________________

E-mail (print neatly!): ________________________

Exam Format and Grading. This take home final exam is open notes and open book. You need to comply with the university regulations for academic integrity.

There are five questions. Partial credit will be given for partially correct answers, so please show all your working.

IMPORTANT: Please see the **class web page for instructions on how to submit your exam paper**. Also, before submitting your exam, check that **every page has been scanned correctly**.

Question	Points	Score
1	20	
2	10	
3	10	
4	10	
5	10	
Total	60	

Question 1: 20 points

Recall from our class lectures that if $[A]$ is an $(n \times n)$ matrix then, in general, it can be factored into a product of lower and upper triangular matrices, i.e., $[A] = [L]\,[U]$ where $[L]$ and $[U]$ are also $(n \times n)$ matrices. Our case study programs with Python and sympy assume that the lower diagonal elements are unity (i.e., $L_{ii} = 1$), but this is only one way of enabling the factorization. A second possibility is to set the upper diagonal elements to unity (i.e., $U_{ii} = 1$). The key point here is that any set of constraints that reduces the total number of unknowns from $\left(n^2 + n\right)$ to n^2 might work.

[1a] (10 pts). Calculate the LU decomposition for the matrix

$$[A] = \begin{bmatrix} 1 & 1 & 0 \\ a & 8 & 4 \\ 0 & 1 & 2 \end{bmatrix}. \tag{1}$$

by assuming that $U_{ii} = 1$. Show all of your working ...

[1a] continued ...

[1b] (3 pts). Hence, write down the det[A]?. Note: Do not calculate the determinate by the method of cofactors - there is a much faster way that is a one line calculation!

[1c] (4 pts). Use forward and backward substitution to show that the general solution to $[L]\,[U]\,[x] = [1,2,b]^T$. can be written:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} (-4-2b)/(a-6) \\ (a+2b-2)/(a-6) \\ (ab-a-8b+2)/(2a-12) \end{bmatrix}. \tag{2}$$

This is a hand calculation, so show all of your working.

[1c] Continued ...

[1d] (3 pts). For what values (a,b) will the system have: (a) a unique solution, (b) zero solutions, (c) infinite solutions?

Question 2: 10 points

This question covers numerical computation of the roots to an equation (i.e., f(x) = 0) using a novel implementation of Newton Raphson iteration. Preliminary work indicates that $f(x)$ has multiple roots at a and, thus, can be factored:

$$f(x) = (x - a)^m h(x). \tag{3}$$

Here $m \geq 2$. Our test equation for this question is:

$$f(x) = x^4 - 12x^3 + 47x^2 - 66x + 18. \tag{4}$$

[2a] (4 pts). Show that the Newton-Raphson update formula for equation 3 can be written:

$$x_{n+1} = x_n - \left[\frac{(x_n - a)h(x_n)}{mh(x_n) + (x_n - a)h'(x_n)} \right]. \tag{5}$$

[2b] (3 pts). Determine appropriate values of a, m and $h(x)$ for equation 4:

[2c] (3 pts). Starting from an initial value, $x_o = 0.0$, compute no more than three iterations of approximation to the lowest root of equation 4. I suggest your organize your computations into a table that shows iteration no, x, $(x - a)^m$, $h(x)$ and $f(x)$.

Question 3: 10 points

This question covers function interpolation with the methods of divided differences and lagrange interpolation, and curve fitting with the method of least squares.

The whole question is motivated by the small dataset:

```
    x |  0.0 |   2.0 |   3.0 |   5.0
-----------------------------------
 f(x) |  0.0 |   8.0 |  27.0 | 125.0
```

[3a] (5 pts). Use the method of **divided differences** to find a polynomial of lowest order that will fit the dataset.

Be sure to show all of your working.

[3b] (5 pts). Check your answer in part 3a by computing the functional form via the method of **Lagrange Interpolation**.

Be sure to show all of your working.

Question 4: 10 points

Figure 1 is a graph of $f(x) = \left[(x-2)^2 - 16\right]$ over the interval $[-2, 8]$.

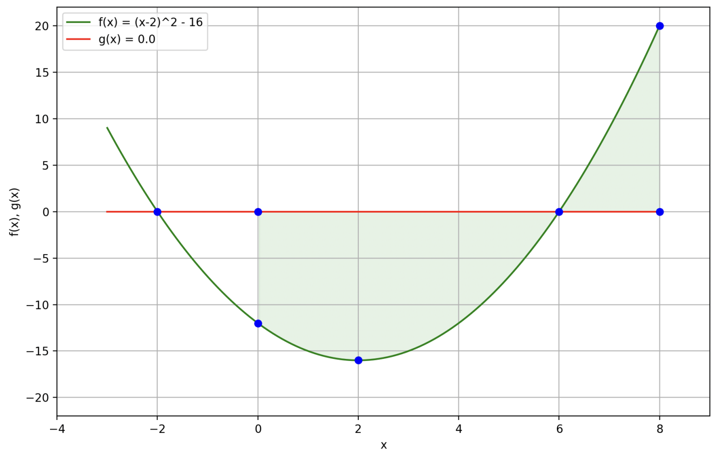

Figure 1. Graph of quadratic curve.

In this question we are interested in numerical approximations to the area of the green shaded region, i.e.,

$$I = \int_0^8 f(x)dx. \tag{6}$$

with the Trapezoid and Simpson's Integration Rules.

[4a] (7 pts). Suppose that equation 6 is estimated by one step of the Trapezoid Rule. Develop an expression for the actual error. Compute the error estimate (which we have covered in class). Is the latter less than the actual error?

[4a] continued ...

[4b] (3 pts). Show that one step of Simpson's Rule integrates the quadratic equation exactly.

Question 5: 10 points

Theoretical considerations indicate that:

$$\int_1^5 x^4 dx = \frac{1}{5}\left[x^5\right]_1^5 = 624.8. \tag{7}$$

[5a] (4 pts). Use the method of Romberg integration to obtain an $O(h^6)$ accurate estimate of equation 7. Be sure to show all steps in your working.

[5b] (3 pts). Evaluate equation 7 using 2-pt Gauss Quadrature. Be sure to show all steps in your working.

[5c] (3 pts). Evaluate equation 7 using 3-pt Gauss Quadrature. Be sure to show all steps in your working.